THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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VOLUME XXXIX

1932

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Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members

$5.00 a Year, Single Copies 60 cents, to Others

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W. D. CAIRNS, *Secretary*

THE MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The twenty-ninth regular meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the University of Richmond, Richmond, Va., on Saturday, May 9, 1931. Sessions were held in the morning and in the afternoon. Professor Clara L. Bacon, chairman of the Section, presided at both sessions.

One hundred and four persons attended the meeting, including the following forty-one members of the Association: O. S. Adams, R. N. Ashmun, Clara L. Bacon, C. C. Bramble, G. R. Clements, A. Cohen, Alexander Dillingham, J. A. Duerksen, Mary Ewin, P. J. Federico, R. E. Gaines, Michael Goldberg, Patricia Gosnell, Isabel Harris, F. E. Johnston, L. M. Kells, W. D. Lambert, Gillie A. Larew, Florence P. Lewis, B. Z. Linfield, J. J. Luck, G. A. Lyle, Florence M. Mears, Ethel I. Moody, Eugenie M. Morenus, F. D. Murnaghan, E. K. Paxton, R. E. Root, Beulah Russell, T. McN. Simpson, Jr., L. W. Smith, Martha L. Smith, J. M. Stetson, Mildred E. Taylor, John Tyler, C. H. Wheeler, III, G. T. Whyburn, John Williamson, E. W. Woolard, R. C. Yates, Oscar Zariski.

A meeting of the executive committee was held before the opening of the morning session, and during the afternoon session a business meeting was held. The nominating committee appointed at the preceding meeting presented its report, and the following officers were unanimously elected for the coming year: Chairman, E. W. Woolard, George Washington University; Secretary-Treasurer, Paul Capron, U. S. Naval Academy; Members of the Executive Committee, Gillie A. Larew, Randolph-Macon Woman's College, and Oscar Zariski, Johns Hopkins University.

Those attending the meeting were entertained at luncheon by the University of Richmond. During luncheon, the chairman presented Dr. Wilhelm Blaschke, who was a guest at the meeting, and who gave an informal talk. Following the afternoon session, many of those present took advantage of a bus tour of historic points in and around Richmond which had been arranged for them. The Section expressed its great appreciation of the opportunity to meet at the University of Richmond, and of the efforts of Professor Wheeler and his colleagues in arranging for the meeting and providing for the entertainment of the group. It is hoped that this meeting may result in bringing the members from the different regions covered by the Section into closer touch with each other.

Following an opening address of welcome by President F. W. Boatwright, of the University of Richmond, the following seven papers were presented, of which the first five were delivered at the morning session and the last two at the afternoon session:

1. "Primitive roots of prime numbers" by F. E. Johnston, George Washington University.

2. "Rotations in four-dimensional space" by B. Z. Linfield, University of Virginia.

3. "The restricted use of the ruler in constructions of projective geometry" by Oscar Zariski, Johns Hopkins University.

4. "Some applications of Carathéodory's method of geodetic equidistance" by Gillie A. Larew, Randolph-Macon Woman's College.

5. "The potential of a homogeneous cylinder" by Walter D. Lambert, U. S. Coast and Geodetic Survey.

6. "A direct analytical proof of Pascal's theorem and related theorems" by T. L. Wade, University of Virginia, by invitation.

7. "Homologous rectifications of curves" by John Tyler, U. S. Naval Academy.

Abstracts of some of these papers follow:

1. The linear congruence $ax \equiv b \bmod p$, where a and p are relatively prime, has a unique solution. If for a given a, b runs through the $p-1$ totitives of p, there result $p-1$ congruences, whose solutions are the totitives again in some order. This set of equations will define a substitution on the b's, if it is assumed that each constant term in an equation is to be replaced by the solution of that equation. If then a runs through the $p-1$ totitives, there result $p-1$ substitutions which form a regular group, simply isomorphic with the group formed when the totitives are combined by ordinary multiplication. The sum of the numbers in each cycle of each substitution is congruent zero modulo p. If d, a divisor of $p-1$, is of the form p^2k, the group contains a substitution such that each cycle which contains a totitive belonging to the exponent d contains only such totitives. If $d=p_1$, the group contains a substitution involving a cycle consisting of unity and the numbers which belong to the exponent d. If $\sigma(d)$ denotes the sum of the numbers belonging to the exponent d, we thus have $\sigma(p_1) \equiv -1 \bmod p$. By induction we have $\sigma(d) \equiv (-1)^\lambda \bmod p$ if $d = p_1 p_2 \cdots p_\lambda$, where the p_i are distinct primes. These results taken together give a new and simple proof of Stern's generalization of Gauss's theorem which states:

$$\sigma(d) \equiv 0 \bmod p \text{ if } d = p_1^2 K,$$
$$\sigma(d) \equiv (-1)^\lambda \bmod p \text{ if } d = p_1 \cdots p_\lambda.$$

2. In 3-dimensional space it is well known that each 3-dimensional orthogonal matrix whose determinant is 1 determines a rotation leaving a line invariant. In 4-dimensional space, Professor Linfield pointed out, not every 4-dimensional orthogonal matrix whose determinant is 1 represents a rotation, but every such matrix is factorable into a product of two orthogonal matrices each of which represents a rotation leaving a 2-dimensional plane invariant. He also indicated extensions to n-dimensional space.

3. The purpose of this paper is to show that all linear constructions of projective geometry, i.e., all constructions which require only the use of the ruler, can also be performed if the use of the ruler is restricted to the drawing of lines through 4 fixed points A, B, C, D, not all on a line. It is only necessary to show

that given two arbitrary pairs of points, X, Y and X', Y', it is possible to find the point of intersection of the lines XY and $X'Y'$ by means of a construction which involves only lines passing through either one of the 4 fixed points. This is done by considering successively the following cases: (1) The line $X'Y'$ passes through one of the 4 fixed points, say C, the point X is on the line joining two of the remaining 3 points, say on the line AB, and the point Y is arbitrary. (2) The line $X'Y'$ subject to the same condition as in the previous case, while the points X and Y are arbitrary. (3) The points X, Y and X', Y' are taken arbitrarily.

The success of the restricted use of the ruler, as described above, is significant if considered from the point of view of the foundations of projective geometry. It shows that 4 pencils of lines in a projective plane whose centers are not all on a line determine completely *all* the lines of the plane, if we require that the theorem of Desargues and the fundamental theorem of projective geometry should hold. In a sense we can say that when we give all the lines of the plane, we give too much. However it is by no means easy to give a set of assumptions, involving only the 4 given pencils of lines, which should assure the validity of the theorem of Desargues and of the fundamental theorem in the projective plane which is thus determined. Certain necessary assumptions are easily obtained, which are quite similar to those which were introduced by Blaschke in his elegant *Textilmathematik* and by Reidemeister. It can also be seen that if of the 4 centers of the given pencils 3 are on a line, then these assumptions are also sufficient. The general case requires a further investigation.

4. Carathéodory's method of geodetic equidistance is briefly sketched in relation to the simplest problem of the calculus of variations. The method is then applied to the Mayer problem, and the Weierstrass necessary condition is established in a simple and direct fashion. The theory is also used to derive a transversality condition for a certain problem with variable endpoints.

5. In recent experiments at the United States Bureau of Standards for determining the Newtonian constant of gravitation, a cylinder was used as the large attracting mass instead of the usual sphere. This was because it was believed that it would be easier for the machinist to approximate to a mechanically perfect cylinder than to a mechanically perfect sphere. The machinist's difficulties were thus unloaded on the mathematician, who was obliged to derive the necessarily complicated formulas for the attraction of a cylinder on a point close to it. In this paper the formulas used by the Bureau of Standards are derived. These formulas offer certain advantages but in spite of the symmetry about the axis of the cylinder they involve three variables, not all, of course, independent; and in differentiating the potential to obtain the attraction it is necessary to attend to the relations among these variables. Other formulas in terms of the more conventional polar coordinates are also derived. These formulas are an infinite series of terms, each consisting of the product of a coefficient, a power of the radius vector and a zonal harmonic. The argument of the zonal harmonic is the angle between the radius vector and the axis of a cylinder. The coefficients involve

zonal harmonics also, but the argument in these cases is independent of the position of the attracted point and depends only on the proportions of the cylinder.

6. If we interpret the three-dimensional vectors A_1, A_2, A_3 and B_1, B_2, B_3 of the matrices A and B in the equation $B = C \times A$ as the homogeneous coordinates of the vertices of two triangles, the matrix C determines six scalars (the elements in the expansion of $|C|$):

$$\begin{aligned} I_1 &= C_{11}C_{22}C_{33}, & J_1 &= -\,C_{13}C_{22}C_{31}, \\ I_2 &= C_{12}C_{13}C_{31}, & J_2 &= -\,C_{21}C_{33}C_{12}, \\ I_3 &= C_{13}C_{32}C_{21}, & J &= -\,C_{11}C_{23}C_{32}, \end{aligned}$$

which, as shown by Mr. Aylor in his University of Virginia thesis, are relative invariants under all linear transformations, and the quotient of an I and a J is an absolute invariant. Furthermore, the necessary and sufficient condition that the six points A_1, A_2, A_3 and B_1, B_2, B_3 lie on a conic is that

$$F(I, J) \equiv I_1I_2 + I_2I_3 + I_1I_3 - J_1J_2 - J_2J_3 - J_1J_3 = 0.$$

We will now use the condition that $F(I, J) = 0$, to prove Pascal's theorem.

If the vertices of the hexagon inscribed in a conic are taken in the order A_1, A_2, A_3, B_1, B_2, B_3, then the opposite sides are paired as below, the points of intersection being P, Q, R:

$$\left.\begin{matrix} A_1A_2 \\ B_1B_2 \end{matrix}\right\} P \qquad \left.\begin{matrix} A_2A_3 \\ B_2B_3 \end{matrix}\right\} Q \qquad \left.\begin{matrix} A_3B_1 \\ B_3A_1 \end{matrix}\right\} R.$$

Solving the equations of the sides of the hexagon for the points of intersection, we have

$$\begin{aligned} P &= C'_{31}A_2 - C'_{32}A_1, \\ Q &= C'_{12}A_3 - C'_{13}A_2 \quad (C' = \text{adj. } C) \\ R &= C_{32}C_{11}A_1 + C_{32}C_{12}A_2 + C_{12}C_{33}A_3. \end{aligned}$$

The condition that these three points P, Q, R be collinear, i.e., that Pascal's theorem be true, is that the determinant, Δ, of the vectors P, Q, R shall vanish. By introducing the I's and J's in the expansion of Δ we can write

$$\Delta = |A| \times F(I, J),$$

and consequently Δ is zero, and Pascal's theorem is proved.

Similar proofs of the related theorems of Steiner and Kirkman are likewise based upon the invariant relation that six points lie upon a conic.

Edgar W. Woolard, *Secretary*

FUNCTIONS OF THE MATHEMATICAL ASSOCIATION OF AMERICA[1]

By J. W. YOUNG, Dartmouth College

The American Mathematical Society is devoted to research—it will voluntarily have nothing to do with anything else. This singleness of purpose is doubtless a source of strength to the Society. It would, however, be a source of weakness to mathematics as a whole, and, indeed, to the development of research itself were not other important phases of mathematical enterprise taken care of elsewhere. It makes the definition of the functions of the Association very easy: Everything that is worth doing for mathematics, other than research, is a function of the Association. The latter is the proverbial "George" of the mathematical family.

George is or should be a very busy person. The Association has during its sixteen years of existence done many valuable things. It is not my purpose to pass them in review. Rather would I consider certain new activities which, in my opinion, need to be inaugurated or stimulated.

It may clarify my purpose if I say a word as to the immediate background from which I approach my present task. At the summer meeting a year ago, the Society and the Association appointed a Joint Committee on Funds, which was charged with the investigation of the financial needs of the two organizations and the consideration of ways and means of satisfying them. The Society was and is facing difficult financial problems. The Association has been operating strictly within its assured income; but there are a number of things it wished to do, which it felt it should do, for which additional funds were necessary. The work of this Committee has raised a number of fundamental questions. What is most worth while and most urgent? What is important and why? The financial problems of the Society are largely, though not wholly, connected with the cost of its publications. The research output has been increasing rapidly during the past years; the publications have been correspondingly expanded. If our editorial policy remains the same, these publications will have to continue their expansion. Their cost must then continue to increase. Is this policy sound—or has the time come to raise materially our editorial standards? Are we publishing too much? Should the Society solve its financial problems by retrenchment?

If not, how shall the additional expense be met? What additional sources of support can mathematics find? What are the underlying conditions under which additional sources of support can be tapped? These are a few of the questions which have arisen and they suggest questions still more fundamental. It is some of these more fundamental questions which I wish to raise this afternoon and to invoke the active help of our membership in finding solutions. The questions offer, as you will see, a large variety; their solution will call for varied abilities. Among the more than two thousand members of the Association there surely can be found the men and women who have the ability, the energy, and the enthusiasm to tackle these various tasks successfully. I am issuing a general call

[1] Retiring presidential address delivered at Minneapolis, Sept. 8, 1931.

to service to every member of the Association—let each one find the particular phase of the work for which he or she is best fitted. There is much that needs to be done, and every one can make his or her contribution somewhere.

In the minds of most the Association is thought of as representing the field of collegiate mathematics. Lying between the work of the graduate school on the one hand and the field of secondary mathematics on the other it would inevitably have contacts with both. But circumstances force it actually to penetrate the adjacent domains. Owing to the singleness of purpose of the Society referred to at the beginning of my address, any problem relating to the organization of graduate instruction falls into the proper domain of the Association. I do not propose to raise such problems on this occasion, although they exist. I mention the fact merely to indicate the scope of the work that George may interest himself in. In the field of secondary education, we might expect the National Council of Teachers of Mathematics to carry on. Unfortunately, from some points of view at any rate, the National Council reflects the point of view of our schools of education rather than that of our departments of mathematics. Their official organ, The Mathematics Teacher, seems to me to be almost altogether "teacher" and hardly at all "mathematics." So that here again any real mathematical interests seem to be the function of the Association.

The first question I wish to raise lies perhaps on the border line between the activities of the Society and those of the Association. When is a piece of research important? Every research man has to answer this question, consciously or subconsciously, with respect to his own research. The editors of our journals are consciously facing it all the time. And yet we have no explicitly formulated canons of critical evaluation. We have certain standards of rigor; we insist on originality of result or of method. But beyond this we appear to be guided by little more than an instinctive sense. Literature, art, and the drama develop their critics; why not mathematics? True, we have a kind of criticism in our book reviews; and occasionally indeed a general principle of evaluation may be inferred between the lines of such a review. But this is far removed from the problem of formulating general criteria of evaluation. We are here facing an almost untouched field of inquiry of, it seems to me, great importance.

The problem is an extraordinarily difficult one, which is doubtless one of the reasons why so little has been done with it. Generality is one of the criteria that first comes to mind—generalization has doubtless been one of the most effective modes of progress. May we then infer that the more general a concept, a theorem, a process is, the more important it is? Hardly. Some of you will recall a paper by the late Professor A. D. Pitcher published some years ago in the Bulletin, in which from the point of view of general analysis he sought a generalization of the idea of "continuity." The means employed was the concept of the "development" of the range of the general variable. On the basis of this concept he arrived very naturally at the definition of "continuity with respect to a given development." He then proved the surprising theorem that *every function is continuous with respect to some development of the range.* The result

may be of interest—I am inclined to think it is—but as a generalization of continuity it is obviously futile.

Generalization to be useful is clearly subject to limitations. About *anything* one can say precisely *nothing*, except that the underlying concept is self-contradictory. Unlimited generalization implies not merely futility but paradox. This particular path to virtue leads ultimately to unforgivable sin. Just where along this path should the would-be righteous stop? What precisely, or even vaguely, are the limitations that should be imposed on generality? I do not know.

And generalization is only one of many principles of progress; others raise similar questions. And one consideration shows that any final evaluation of contemporary research is quite impossible. The importance of a piece of research often depends on the unforeseeable future. History is full of examples which show that later developments have resurrected results or processes which for a shorter or longer period had been neglected or forgotten and exhibited them as of prime importance. A notable example will appear presently in a quotation.

Some may feel that our lack of a literature of criticism is a healthy sign. As long as mathematical research is vigorous and progress is rapid, there is no impulse toward critical thought. To such, criticism is a sign of decadence. I do not agree. I think the problem I have suggested is one of great interest and great potential value. Are there not some of our members who will seriously attempt its solution?

In any case the bearing of what I have said on the immediate problems facing the Society should be obvious. Without any more valid canons of evaluation than we now possess, the cutting down on the volume of our publication would be a calamity, which can be justified only on the ground of financial necessity. This, incidentally, is the opinion of the overwhelming majority of those consulted by the Joint Committee on Funds.

The following quotation from a letter received by the Committee from Professor Bell shows how strongly some feel on the proposal, made by a small but responsible group in the Society, to reduce materially the volume of publication by raising the standard of editorial acceptance and relegating the manuscripts which do not meet these higher standards to the waste basket. Professor Bell writes:

"This is the Jehovah complex in its most dangerous form; it is arrogating to oneself prescience and wisdom which some of us still like to think belong only to Almighty God. Who of these men, or of any one else for that matter, now knows what will be looked at fifty years hence, or even ten years hence." Professor Bell after citing a number of other examples refers to the recent work of Ricci. "Until Einstein rubbed their noses in it, the differential geometers ignored Ricci's work and everything that has come out of it, secure in their belief that it was mere formalism because some pundit had so stigmatized it."

In view of some recent discussions I feel that I should add at this point, in order to avoid possible misunderstanding, that what I have said refers solely to

standards of content and not at all to standards of form. It may well be that editors should insist on more careful methods of exposition and that thereby not only could space in our journals be saved but also the value of the papers enhanced.

Related to what I have just been discussing is historical research in mathematics, the fostering of which has long been recognized as a function of the Association. The latter is all set to revive the Bibliotheca Mathematica—it is prevented only by a lack of the necessary funds. May I take this opportunity to voice the hope that our historians and those of us who are interested in historical developments without having as yet qualified as historians, will in the future not confine themselves so exclusively to what might be called the archeology or the ethnology of mathematics. The history of mathematics among the ancients or in the middle ages and the renaissance and among the Chinese and Hindus is doubtless of interest. But I venture the assertion that the history of modern mathematical disciplines is of equal interest and of much greater potential value to present progress.

Here also I may quote to advantage a paragraph from the same letter of Professor Bell: "I am fairly well acquainted," he writes, "with some branches of technical scientific history, and I can state with fair confidence that there is nowhere a work approaching in completeness and usefulness Dickson's *History of the Theory of Numbers*. Americans have made a splendid beginning in this sort of work and, as it was their own idea, they should keep it up . . .. The great usefulness of such histories to research workers is obvious to any one who has used such a history." Here is another important field of work for some of our members to cultivate. Won't some of them enter it?

I come now to the functions of the Association relating principally to collegiate mathematics. The improvement of teaching is largely a question of improving the teachers. The meetings of the Association and of its various sections serve to stimulate enthusiasm and offer opportunity for the exchange of ideas. It would like to expand its missionary work in the less favored sections of the country; it would like to send speakers and lecturers into those districts where stimulation is most needed. This requires additional funds not at present available. The publication of the series of Carus monographs is making brief expository treatment of important topics available. The American Mathematical Monthly through its articles, discussions, and problems provides an additional and effective method of stimulation. The evolution of the Monthly from a small struggling pamphlet to its present size and standing forms an interesting chapter in the development of mathematics in this country. We can all rejoice in its progress and its increased influence. As we look back over this evolution, however, we note a marked shift in its appeal. The Monthly at present is addressed primarily to the college teacher; it used to be of interest to the college student, as well. Nowadays, it has little to offer the undergraduate, except possibly the senior who is majoring in mathematics. Its problem department is in general beyond the ability and training of the undergraduate in his earlier years.

This shift of appeal was doubtless inevitable. The present Monthly is occupying a well defined position and is meeting a well-defined need. I would not be understood as criticizing in any way its editorial policy. I am merely calling attention to the fact that it has, in its progress, left behind a field of usefulness which it used to occupy and which is now being neglected. In other words, I believe there is need for a new periodical devoted to the mathematical interests of undergraduate students.

The embryology of a mathematician is an undeveloped branch of science. Somewhere during his student days some stimulus fertilizes his latent mathematical interests. In rare instances this important event may take place before he enters college; in most cases, however, it occurs during his freshman or sophomore years or later. The nature of this stimulus doubtless shows great variety. It would be interesting and I think not without value, if we could know just what agency kindled the spark and what influences nurtured the flame that started our members on their careers and kept them there, in spite of all rival attractions.

In any case, to attract men and women of ability to our subject is one of our important functions; we do so, presumably, by attempting to arouse and maintain a vital interest in mathematics among those properly qualified. The establishment of undergraduate mathematical clubs is a means to this end. Some of us, perhaps many of us, can testify that the problem department of the Monthly a generation ago was a powerful stimulus in this direction. During my own undergraduate days I was a subscriber to the Monthly and used to have great fun in attempting to solve the problems proposed. The programs of the clubs just referred to show that there are many topics of interest to undergraduates. Occasionally, indeed, papers presented at such meetings would merit publication if a suitable organ were available. Think of the added stimulus in the preparation of such a paper, if the author could look forward to the possibility of its appearing in print.

And so I suggest the publication under the auspices of the Association of a new magazine devoted to the mathematical interests of undergraduates and to others, as will appear presently. A prominent feature of this new magazine would be a department of problems within the range of ability of undergraduates, including some to the solution of which a freshman could aspire. They should be interesting, unusual, not of the conventional text book variety. Such problems exist, though they may not always be easy to find or to invent. The magazine would contain articles of interest to undergraduates, some perhaps, as has been suggested, written by undergraduates. It would take too long to elaborate in detail the plans for such a magazine—nor am I qualified to do so. A further source of articles for such a magazine will appear presently. At this point I wish merely to venture the assertion that this project would probably not require any additional funds. Such a magazine would I believe become self-supporting and might indeed be a source of additional income to the Association. It need not, in fact I think it should not, be a large magazine. Its subscrip-

tion price should be low, not more than \$2.00 annually for say ten numbers published during the college year. But its subscribers should be many. We should expect subscribers among the students and faculties of every college of reasonable standing in the country; the mathematical clubs would be our agents. It should get subscribers from among the more enlightened, progressive, ambitious secondary school teachers; it should do for the mathematical interests of such teachers what the Mathematics Teacher does for their pedagogical interests. It would, I venture to say, get subscribers from the general public. The kind of magazine I have in mind would appeal to all those, relatively few in number but in the aggregate numerous, who have mathematical interests. It should be, according to my vision, an agency for the popularizing of mathematics. But of that more presently—I am getting a bit ahead of my story.

Does the project strike a responsive chord in the minds and hearts of some of you? Among our more than two thousand members does the man or woman exist who has the ideals, the enthusiasm, the courage, the tact, the organizing ability to carry this project to success as editor-in-chief? Do we have in our membership the men and women able and eager to cooperate? It is a large and difficult project; it will have to command the enthusiastic and active support of a considerable number to make it go.

I referred a moment ago to the popularizing of mathematics. I fancy that some will be repelled by the suggestion. They will feel that it can't be done; or at least that it can't be done without loss of dignity, without loss of scientific accuracy. And mathematics, they will say, without precision is not mathematics. I will grant the latter. I will also grant that popularization must be used in a relative sense; some degree of familiarity with and interest in elementary mathematics must be presupposed on the part of the prospective reader. But this much being granted the production of a relatively popular mathematical literature is possible and highly desirable. That it is possible in such abstract domains as modern mathematical physics is shown by the recent books of Whitehead and Eddington; in the domain of pure mathematics, by that of Professor Danzig. That there is a considerable demand for such books on the part of the general public is shown by the sale which these books have enjoyed.

Some of you are perhaps familiar with the recent delightful little book by Toeplitz and Rademacher entitled "Von Zahlen und Figuren"—in which a large variety of topics from the domain of advanced mathematics are discussed with accuracy and clarity, on the basis of a minimum amount of technical mathematical training. You are all familiar with Klein's "Elementarmathematik vom höheren Standpunkte aus." I am here suggesting the opposite point of view as offering a valuable and important field of enquiry: "*Advanced mathematics from the elementary point of view.*" We have done little with this field in this country —and it is worthy of our best efforts.

It is, indeed, a field not without real scientific interest. May I illustrate what I mean by a personal experience. At a recent meeting in New York the program committee organized a symposium on "Recent work in Differential Geometry;"

one of the lecturers on that occasion was Professor Struik and his topic was "Differential Geometry in the Large"—a topic that has been developed since about 1912. It is, as most of you know, concerned with differential properties of curves and surfaces as a whole rather than at a point—it is in fact in its present stage largely concerned with properties of ovals and their counterpart in space. The theorem that every oval has at least four vertices—i.e. points at which the curvature is either a maximum or a minimum—is typical. Many interesting theorems relate to ovals of constant width. Professor Blaschke, who spent some months in this country last year, is one of the principal proponents of this domain.

I was greatly interested in Professor Struik's lecture at the symposium. He gave a very elegant proof of the four-vertex-theorem using the idea of a vector integral. I had an instinctive feeling that this and other theorems on ovals should, on account of their simplicity, be capable of more elementary treatment. A few weeks later I had succeeded in proving the four-vertex-theorem and a considerable number of theorems about ovals of constant width as limiting cases of equiangular polygons. Most of the latter theorems were new to me and they seemed distinctly amusing. An examination of the literature showed that they had all been derived by more advanced methods between the years 1912–1916. My methods were almost ridiculously elementary—I used nothing more advanced than elementary trigonometry to prove my theorems on polygons—a college freshman could follow the argument—and as to the limiting processes I needed only the fact that the limit of sin θ/θ is equal to unity. It was originally my intention to make these developments the subject of my present address—I think you would have found them entertaining.

What I have just described suggests another remark concerning the subject of critical evaluation which I discussed earlier. I wonder if we do not often measure the importance of a result by the difficulty of its derivation. Suppose, what is quite possible, that some obscure teacher in a college or high school had by the methods I used derived the theorems just referred to, say twenty years ago, when they were new. Would our editors have considered them important? Would the Transactions, for example, have accepted the paper? I seriously doubt it. Would they not have said "Very pretty—but hardly up to our standard. Send it to the Monthly," or words to that effect. Perhaps they would have been right—perhaps these results are merely pretty and not really important. How are we to tell? In any case, however, they are important or not in themselves and not on account of the method of their derivation.

To return to my original topic. My illustration is a good example of what I mean by "advanced mathematics from an elementary point of view." Does anyone believe for a moment that it is an isolated example? There must be many others awaiting discovery. Some years ago Professor Coolidge gave an address to our undergraduate club at Dartmouth in which he derived many of the geometric properties of the cycloid very easily by elementary geometric methods without the use of the calculus.

Here then is another large field of inquiry which I wish to call to the attention of our members as very worthy of cultivation. As I have indicated it seems to be of real scientific value—especially to those of us who have a perhaps irrational faith that mathematics is essentially simple—that complexities are temporary blemishes which future insight will remove. Its bearing on the popularization, relatively, of mathematics is obvious.

There is another aspect of popularization that is of tremendous importance from the practical point of view of securing financial support for mathematical enterprise. I refer to the popularization or perhaps, better, publicising of the value of mathematical research. The greatest obstacle that lies in the way of securing funds is popular ignorance on this subject. The educated layman knows that mathematics is an important subject, that it has important applications in engineering, in applied science, and presumably in other fields. A small proportion are perhaps sufficiently well informed to realize, if they happen to give the matter any thought, that our civilization is largely built on a mathematical foundation; that, in its material manifestations at least, it could not have arisen and could not continue to exist without mathematics. But there are very few who have any conception of mathematics as alive and growing. To practically every "man-on-the-street"—including of course Wall Street—the practical aspects of mathematics were developed long ago. He has no conception of the fact that the research of today becomes the application of tomorrow, if indeed he has any realization that there is any research today.

This ought not to be. Whatever may be the opinion as to the feasibility of popularizing modern mathematics itself, there can be little doubt, it seems to me, that the tremendous importance of mathematics can be demonstrated even to the man-on-the-street if sufficient effort is made by those qualified to make the demonstration. The influence of mathematics on the various aspects of our evolving civilization, cultural as well as technological, should make a fascinating story if told by men of vision and imagination with the necessary literary gifts. May we not hope that such individuals will emerge from among our members, if the need is once realized. There is unquestionably a market for works dealing authoritatively and interestingly with the various aspects of this subject. The author of such a work could look forward to financial rewards as well as to the satisfaction of contributing materially to the advancement of mathematics. Articles discussing limited portions of this vast subject would be appropriate for inclusion in the proposed new periodical.

Would you consider me quite mad, if I suggest that this new periodical might find a sale among the general public; that we might find it ultimately on the news stands side by side with The Scientific Monthly and The Scientific American? I wonder. There is evidence to indicate that there is a far greater popular interest in our subject than we are generally inclined to believe.

Articles on the influence of mathematics could also find a place in magazines of general appeal and even in our Sunday newspapers. Numerous specific problems that engage the interest of mathematicians are also capable of being pre-

sented in such a way as to arouse popular interest. Dr. Fry has recently expressed his conviction that the problem of the man, the tree and the automobile, which was the subject of a recent paper in the Monthly by my colleague C. E. Wilder, could have commanded a full half-page in the Sunday edition of our metropolitan newspapers, if properly written up for this purpose.

Mathematics needs more publicity. The American Mathematical Society and the Mathematical Association of America should have more publicity. Dr. Fry in a recent letter called attention to the fact that, while Professor Einstein recently spent several months in this country, neither the Society nor the Association seems to have taken any official notice of the fact.

I believe the officers of the Association are at Dr. Fry's suggestion considering the possibility of remedying this situation by the appointment of a permanent committee on publicity.

I come to my final suggestion. Some of you doubtless heard the address delivered at the last Christmas meeting of the Association in Cleveland by Professor Radó on "Mathematics in Hungary." He there told of a mathematical competition held annually among selected graduates of Hungarian secondary schools. It is a national event which is given much publicity and attracts much attention. Moreover it has real scientific value—about half of the winners of this competition, I am told, have become mathematicians of note—a remarkable proportion. Could we not, in this country, emulate the example of Hungary?

The problems proposed for such a competition must be such that they are solvable on the basis of the information and training acquired by a high school graduate—i.e., they should require for their solution, in the way of technical training, nothing beyond plane trigonometry and advanced algebra—but they should be such as to call for unusual originality, power and insight.

I know nothing as to the machinery employed in Hungary for the purpose of this competition. Hungary is a relatively small country; the problem there would be simpler than with us. But I wonder if the already existing machinery of the College Entrance Examination Board could not be utilized for such a purpose? As to the cost of such a nation-wide competition, would not each secondary school from which candidates come be willing to pay the necessary fee? This is not the place to consider the project in detail—it seems to me well worth the most careful consideration as to its feasibility. It would give mathematics and the organization which sponsored it wide and favorable publicity. Would it not be worth while to appoint a committee to study its possibilities?

Appreciation on the part of the general educated public of the importance of mathematical activity would not only make easier the securing of material support for our enterprise—it would unquestionably help in recruiting our ranks with promising material. I have often been struck with the number of students of unusual ability and promise who are drawn away from a mathematical career by the rival attractions of business and the other professions. This is due, in part at least, to the present attitude of the public toward mathematics. The right sort of publicity would materially improve this situation.

I am at the end of my address. You have doubtless observed that my title, The Functions of the Association, is to some extent a misnomer. While I have suggested some projects that the Association as an organization might sponsor, a good deal of what I have said is directed to individuals and calls for individual effort. These portions of my remarks might have been entitled "Unsolved problems of mathematics—other than research."

We can't all be research men. Some of us do not even want to be. I have sought to show that there exist wide fields of enquiry and activity other than research that are important, interesting and worthy of intensive cultivation, and that are being sadly neglected. I have incidentally attempted to combat the attitude, if and wherever it exists, that would make of research a fetish, that proclaims that the only worthy function of a mathematician is research and that other activities are to be looked on with contempt. There is fortunately very little of this sort of self-righteous snobbery in our two organizations. The great majority of our research men themselves realize that the roots and trunk and branches of the tree of mathematics are quite as important as the blossom or the fruit; and that the former must exhibit healthy life if the latter are to be produced at all. Any other attitude is so utterly stupid as hardly to merit attention. To change the simile the star back on a football team who advances the ball and receives the plaudits of the multitude knows, unless his head has been completely turned by an exaggerated sense of his own importance, that he would be quite helpless were it not for a strong line and able interference.

It is probably impossible to determine the relative value of a guard and a back on a football team. It is probably even more difficult to determine the relative value, to the mathmatical organism as a whole, of the research man and the man who labors to improve the conditions which make research possible and which give it significance. Both are essential. But I am quite clear in my own mind, that if entry into the mathematical heaven depends on what a man has done for mathematics during his life on earth, the record of such a man as our good friend Slaught will far outweigh that of most mere research men. There is important work for all of us. The sin of the mathematician is not that he doesn't do research, the sin is idleness, when there is work to be done. If there be sinners in my audience I would urge them to sin no more. If your interest is in research, do that; if you are of a philosophical temperament, cultivate the gardens of criticism, evaluation, and interpretation; if your interest is historical, do your plowing in the field of history; if you have the insight to see simplicity in apparent complexity, cultivate the field of advanced mathematics from the elementary point of view; if you have the gift of popular exposition, develop your abilities in that direction; if you have executive and organizing ability, place that ability at the disposal of your organization. Whatever your abilities there is work for you to do—for the greater glory of mathematics. And this, I think, is the nearest I have ever come to preaching a sermon.

A NOTE ON THE CIRCULAR CUBIC REGARDED AS THE ENVELOPE OF FAMILIES OF CIRCLES

By H. G. GREEN, and L. E. PRIOR, University College, Nottingham, England

The investigation of the circular cubic and its enveloping circles usually consists of a direct attack which involves heavy and rather artificial analytical processes,[1] or else takes the form of a deduction from the properties of a quartic curve.[2] A third powerful method is that of a projection from three dimensions,[3] but all these methods have the slight disadvantage that they may lead some to believe that the problem is incapable of simple direct treatment. We give a discussion based on the methods of pure geometry and involving only the elementary properties of the cubic curve. It also shows clearly the development of the circles.

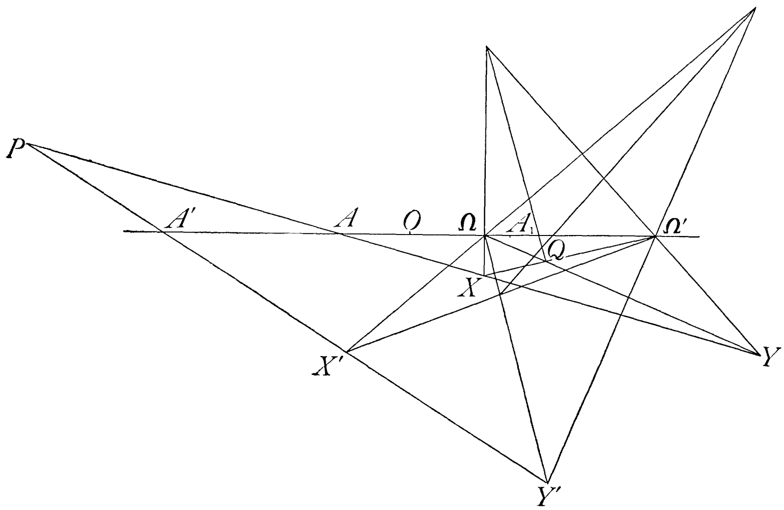

FIG. 1

We consider only the general case of the circular cubic. The modifications for special cases can be followed through on the same lines. Taking Ω, Ω' as the circular points let the circular cubic cut the line at infinity again at O; let P be the point of contact of one of the four tangents from O to the curve. Through P draw two straight lines cutting the curve again in X, Y; X', Y', respectively. Then since the three sets of points on the cubic Ω, Ω', O; X, Y, P; X', Y', P are collinear, and OP is a tangent, the points X, Y, X', Y', Ω, Ω' lie on a conic, or in other words the points X, Y, X', Y' are concyclic. If we keep the line PXY fixed and allow $PX'Y'$ to approach it indefinitely, the circle becomes in the limit the circle which has double contact with the cubic at X and Y.

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[1] Hilton, *Plane Algebraic Curves*, p. 217 et seq.

[2] Basset, *Cubic and Quartic Curves*, pp. 152–159.
Salmon, *Courbes Planes*, p. 350.

[3] Baker, *Principles of Geometry*, Vol. 4, p. 97.

Taking any position of $PX'Y'$, the power of P for the circle of contact on this chord is equal to that for the circle of contact on XY, since X, Y, X', Y' are concyclic. Hence for all positions of $X'Y'$, the circles of contact are orthogonal to a circle with centre P, and radius $\sqrt{(PX \cdot PY)}$. The cubic is therefore the double envelope of a family of circles cutting the fixed circle with centre P orthogonally, and further both the cubic and enveloping circles are self-inverse with respect to the fixed circle.

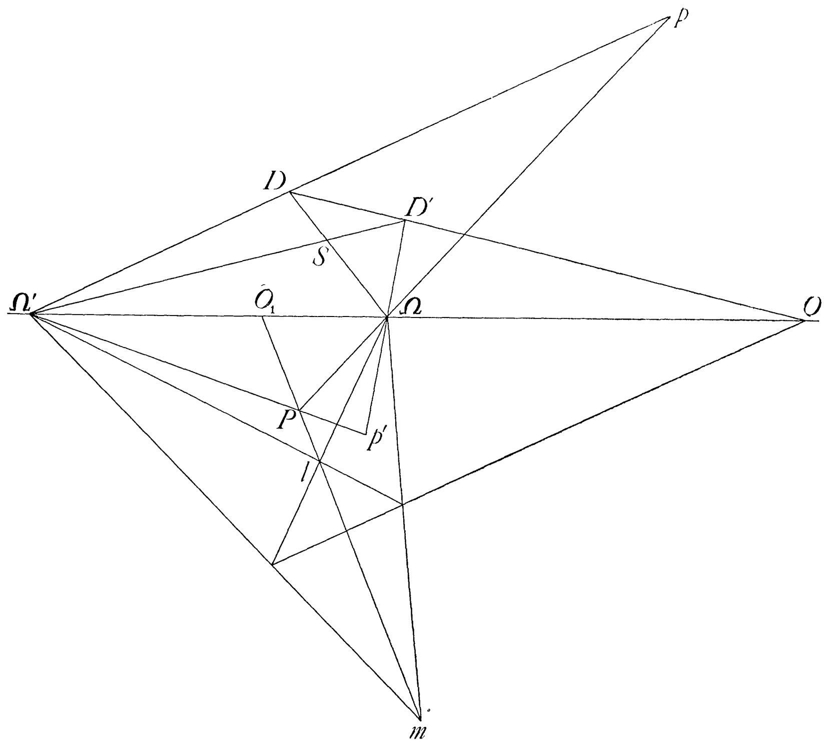

FIG. 2

In the non-singular case four families of enveloping circles can be built up in this way. If P_1 and P_2 are two positions of P, r_1 and r_2 being the radii of their circles of inversion, and if P_1P_2 meets the cubic again in Z, then from the inversion properties

$$r_1^2 = P_1Z \cdot P_1P_2, \quad r_2^2 = P_2Z \cdot P_1P_2,$$

and therefore

$$r_1^2 + r_2^2 = P_1P_2^2,$$

or the circles of inversion are mutually orthogonal.

Consider two near positions PXY, $PX'Y'$ cutting $\Omega\Omega'$ in A and A'. The polar of A with respect to the circle $\Omega\Omega'XYX'Y'$ is the harmonic line of the

quadrangle $\Omega\Omega' XY$ not passing through A. Hence the centre of the circle (the pole of AA') is the intersection Q of the two harmonic lines of this type (see Figure 1). In the limiting case, Q is the point of contact of the harmonic line with its envelope, and is also the centre of the circle touching the cubic at X and Y. Consider now the tangents to the locus (not coinciding with $\Omega\Omega'$) at any point A_1 on $\Omega\Omega'$. The corresponding position of PXY must pass through its conjugate A with respect to Ω, Ω', and is therefore fixed. Hence the tangent from A_1 to the locus is unique, excepting the possibility of the tangency of $\Omega\Omega'$. Take positions of PXY approximating to PO. As it approaches PO the tangent to the envelope considered above approaches $\Omega\Omega'$, and the point of contact approaches O_1, the conjugate of O with respect to Ω, Ω'. Hence the locus of Q is a parabola, whose axis passes through O_1 (see Figure 2). If we take $P\Omega$ as the line PXY, the harmonic line of the quadrangle becomes ΩS, where S is the singular focus of the cubic. Similarly $\Omega' S$ is a tangent to the Q locus, or S is the focus of the parabola. Further if $P\Omega$ cut the cubic again in p and $\Omega' p$ meets $S\Omega$ in D, then D is the point of contact of the parabola with $S\Omega$. Similarly D' may be constructed on $S\Omega'$, and the directrix is DD', passing through O since it is at right angles to SO_1. For the tangent at the vertex let PO_1 meet the cubic in l and m. Then the tangent at the vertex is the harmonic line, passing through O, of the quadrangle l, m, Ω, Ω'.

If two tangents to the cubic from Ω and Ω' be so chosen that the join of the points of contact passes through P, (which is possible in four ways) and if these tangents intersect in s, then Ωs, $\Omega' s$, or equally well the point circle s, can be considered as a circle of the enveloping system. Since the enveloping circles cut the circle of inversion of the same system orthogonally it follows that the ordinary foci s lie four by four on the four circles of inversion.

THE LEMOINE CENTER IN THE GEOMETRY OF THE TETRAHEDRON

By ALBERT A. BENNETT, Brown University

§1. In the geometry of the triangle among the points appropriately called centers of the triangle are:

1. The centroid, which is the point of intersection of medians of the base triangle.

2. The circumcenter, which is the center of the circle circumscribed about the triangle.

3. The incenter, which is the center of the inscribed circle.

4. The orthocenter, which is the point of intersection of the altitudes.

5. The symmedian or Lemoine center, which is, simultaneously, (i) the isogonal conjugate of the centroid, (ii) the center of perspectivity of the given triangle and the triangle whose sides are tangent to the circumcircle at the vertices of the base triangle.

While the definitions in 2, 3, 4, 5(i) are essentially metrical, it is significant that the definitions 1, and 5(ii) remain covariant under more extensive groups. The centroid is determinate under the general affine or "parallel" group of transformation upon the base triangle, and the Lemoine center defined in 5(ii) is covariant under the most general projective group which leaves the base triangle and its circumcircle unaltered.

For a study of the triangle and its circumcircle in relation to purely projective properties, it is convenient to make use of known Euclidean results so as to render intuitively obvious, relations otherwise unfamiliar. The Lemoine center is the center of a perspectivity whose associated axis of perspectivity is called the Lemoine axis. If Euclidean language be used appropriate to treating this line as the line at infinity, then the triangle may be interpreted as a regular triangle, with Lemoine center as its center of symmetry, while the circumscribed circle, continues to be treated as a circle in this metric. The topic is almost devoid of interest due to its extreme simplicity.

The study of the properties of three distinct points under the inversive group leads to the isodynamic point-pair, the Apollonian circles and other elements of interest, but does not provide a unique "center," and for this reason neither this question nor its space analogues will be discussed here.

§2. In studying the geometry of the tetrahedron,[1] it is natural to look for extensions to three-space of relations which have proved significant in the plane. Taking up in turn the five centers mentioned above, it is easily verified that for the tetrahedron one has:

1. The centroid. This is the point of intersection of the four medians, each drawn from a vertex of the tetrahedron to the centroid of the opposite face and

[1] For Desmic tetrahedra, see

R. W. H. T. Hudson, *Kummer's Quartic Surface*, Cambridge, University Press, 1905. Chapter 1, and references.

For mensurational relations of the metrical tetrahedron and some other relations also, see

Hermann Thieme, *Die Elemente der Geometrie*, Leipzig, Teubner, 1909, pages 360–375.

Richard Baltzer, *Déterminants* (Translation by Hoüel), Paris, 1861, pages 189–211.

G. Doster, *Déterminants*, Paris, 1883, pages 252–293.

Gustav Holzmüller, *Elemente der Stereometrie*, vol. 2, Leipzig, 1900, pages 186–251.

For a detailed discussion of the "parallel geometry" in space, see

Lothar Heffter, *Lehrbuch der analytischen Geometrie*, vol. 2, Leipzig, Teubner, 1923, pages 196–270.

For the classical geometry of the tetrahedron, see

J. J. Neuberg, *Zur Tetraedergeometrie*, Leipzig, 1909, (Reprinted from Archiv der Mathematik und Physik (3) vol. 16 (1909).

Eugène Rouché et Ch. de Comberousse, *Traité de Géométrie* (8th edition), Paris, 1912, vol. 2, note 4. *Sur la Géométrie récente du tétraèdre*, pages 643–564.

J. L. Coolidge, *A Treatise on the Circle and the Sphere*, Oxford, 1916, pages 233–240.

George Salmon, *Geometry of Three Dimensions*, 5th edition, 2 vols. (1912 and 1915), London, Longmans, Vol. 1, pages 138–145.

E. Duporq, *Premiers principes de géométrie moderne*, Paris, 1912, pages 101–110.

Cl. Servais, *Sur la géométrie du tétraèdre* (5 articles), Acad. Roy. de. Belg. (Bull Cl. Sci.) (5) vol. 15 (1929), and vol. 16 (1930).

is situated one-fourth of the distance from the face. It is also the point of intersection of six planes, one through each edge of the tetrahedron and through the mid-point of the opposite edge. It is also the common mid-point of each of the three lines joining mid-points of opposite edges. It is also the mid-point of each of the three parallelograms with vertices at the mid-points of two pairs of opposite edges.

2. The circumcenter, which is the center of the sphere circumscribed about the base tetrahedron.

3. The incenter, which is the center of the sphere, inscribed in the base tetrahedron.

4. The orthomedial center. The four altitudes of the base tetrahedron are not in general concurrent. They are four rulers of an equilateral hyperboloid of one sheet, whose center is the orthomedial center or Monge Point (Monge 1808). It is also the point of intersection common to the six planes, each through the mid-point of an edge, and perpendicular to the opposite edge. Save for the specialized orthocentric tetrahedra for which the hyperboloid reduces to a cone, there is no "orthocenter," although the orthomedial center is an appropriate generalization to space.

Even the fact that for the plane case, the centroid, circumcenter and orthocenter are collinear (on the Euler line) with simple numerical relations for their mutual distances has a space analogue since the distance of the orthomedial center of the tetrahedron, to the circumcenter is bisected by the centroid.

5. The symmedian center, which is the isogonal conjugate of the centroid. However the symmedian center so defined does not enjoy projective properties analogous to those holding in the plane. The tetrahedron circumscribed about the circumsphere and tangent at the vertices of the base tetrahedron, will ordinarily fail to be centrally perspective with the base tetrahedron. The four corresponding joins are rulers of an hyperboloid.

The existence of a projectively determined point, which we shall call the Lemoine center of a tetrahedron, which enjoys properties generalizing in a sense the definition 5(ii) in the plane, has not been hitherto noted. The symmedian center, certainly fails to fulfill these demands. Thus we shall for tetrahedra distinguish between the symmedian and the Lemoine centers, although for triangles these coincide.

§3. Before discussing the Lemoine center, it is well to call to mind some elementary projective properties of the tetrahedron with reference to a general point P not on one of its face-planes. Let the vertices of the base tetrahedron be called the base points, in tetrahedral coordinates we have $B_1=(1, 0, 0, 0)$, $B_2=(0, 1, 0, 0)$, $B_3=(0, 0, 1, 0)$, $B_4=(0, 0, 0, 1)$. Let $P=(p_1, p_2, p_3, p_4)$. The projections of P from the base points upon the base planes will be $P_1=(0, p_2, p_3, p_4)$, $P_2=(p_1, 0, p_3, p_4)$, $P_3=(p_1, p_2, 0, p_4)$, $P_4=(p_1, p_2, p_3, 0)$. The following are the vertices of the P-complementary tetrahedron. $P^1=(-p_1, p_2, p_3, p_4)$, $P^2=(p_1, -p_2, p_3, p_4)$, $P^3=(p_1, p_2, -p_3, p_4)$, $P^4=(p_1, p_2, p_3, -p_4)$. The base tetrahedron and the P-complementary tetrahedron are quadruply perspective, with centers

of perspectivity at P, and at $P'=(p_1, p_2, -p_3, -p_4)$, $P''=(p_1, -p_2, p_3, -p_4)$, $P'''=(p_1, -p_2, -p_3, p_4)$. These form a desmic system. The plane containing P', P'', P''', namely

$$\frac{x_1}{p_1}+\frac{x_2}{p_2}+\frac{x_3}{p_3}+\frac{x_4}{p_4}=0$$

is called the P-double plane.

The vertices of the base tetrahedron and of its P-complementary tetrahedron together constitute the vertices of the P-hexahedron, whose six face planes are $(x_i/p_i)+(x_j/p_j)=0, j\neq i$. Each of the face planes contains one edge of the base tetrahedron and one edge of the P-complementary tetrahedron.

The three principal planes of the P-hexadron are

$$\frac{x_1}{p_1}+\frac{x_2}{p_2}-\frac{x_3}{p_3}-\frac{x_4}{p_4}=0,\qquad \frac{x_1}{p_1}-\frac{x_2}{p_2}+\frac{x_3}{p_3}-\frac{x_4}{p_4}=0,\qquad \frac{x_1}{p_1}-\frac{x_2}{p_2}-\frac{x_3}{p_3}+\frac{x_4}{p_4}=0.$$

These pass through P. Each contains two of the points P', P'', P'''. Each contains four of the following six points of intersections of edges of the base tetrahedron with edges of the P-complementary tetrahedron: $P_{12}=(0, 0, p_3, p_4)$, $P_{13}=(0, p_2, 0, p_4)$, $P_{23}=(p_1, 0, 0, p_4)$, $P_{14}=(0, p_2, p_3, 0)$, $P_{24}=(p_1, 0, p_3, 0)$, $P_{34}=(p_1, p_2, 0, 0)$. The study of the desmic properties may be based upon the identity

$$\left(-\frac{x_1}{p_1}+\frac{x_2}{p_2}+\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)\left(\frac{x_1}{p_1}-\frac{x_2}{p_2}+\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)\left(\frac{x_1}{p_1}+\frac{x_2}{p_2}-\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)\left(\frac{x_1}{p_1}+\frac{x_2}{p_2}+\frac{x_3}{p_3}-\frac{x_4}{p_4}\right)$$
$$+\left(\frac{x_1}{p_1}+\frac{x_2}{p_2}+\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)\left(\frac{x_1}{p_1}+\frac{x_2}{p_2}-\frac{x_3}{p_3}-\frac{x_4}{p_4}\right)\left(\frac{x_1}{p_1}-\frac{x_2}{p_2}+\frac{x_3}{p_3}-\frac{x_4}{p_4}\right)\left(\frac{x_1}{p_1}-\frac{x_2}{p_2}-\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)$$
$$=16\, x_1x_2x_3x_4/p_1p_2p_3p_4.$$

Analogous identities whose geometric import might be discussed are,

$$\left(\left(\frac{x_1}{p_1}\right)^2-\left(\frac{x_2}{p_2}\right)^2\right)\left(\left(\frac{x_3}{p_3}\right)^2-\left(\frac{x_4}{p_4}\right)^2\right)+\left(\left(\frac{x_1}{p_1}\right)^2-\left(\frac{x_3}{p_3}\right)^2\right)\left(\left(\frac{x_4}{p_4}\right)^2-\left(\frac{x_2}{p_2}\right)^2\right)$$
$$+\left(\left(\frac{x_1}{p_1}\right)^2-\left(\frac{x_4}{p_4}\right)^2\right)\left(\left(\frac{x_2}{p_2}\right)^2-\left(\frac{x_3}{p_3}\right)^2\right)=0,$$

and

$$4\left(\left(\frac{x_1}{p_1}\right)^2+\left(\frac{x_2}{p_2}\right)^2+\left(\frac{x_3}{p_3}\right)^2+\left(\frac{x_4}{p_4}\right)^2\right)$$
$$=\left(-\frac{x_1}{p_1}+\frac{x_2}{p_2}+\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)^2+\left(\frac{x_1}{p_1}-\frac{x_2}{p_2}+\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)^2+\left(\frac{x_1}{p_1}+\frac{x_2}{p_2}-\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)^2$$
$$+\left(\frac{x_1}{p_1}+\frac{x_2}{p_2}+\frac{x_3}{p_3}-\frac{x_4}{p_4}\right)^2=\left(\frac{x_1}{p_1}+\frac{x_2}{p_2}+\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)^2+\left(\frac{x_1}{p_1}+\frac{x_2}{p_2}-\frac{x_3}{p_3}-\frac{x_4}{p_4}\right)^2$$
$$+\left(\frac{x_1}{p_1}-\frac{x_2}{p_2}+\frac{x_3}{p_3}-\frac{x_4}{p_4}\right)^2+\left(\frac{x_1}{p_1}-\frac{x_2}{p_2}-\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)^2.$$

The six diagonal planes of the P-hexahedron are given by $(x_i/p_i)-(x_j/p_j)=0, j\neq i$. Each contain one edge of the base tetrahedron, and one of the P-complementary tetrahedron.

The P-circumquadric $\sum x_ix_j/p_ip_j=0$ passes through the eight vertices of the P-hexahedron. The point P, and the P-double-plane are pole and polar with respect to this quadric.

The three P-edge quadrics

$$\frac{x_1x_2}{p_1p_2}-\frac{x_3x_4}{p_3p_4}=0,\quad \frac{x_1x_3}{p_1p_3}-\frac{x_2x_4}{p_2p_4}=0,\quad \frac{x_1x_4}{p_1p_4}-\frac{x_2x_3}{p_2p_3}=0$$

are such that each contains four edges of the base tetrahedron, and contains two lines joining opposite points among the set P_{ij} and hence also contains P. Each also contains one of the three lines in the P-double-plane joining P', P'', P''' in pairs.

The three P-hexahedral ruled edge quadrics are

$$\frac{x_1x_2}{p_1p_2}+\frac{x_3x_4}{p_3p_4}=0,\quad \frac{x_1x_3}{p_1p_3}+\frac{x_2x_4}{p_2p_4}=0,\quad \frac{x_1x_4}{p_1p_4}+\frac{x_2x_3}{p_2p_3}=0.$$

Each contains two pairs of opposite edges of the base tetrahedron and two pairs of opposite edges of the P-complementary tetrahedron, and two of the three lines $P'P''$, $P'P'''$, $P''P'''$.

The P-pencil of quadrics consists of all quadrics whose equations are expressible in the form

$$m\sum_i x_i^2/p_i^2 + n\sum_{i\neq j} x_ix_j/p_ip_j = 0.$$

This pencil contains among others

1. The P-double plane counted twice, $m=1$, $n=2$.
2. The P-circumquadric, $m=0$, $n=1$.
3. The degenerate quadric whose only real point is P, $m=3$, $n=-2$.
4. The P-inscribed quadric, inscribed in the base tetrahedron and in the P-complementary tetrahedron, $m=1$, $n=-1$.
5. The quadric touching the six edges of the base tetrahedron and the six edges of the P-complementary tetrahedron, and inscribed in the P-hexahedron, $m=2$, $n=-1$.

For each point $(x_1, x_2, x_3, x_4,)$ not on the face of the base tetrahedron, there is a unique P-reciprocal point, given by

$$\left(\frac{p_1^2}{x_1},\frac{p_2^2}{x_2},\frac{p_3^2}{x_3},\frac{p_4^2}{x_4}\right).$$

The P-circumquadric, the three P-edge quadrics and the three P-hexahedral ruled edge quadrics are self reciprocal under P-reciprocation. The points P, P', P'', P''', P^1, P^2, P^3, P^4, are invariant under P-reciprocation and are the only such invariant points.

§4. It is sometimes convenient to introduce new coordinates determined by P, as follows:

$$2y_0 = \frac{x_1}{p_1} + \frac{x_2}{p_2} + \frac{x_3}{p_3} + \frac{x_4}{p_4}, \qquad 2\frac{x_1}{p_1} = y_0 + y_1 + y_2 + y_3$$

$$2y_1 = \frac{x_1}{p_1} + \frac{x_2}{p_2} - \frac{x_3}{p_3} - \frac{x_4}{p_4}, \qquad 2\frac{x_2}{p_4} = y_0 + y_1 - y_2 - y_3$$

$$2y_2 = \frac{x_1}{p_1} - \frac{x_2}{p_2} + \frac{x_3}{p_3} - \frac{x_4}{p_4}, \qquad 2\frac{x_3}{p_3} = y_0 - y_1 + y_2 - y_3$$

$$2y_3 = \frac{x_1}{p_1} - \frac{x_2}{p_2} - \frac{x_3}{p_3} + \frac{x_4}{p_4}, \qquad 2\frac{x_4}{p_4} = y_0 - y_1 - y_2 + y_3.$$

Then the P-double plane becomes $y_0 = 0$. The base points become $B_1 = (1, 1, 1, 1)$, $B_2 = (1, 1, -1, -1)$, $B_3 = (1, -1, 1, -1)$, $B_4 = (1, -1, -1, 1)$. The vertices of the P-complementary tetrahedron are then in y-coordinates, $P^1 = (-1, 1, 1, 1)$, $P^2 = (1, -1, 1, 1)$, $P^3 = (1, 1, -1, 1)$, $P^4 = (1, 1, 1, -1)$, while $P = (1, 0, 0, 0)$, $P' = (0, 1, 0, 0)$, $P'' = (0, 0, 1, 0)$, $P''' = (0, 0, 0, 1)$. The principal planes become $y_2 = 0$, $y_3 = 0$, $y_4 = 0$. The diagonal planes of the P-hexahedron become $y_1 - y_2 = 0$, $y_2 - y_3 = 0$, $y_3 - y_1 = 0$, $y_1 + y_2 = 0$, $y_2 + y_3 = 0$, $y_3 + y_1 = 0$. The P-circumquadric becomes $y_1^2 + y_2^2 + y_3^2 - 3y_0^2 = 0$. The three P-edge quadrics become respectively, (in the order given)

$$y_0y_1 - y_2y_3 = 0, \quad y_0y_2 - y_1y_3 = 0, \quad y_0y_3 - y_1y_2 = 0.$$

The three P-hexahedral ruled edge quadrics become $y_0^2 + y_1^2 - y_2^2 - y_3^2 = 0$, $y_0^2 - y_1^2 + y_2^2 - y_3^2 = 0$, $y_0^2 - y_1^2 - y_2^2 + y_3^2 = 0$. The P-pencil of quadrics becomes

$$(2m - n)\sum_{i=0}^{3} y_i^2 + 4ny_i^2 = 0.$$

The P-reciprocal of the point (y_0, y_1, y_2, y_3) is

$$[y_0^3 - y_0(y_1^2 + y_2^2 + y_3^2) + 2y_1y_2y_3,\ y_1^3 - y_1(y_0^2 + y_2^2 + y_3^2) + 2y_0y_2y_3,$$
$$y_2^3 - y_2(y_0^2 + y_1^2 + y_3^2) + 2y_0y_1y_3,\ y_3^3 - y_3(y_0^2 + y_1^2 + y_2^2) + 2y_0y_1y_2].$$

§5. The equation of the circumsphere will be of the form

$$\sum_{i \neq j} e_{ij}^2\, x_i\, x_j = 0$$

for a general system of tetrahedral coordinates. If special barycentric coordinates be used so that the centroid is given by (1, 1, 1, 1) then the coefficients e_{ij}^2 may be interpreted as the square of the length of the edge joining B_i to B_j. It is often convenient to associate a direction with each edge so that the directed length B_iB_j is given by e_{ij}. Then $e_{ji} = -e_{ij}$. While the choice of directions might be made at random, a "regular" choice will be said to be any choice for which the following three products are of the same sign: $\beta' = e_{12}e_{34}$, $\beta'' = e_{13}e_{42}$, $\beta''' = e_{14}e_{23}$.

The matrix[2]

$$\begin{pmatrix} 0 & e_{12} & e_{13} & e_{14} \\ e_{21} & 0 & e_{23} & e_{24} \\ e_{31} & e_{32} & 0 & e_{34} \\ e_{41} & e_{42} & e_{43} & 0 \end{pmatrix}$$

is then skew symmetric, with determinant $4\sigma^2$, where $2\sigma=\beta'+\beta''+\beta'''$. The inverse matrix is proportional to the skew symmetric matrix,

$$\begin{pmatrix} 0 & e_{34} & e_{42} & e_{23} \\ e_{43} & 0 & e_{14} & e_{31} \\ e_{24} & e_{41} & 0 & e_{12} \\ e_{32} & e_{13} & e_{21} & 0 \end{pmatrix}.$$

The products of the non-zero elements of the several rows will be denoted as follows: $k_1=e_{34}e_{42}e_{23}$, $k_2=e_{43}e_{14}e_{31}$, $k_3=e_{24}e_{41}e_{12}$, $k_4=e_{32}e_{13}e_{21}$. They will be of the same sign. For instance, $k_1k_2=e_{34}e_{42}e_{23}e_{43}e_{14}e_{31}=e_{34}^2e_{13}e_{42}e_{14}e_{23}=e_{34}^2\beta''\beta'''$, which is positive. Hence k_1 and k_2 have the same sign.

The equation of the circumsphere may now be written in the form

$$\beta'\left(\frac{x_1x_2}{k_1k_2}+\frac{x_3x_4}{k_3k_4}\right)+\beta''\left(\frac{x_1x_3}{k_1k_3}+\frac{x_2x_4}{k_2k_4}\right)+\beta'''\left(\frac{x_1x_4}{k_1k_4}+\frac{x_2x_3}{k_2k_3}\right)=0.$$

Under real projective transformations one may distinguish between the interior and exterior of the circumsphere. In particular each real point (x_1, x_2, x_3, x_4) for which all the coordinates have the same sign is in that one of the eight tetrahedral regions determined by the base points, which is wholly interior to the circumsphere.

By taking $K=(k_1, k_2, k_3, k_4)$ in place of the general point P, and by using the corresponding y-coordinates, the equation of the circumsphere assumes the form

$$(\sigma-\beta')y_1^2+(\sigma-\beta'')y_2^2+(\sigma-\beta''')y_3^2=\sigma y_0^2.$$

While under real projective transformation in the plane, any nondegenerate real conic is transformable into any other, not all nondegenerate real quadrics are equivalent under real projective transformations of space. Since the plane $y_0=0$ does not cut the circumsphere in real points, and yet there are real points upon this circumsphere, we may conclude that not only are β', β'', β''', of the same sign, but σ, $\sigma-\beta'$, $\sigma-\beta''$, $\sigma-\beta'''$, are also of this same sign. It may be noted that

$$[\sigma(\sigma-\beta')(\sigma-\beta'')(\sigma-\beta''')]^{1/2}$$

[2] The rows or columns may be used to identify a tetrahedron inscribed and circumscribed to the base tetrahedron giving a pair of so-called Möbius tetrahedra.

which is analogous to the formula for the area of a triangle in terms of its edges, gives in the metrical interpretation of e_{ij} as the edge B_iB_j, six times the product of the volume of the base tetrahedron by the radius of its circumsphere.

The point (k_1, k_2, k_3, k_4) will be called the Lemoine center, K, of the tetrahedron. One has, as for the general point, P, the associated points, K^1, K^2, K^3, K^4, K', K'', K''', and the Lemoine double plane

$$\frac{x_1}{k_1} + \frac{x_2}{k_2} + \frac{x_3}{k_3} + \frac{x_4}{k_4} = 0$$

and so forth.

The Lemoine center and Lemoine double-plane are pole and polar with respect to the circumsphere, which serves to characterize the Lemoine system.

§6. To show directly that the point K is projectively identified, it suffices to show that the set of four points, K^1, K^2, K^3, K^4, is projectively determined. This set is quadruply perspective with the base points. Three of the perspective centers namely K', K'', K''', lie in an exterior plane, polar to the interior point, K, which for real projective purpose is thus specialized.

In each face of the base tetrahedron there is a symmedian center. Thus in $x_1=0$, there is a triangle with sides given by $x_2=0$, $x_3=0$, $x_4=0$, and with circumcircle given by

$$e_{23}{}^2x_2x_3 + e_{24}{}^2x_2x_4 + e_{34}{}^2x_3x_4 = 0.$$

The symmedian center is $(0,\ e_{34}{}^2,\ e_{24}{}^2,\ e_{23}{}^2)$. The four lines each joining a base point to the symmedian center of the opposite face lie in the ruled quadric

$$(\beta''^2 - \beta'''^2)(e_{12}{}^2x_1x_2 + e_{34}{}^2x_3x_4) + (\beta'''^2 - \beta'^2)(e_{13}{}^2x_1x_3 + e_{24}{}^2x_2x_4) \\ + (\beta'^2 - \beta''^2)(e_{14}{}^2x_1x_4 + e_{23}{}^2x_2x_3) = 0.$$

In each face there is a Lemoine axis. Thus in $x_1=0$ the Lemoine axis is given by

$$e_{23}{}^2e_{24}{}^2x_2 + e_{23}{}^2e_{34}{}^2x_3 + e_{24}{}^2e_{34}{}^2x_4 = 0.$$

This meets the line of intersection with $x_1=0$ of the plane tangent to the circumsphere at B_1, namely the plane $e_{12}{}^2x_2+e_{13}{}^2x_3+e_{14}{}^2x_4=0$, in the point

$$\left[0,\ e_{34}{}^2(\beta''^2 - \beta'''^2),\ e_{24}{}^2(\beta'''^2 - \beta_1{}^2),\ e_{23}{}^2(\beta'^2 - \beta''^2)\right].$$

The line joining this point to the base point B_1 is one of four such lines all of which lie on the ruled quadric.

$$(2\beta''^2\beta'''^2 - \beta'^2\beta''^2 - \beta'^2\beta'''^2)(\beta''^2 - \beta'''^2)(e_{12}{}^2x_1x_2 + e_{34}{}^2x_3x_4) \\ + (2\beta'^2\beta'''^2 - \beta'^2\beta''^2 - \beta''^2\beta'''^2)(\beta'''^2 - \beta'^2)(e_{13}{}^2x_1x_3 + e_{24}{}^2x_2x_4) \\ + (2\beta'^2\beta''^2 - \beta'^2\beta'''^2 - \beta''^2\beta'''^2)(\beta'^2 - \beta''^2)(e_{14}{}^2x_1x_4 + e_{23}{}^2x_2x_3) = 0.$$

These two ruled quadrics meet the circumsphere in a self-associated set of eight points namely the vertices of the Lemoine hexahedron, which latter is thus projectively determined. As in the case of the triangle in the plane, it is convenient to introduce a special Euclidean metric not related to the metric used for the study of other properties of the tetrahedron, but merely used to make available in familiar language projective relations.

The Lemoine double plane, $y_0=0$, will be taken as the plane at infinity. The Lemoine hexahedron is now the cube with vertices $(1, \pm 1, \pm 1, \pm 1)$. The Lemoine principal planes are the finite coordinate planes, $y_1=0$, $y_2=0$, $y_3=0$. The Lemoine circumquadric becomes the Euclidean sphere $y_1^2+y_2^2+y_3^2=3y_0^2$, while the original circumsphere becomes interpreted as an ellipsoid whose three (in general unequal) axes, are the finite coordinate axes.

§7. In connection with the special metric, the notion of three mutually orthogonal planes, describes what may be called a Lemoine trihedron, abstractly defined as a trihedron which meets the Lemoine double plane in a triangle self-polar with respect to the imaginary conic in which the Lemoine circumquadric meets this plane. Numerous examples of Lemoine trihedra present themselves. Thus one has in particular

1. The set of the three Lemoine principal planes.

2. The three face planes of the Lemoine hexahedron at any one of its vertices.

3. Two Lemoine principal planes and the plane tangent to the circumsphere at its intersection with these principal planes.

4. Two Lemoine principal planes and the plane tangent to the Lemoine circumquadric at its intersection with these principal planes.

5. Three tangent planes (no two intersecting the Lemoine double plane in a common line) to the circumsphere at its intersections with the Lemoine axes.

6. Three tangent planes (no two intersecting the Lemoine double plane in a common line) to the Lemoine circumquadric at its intersections with the Lemoine axes.

It is known that the locus of the vertex of a variable trirectangular trihedron tangent to an ellipsoid is a sphere concentric with it. Hence the locus of the vertex of a Lemoine trihedron tangent to the circumsphere is a quadric in the Lemoine pencil.

Interpreting the facts concerning a system of confocal quadrics, we conclude that there are two quadrics which in the special metric sense are confocal with the initial circumsphere and passing through the eight vertices of the Lemoine hexahedron and whose tangent planes at each of the eight vertices of this hexahedron form Lemoine trihedra.

§8. The correspondence

$$z_1 = 1\Big/\left(\frac{x_1x_2}{k_1k_2}+\frac{x_3x_4}{k_3k_4}\right),\quad z_2 = 1\Big/\left(\frac{x_1x_3}{k_1k_3}+\frac{x_2x_4}{k_2k_4}\right),\quad z_3 = 1\Big/\left(\frac{x_1x_4}{k_1k_4}+\frac{x_2x_3}{k_2k_3}\right)$$

carries the Lemoine circumquadric into $z_2z_3+z_3z_1+z_1z_2=0$, and the circumsphere into $\beta' z_2z_3+\beta'' z_3z_1+\beta''' z_1z_2=0$.

A linear equation in the z's corresponds to a quartic surface. This quartic has the special properties of meeting each face of the base tetrahedron in four lines of which three constitute the base triangle. Thus in $x_4=0$, the equation $a_1z_1+a_2z_2+a_3z_3=0$, reduces to $a_1k_2k_3x_1+a_2k_3k_1x_2+a_3k_1k_2x_3=0$ save for the extraneous factors.

A quadratic equation in z_1, z_2, z_3, without squared terms corresponds to a quadratic in x_1, x_2, x_3, x_4 without squared terms. Thus a line in the z-plane corresponds to a line in each face of the base tetrahedron. A conic circumscribed about the reference triangle in the z-plane corresponds to a conic circumscribed about the reference triangle in each of the faces of the base tetrahedron. A point in the z-plane corresponds save for extraneous fixed points to a point in each face of the base triangle. When reference is made to the whole of space, a point in the z-plane corresponds to a space quartic through the eight vertices of the Lemoine hexahedron. One may interpret each element in the metric geometry of the triangle as an element projectively determined in the geometry of the tetrahedron, where points correspond to quartic curves. For the special point (1, 1, 1) in the z-plane, the quartic reduces to the four lines joining K to the respective base points.

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

QUESTION CONCERNING THE MAXIMUM TERM IN THE DIATOMIC SERIES

By ALBERT A. BENNETT, Brown Univers'ty

Given k, let π_k denote the continued product of the first k primes, p_i, starting with $p_1=2$. For example, $\pi_5=2\cdot 3\cdot 5\cdot 7\cdot 11=2310$. Let π_0 be 1.

Question: *Given k, what is the maximum number of consecutive integers each of which is divisible by one of the primes $p_1, p_2, \cdots, p_k$?*

It is necessary to consider only the sequence of residues modulo π_k. The question asks for the maximum interval between adjacent entries in the sequence of all those numbers which are relatively prime to π_k. Otherwise stated it asks for the maximum term in the diatomic series[1] for p_k.

[1] These periodic series were introduced by de Polignac in 1851 in partial application of the Sieve of Eratosthenes. A description with references is given in L. E. Dickson, *History of the Theory of Numbers*, vol. 1, (1919), p. 439. Among further properties, one may readily show that for $k>1$, in any period the number of ones is equal to the number of threes, $=(p_2-2)(p_3-2)\cdots(p_k-2)$, while each other odd number that occurs, appears an even number of times in a period.

It is obvious that a sequence of at least $p_{k+1}-2$ successive integers of the sort desired is always available since 2, 3, 4, 5, 6, $\cdots$, $p_{k+1}-1$, form such a sequence, and this type of sequence is maximal for $k=1$, 2, 3. But it is also true that a sequence of $2p_{k-1}-1$ (ordinarily greater than $p_{k+1}-2$) is always available for $k>1$. Indeed let x be a positive solution of the three simultaneous congruences:

$$x \equiv 0 \pmod{\pi_{k-2}},\ x \equiv \pm 1 \pmod{p_{k-1}},\ x \equiv \mp 1 \pmod{p_k},$$

then $x+i$, $(-p_{k-1}<i<p_{k-1})$ denotes a sequence of $2p_{k-1}-1$ numbers each of which is divisible by some factor of π_k. Is there ever (for k given) a longer sequence? For example, for $k=5$ the sequence of thirteen consecutive numbers, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, is such that each is divisible by one of the first five primes, 2, 3, 5, 7, 11, and is a maximal sequence.

NOTE: If it can be proved that for each $k>1$, no sequence of $2p_k+2$ consecutive integers exists each of which is divisible by at least one factor of π_k, it would also hold that no sequence of $2m+2$ consecutive integers exists each of which is divisible by some integer not exceeding m, and hence, as one result, in the sequence $m^2-1, m^2, m^2+1, \cdots, m^2+2m$, at least one term could not contain a factor less than $m+1$. Since $m+1$ exceeds the square root of m^2+2m, the existence of at least one prime number between m^2 and $(m+1)^2$ would then be established (one of those numerous, plausible, and "asymptotically true," but as yet unproven relations!).

A CORRECTION

By MORGAN WARD, California Institute of Technology

Mr. Hans Rohrbach of the University of Berlin has kindly called my attention to an error in a paper of mine in the Monthly (A Simplification of Certain Problems in Arithmetical Division, Vol. XXXV, No. 1, Jan. 1928) which I should like to correct here.

On page 11 of the paper cited, the last summation in the last line should read

$$\sum_{\sigma_1=1}^{r-1-\alpha} a^{\alpha+\sigma_1} V_{r-\sigma_1}$$

instead of

$$\sum_{\sigma_1=1}^{r-1-\alpha} a^{\alpha+\sigma_1} V_{\alpha+\sigma_1}.$$

Also on page 12, the last line of Formula IV should read

$$\frac{a^{kr}-1}{a^r-1}(a^{r-1}V_{\alpha+1}+a^{r-2}V_{\alpha+2}+\cdots+a^{\alpha+1}V_{r-1})$$

instead of

$$\frac{a^{kr}-1}{a^r-1}\, a^{\alpha}(aV_{\alpha+1}+a^2V_{\alpha+2}+\cdots+a^{r-\alpha-1}V_{r-1}).$$

These corrections entail changes in the numerical examples; for instance, we have

$$\left[\frac{2^{257}-1}{1023}\right]=\frac{2^{250}-1}{2^{10}-1}\cdot 2^{7}=2^{7}+2^{17}+2^{27}+\cdots+2^{247}.$$

A Simple Geometrical Paradox—Proposed by J. L. Coolidge[1]

Replies by Grace Bareis and Vladimir F. Ivanoff

The Paradox

Question: A point P in three-dimensional Euclidean space is required to be collinear with two given points A and B; how many independent algebraic conditions are thereby imposed upon its coordinates?

First answer: In order to be collinear with A and B it is necessary and sufficient that P should be coplanar with A and B and each of two arbitrary points not in the same plane through A and B; *two conditions.*

Second answer: If P be collinear with A and B it is necessary and sufficient that the sum of two of the distances determined by two pairs from these three points should be equal to the distance determined by the third pair: hence but one condition is imposed.

Which answer is right? How many conditions are imposed in a Euclidean space of n dimensions?

1. *A Reply by Grace Bareis, Ohio State University.*

That the first answer is the correct one is easily seen by making a proper choice of axes, which in no way affects the generality of the conclusion.

We select A as origin and AB as x-axis; then, using the points

$$A(0,0,0),\ B(b,0,0),\ P(x,y,z),$$

the condition

$$AB^4+BP^4+PA^4-2BP^2\cdot PA^2-2PA^2\cdot AB^2-2AB^2\cdot BP^2=0$$

reduces at once to $y^2+z^2=0$, or $y=0$ and $z=0$ for real points. Therefore two conditions are imposed upon the coordinates of P.

Similarly, in Euclidean space of n dimensions, we may choose

$$A(0,0,0,\cdots,0,)B(b,0,0,\cdots,0)P(x_1,x_2,x_3,\cdots,x_n).$$

The above condition then becomes

$$x_2^2+x_3^2+\cdots+x_n^2=0,\ \text{or}\ x_2=0,\ x_3=0,\cdots x_n=0.$$

Therefore $n-1$ conditions are imposed upon the coordinates of the point P if it is required to be collinear with the two given points A and B.

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[1] This paradox was proposed by Professor Coolidge in the April issue of this Monthly, vol. 38 (1931) p. 22.

2. *A Reply by Vladimir F. Ivanoff, San Francisco, California.*

The paradox proposed by Professor Coolidge may be explained in the following way.

We take

$$PA^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = a_{11}^2 + a_{12}^2 + a_{13}^2,$$
$$BP^2 = (x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2 = a_{21}^2 + a_{22}^2 + a_{23}^2,$$
$$AB^2 = (x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2 = a_{31}^2 + a_{32}^2 + a_{33}^2,$$

whence

$$(1) \qquad a_{11} + a_{21} + a_{31} = 0, \; a_{12} + a_{22} + a_{32} = 0, \; a_{13} + a_{23} + a_{33} = 0,$$

no matter whether P lies on AB or not.

The condition

$$AB^4 + BP^4 + PA^4 - 2BP^2 \cdot PA^2 - 2PA^2 \cdot AB^2 - 2AB^2 \cdot BP^2 = 0$$

takes the following form:

$$(2) \qquad \sum_1^3 a_{ij}^4 + \sum_{\substack{1 \\ (j \neq k)}}^3 a_{ij}^2 a_{ik}^2 - \sum_{\substack{1 \\ (i \neq k)}}^3 a_{ij}^2 a_{kl}^2 = 0.$$

From (1) we have

$$(a_{1j} + a_{2j})^2 = a_{3j}^2 \text{ or } a_{1j}^2 + a_{2j}^2 - a_{3j}^2 = -2a_{1j}a_{2j};$$

and squaring both sides of the last expression,

$$(3) \qquad a_{1j}^4 + a_{2j}^4 + a_{3j}^4 - 2a_{1j}^2 a_{2j}^2 - 2a_{2j}^2 a_{3j}^2 - 2a_{3j}^2 a_{1j}^2 = 0; \; j = 1, 2, 3.$$

Again, using (1) we may prove the following identities:

$$(4) \qquad \begin{aligned} &a_{1i}^2 a_{1j}^2 + a_{2i}^2 a_{2j}^2 + a_{3i}^2 a_{3j}^2 - a_{1i}^2 a_{2j}^2 - a_{2i}^2 a_{1j}^2 \\ &\quad = 2a_{1i}a_{1j}a_{3i}a_{3j} + 2a_{2i}a_{2j}a_{3i}a_{3j}; \qquad i \neq j \end{aligned}$$

Taking into account (3) and (4) we get from (2):

$$\begin{aligned} &(a_{11}a_{32} - a_{12}a_{31})^2 + (a_{12}a_{33} - a_{13}a_{32})^2 + (a_{13}a_{31} - a_{11}a_{33})^2 \\ &\quad + (a_{21}a_{32} - a_{22}a_{31})^2 + (a_{22}a_{33} - a_{23}a_{32})^2 + (a_{23}a_{31} - a_{21}a_{33})^2 = 0, \end{aligned}$$

which gives us six conditions, only two being independent. As such we may take

$$(x_1 - x_2)/(x_3 - x_1) = (y_1 - y_2)/(y_3 - y_1) = (z_1 - z_2)/(z_3 - z_1).$$

It is evident that in n-dimensional space we have $(n-1)$ conditions.

Replies were also received from J. Rosenbaum and W. R. Ransom.

A Note by the Editor

This "paradox" devised by Professor Coolidge emphasizes an important point, undoubtedly well known but often overlooked, namely, that if one limits the discussion to real functions of real variables no significance can be attached to the process of counting analytic conditions imposed on them. For under these restrictions of reality, any geometric configuration which can be characterized by a *set of several* analytic equations can also always be completely characterized by *just one* analytic equation, whether the configuration be in two, three or n-dimensional Euclidean space or indeed in any metric space of a denumerably infinite number of dimensions. It is very easy to see that this is the case, for suppose that

$$f_i(x_1, x_2, \cdots, x_m, \cdots) = 0 \qquad i, m = 1, 2, 3, \cdots$$

be the set of analytic conditions.

This set is equivalent to the single equation

$$\sum_i [f_i(x_1, \cdots, x_m, \cdots)]^2 = 0,$$

where the f's and x's are real.

And thus the explanation of Professor Coolidge's paradox may be said to be that his second formulation is valid only in real Euclidean three space while the first is equally valid in real or complex space.

R.E.G.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Humanism and Science. By Cassius Jackson Keyser. New York, Columbia University Press, 1931. xx+243 pages. $3.00.

The logical structure of this book is exceedingly simple and apparent. If one disagrees with its thesis it will be very easy to put his finger on the exact spot where the disagreement occurs. After reading "literary" and "philosophical" productions dealing with "humanism," weltering, as many of them do, in vague words and confused sequences of statements and paragraphs, it is a relief as from a nightmare to come upon this clear and beautiful exposition.

The book is in Professor Keyser's well known style, a style that certainly is in great contrast to the indignities upon our language that are all too common in the writings of specialists in the sciences. There is moreover much good poetry, though it is written in the form of prose.

The contents of the book can be put before the reader most effectively by quoting its own words, and that is done freely in this review.

In the Foreword we read:

"Somebody ought to tell, or try to tell, the truth about the relations of Humanism to Science.

"In the extensive literature that the so-called new 'humanistic movement' has recently produced in England, in France and especially in America, the relations of Humanism— or what the writers have called Humanism—to literary criticism, to the criticism of art, to education, to politics, to psychology, to philosophy, to ethics, to religion, and to other cardinal interests, have been pretty fully discussed, but the task of setting forth its relations to the most momentous and characteristic intellectual enterprise of the modern world has not even been seriously undertaken, much less performed. And so I have decided to try it myself."

In the literature referred to there are frequent indications that Humanism and Science are believed to be antagonistic—that this movement is directly at variance with the spirit and the aims of what Keyser rightly calls the "most momentous and characteristic intellectual enterprise of the modern world." The main thesis of Professor Keyser's book is a denial of this belief. To maintain this thesis it is necessary to inquire into the exact meaning of the terms "Humanism" and "Science," and this Professor Keyser at once proceeds to do. What meaning do those who regard themselves as identified with this movement attach to the word humanism? Professor Irving Babbitt, who has been said to be at the center of the humanist movement, says: "Humanists are those who, in any age, aim at proportionateness through a cultivation of the law of measure." This meaning of the word, "when understood in accordance with its historical significance and its philosophical implications," Professor Keyser finds far too narrow.

"In the light of the foregoing considerations it is, I think, abundantly evident that the 'strict' doctrine of Professor Babbitt and his kind is too narrow, too sectarian, too prim, too supercilious, too dogmatic, too pharisaical, too lacking in catholicity, in sympathy, and in magnanimity, to bear worthily the historically great name Humanism."

Other meanings currently assigned to humanism are examined and then Professor Keyser says:

"For brevity, penetration, clarity and comprehensiveness the best description I have seen is in the following words of Mr. Walter Lippmann: 'Humanism signifies the intention of men to concern themselves with the discovery of a good life on this planet by the use of human faculties.' What is here so compendiously described is no cult, no doctrine, no sect, no dogma, but a human 'intention.' Not the intention of depraved men to escape, by humble submission to ecclesiastical authority and by means of divine grace, from the eternal wrath of an angry God; not the intention of superstitious men to propitiate, by purchase or prayer

or sacrifice, the countless malignant spirits of an imaginary environment; not the intention of dupes to submit supinely to the indignities of the present world in the hope of everlasting bliss in a world beyond the grave; but the intention of perfectly natural men 'to concern themselves with the discovery'—or as I should prefer to say, with the creation—'of a good life' here upon this mundane sphere where they actually live, and to do it by the application of native powers resident in themselves to the waiting resources of the actual world."

"In respect of spirit and aim and implicit scope the conception thus indicated accords perfectly with the Humanism of antiquity · · · And it accords completely with the Humanism which, after the lapse of more than a thousand years including the Dark Ages, began in the fourteenth century to emerge again, sprang into full life in the fifteenth century and greatly flourished as an essential and momentous part of the Rennaissance first in Italy and later in other countries of Europe. Humans are, as human, naturally endowed with the dignity of autonomous beings, potentially qualified by native inheritance to judge, individually and independently for themselves, in all the great matters of human concern, and, by the exercise of their own faculties, to order and fashion their lives worthily."

"The living sense of personal autonomy is absolutely essential to the dignity of man as man. Whenever and wherever it has been dormant or dead, Humanism has declined or perished. It is from the living sense in men of their personal autonomy that Humanism derives its existence, its character and its power. The fact was continuously manifest in the activity of the great humanists of the Renaissance and frequently became articulate in their words."

So much for the meaning of "Humanism."

The definition of "Science" used by Professor Keyser is that developed in his recent book "Pastures of Wonder," which was reviewed in this journal, vol. 37, p. 81, (see also "Review of a Review," this journal, vol. 38, p. 39).

"Science is the enterprise having for its aim to establish Categorical propositions; in other words, it is the enterprise having for its aim to answer questions relating to the Actual world.

"According to that definition the scope of Science excludes no subject-matter of the Actual world but includes any element thereof, whether material or mental, physical or psychical, with respect to which one can significantly say categorically that it has, or that it has not, such-and-such properties or relations."

Professor Keyser's distinction between "Science" and "Mathematics," which has aroused some controversy[1] is not essential in the discussion of this book, but the extension of the term "Science" to cover a much larger field than the so-called "experimental natural sciences" greatly strengthens his main thesis.

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[1] After some reflection I now wish to record my belief that Professor Keyser's definitions of the words "science" and "mathematics" as given in his "Pastures of Wonder" will be adopted increasingly by those who use these words with discrimination.

It is then maintained that the spirit of science is in the closest accord with the spirit of humanism, and that the achievements of science are the most important means whereby humanism in modern times is moving towards its goal. The argument runs smoothly and irresistibly.

"Science springs from humanistic soil. By that I mean that Science has its base and rootage in human interests, propensities and capacities that, save for variations in degree, are common to all mankind. · · · The spirit of Science—the spirit that seeks Truth, the spirit that leads to wisdom, knowledge, understanding of the Actual—is the animating principle of Man, and it is the soul of Humanism."

Regarding the achievement which science has already made in furthering the cause of humanism we read:

"Here it must suffice to remind ourselves in the most general terms of the immense transformations in our conception of the world and of the countless contributions to the art of life that scientific inventions and other applications of Science have made. It is sufficient to remind ourselves of the fact that such inventions and applications, by conquering not only the continents but also the oceans and the air, and by multiplying many million times our human powers of locomotion, hearing, vision, and speech, thus abolishing what were once insuperable difficulties both of time and of space, have resulted in virtually reducing the entire modern world to the dimensions of an ancient province, thereby bringing the divers people of the globe more and more into the mutual relationship of so many families of one vast community. Who can be so blinded by what remains to be done as not to see that, owing to the agency of Science, immeasurable strides have been taken forward on the endless path towards the realization of Humanism's dream? It is indeed a far cry from a planet inhabited by head-hunting tribes to a world of great nations assembled in friendly council to promote the achievement of a good life for all mankind by the use of human faculties."

Certain difficulties do arise in Professor Keysor's argument: Humanism is said to postulate the "autonomy" of man. This implies freedom of choice, of will. Does this exclude from the ranks of humanists those who hold deterministic views of nature, including man—those who do not believe in the freedom of the will?

Humanists, we are told, seek their ends here on earth and by use of human faculties. Does this exclude those who seek the same ends, in part, by divine help, and does it exclude those who make part of their purpose so to live that they will be placed more advantageously in a future life, or may these be classed as humanists plus something besides?

Does modern science disclose a universe so vast in space and time as to leave man an utterly insignificant mote and his life a flash of movement, "A lantern between dark and dark," and thus lead to despair?

To learn Professor Keyser's way of dealing with these questions one must read his book.

The last chapter of the book is entitled "The Humanistic Bearings of Mathematics." Every mathematician should read this chapter. Frankly, I know of no printed statement which in the space of fifty-odd short pages says so much (and says it so well and so simply) about the essential nature of mathematics and its general human significance. An attempt to epitomize this chapter would be tedious and superfluous.

Some have questioned whether a mathematician should write a book such as this. My answer is emphatically, yes; yes, if he has the wit and taste and information that are necessary to do it. The counter question we may put, why should he not do it? Why should one who has spent a long life in close contact with mathematics and science refrain from a study of the relations of these great fields to more general human interest? Is it not a worthy enterprise to attempt to understand their signifigance as a part of modern life and culture, when these are viewed as a whole? Is it not worth while to attempt to broaden the understanding of specialists in other fields so that instead of being antagonistic to the work of their colleagues and fellows in arms in the common assault on the great unknown, they may understand and appreciate and make common cause? May it not be that the weakest element in the intellectual life of our universities and research institutions is a tendency to fail to evaluate what other so-called "departments" are doing—a tendency not to question the more general significance of special results? A few have disregarded the traditional arrangement into "compartments of thought." In America Professor Keyser is conspicuous among these. England has her Bertrand Russell and her A. N. Whitehead (now of Harvard). In our time, which is supposed to have brought the widest freedom, a freedom so great as to permit even interdepartmental migration of workers, censure for such men is certainly not justified, and if new light is the result of their labors our thanks and appreciation are deserved.

N. J. LENNES

Mathematics for Junior High School Teachers. By William L. Schaaf. Richmond, Johnson Publishing Company, 1931. xiv+440 pages.

This book fills a want that certainly has long been indicated under the curricular arrangement of our present educational system. The reviewer has, till now, read nothing that gives to the candidate, as well as to the present occupant of the position of mathematics teacher so concise and complete a background of essential content.

At the same time practical methods of presentation are not neglected, and the wealth of material for pedagogical study gives the reader plenty of scope to exercise his ability in the direction of original work in this field.

The author's arrangement, which follows a natural tendency of scholastic development along mathematical lines, is logical. He starts with direct measurement, a practical subject which can be checked by simple appeal to the senses; follows this with a chapter on intuitive geometry, goes on to geometrical men-

suration, and then takes up demonstrative geometry and numerical trigonometry. This is followed by a chapter on the statistical graph, emphasizing its value in the field of pedagogy. Then come chapters on arithmetic which in the beginning deal with the common matters that enter a child's life. Subsequently the author analyzes the relations of arithmetic to community business, including banking, investments, and insurance; things with which the reader is bound to come in contact some time or other.

Much attention is given to arithmetic, the attempt obviously being made to give the prospective teacher a very complete survey of this subject and also to prepare the way for algebraic analysis.

The teacher who enters a class room with the background obtained from intelligent use of this book, ought never be at a loss for a practical illustration of the principles he wishes to develop.

The directed number, the formula and the general number, are subject matter for the next chapters and these form a natural introduction to formal algebra and the algebraic graph.

The book is full of meat; there is no waste material in its 400 pages, and as a reference book it should prove invaluable long after it has been thoroughly studied. It seems almost impossible, in a short review, to do it justice; for the table of contents, alone, five pages of condensed printing, gives only the most cursory index of the material summarized.

A course for teachers of mathematics, with this as a text book, should without doubt prove most valuable; and this applies to the aspirant for a collegiate chair almost as much as it does to the Junior High School applicant. To do it justice, however, such a course should be carefully digested and not be hurriedly skimmed over. The questions for discussion at the end of each chapter should be fertile material for making the students think over and develop the modern attitude toward content in mathematics as well as toward practice and theory of teaching.

However, the reader should be warned that this book is one to be *studied*, and does not lend itself to casual reading. He must be prepared to give it undivided and concentrated attention.

From the standpoint of manufacture, like all other publications of the Johnson Publishing Company, it has the merits of being well-bound, well-printed and seems to have been carefully proof-read and edited. A somewhat wider margin might add to the appearance of its pages, and at the same time give opportunity for marginal notes.

ARTHUR HAAS

The Dynamic Universe. By James Mackaye. New York, Charles Scribner's Sons, 1931. 308 pages. $3.50.

The ability and integrity of Einstein have been freely recognized by the early opponents of his theory. The book under review, however, gives the im-

pression of impugning the character of the inventor of the relativity theory. According to this work Einstein is merely adorning himself with borrowed plumes and has succeeded in duping all scientists through "mathematical fictions manipulated at his option." He is even charged overtly with juggling. "Einstein has succeeded, somehow, in conjuring explanations out of the hat by simply juggling definitions and units." He is accused of appropriating ideas of others and putting them forth as his own under some disguise. "Einstein's theory is only a disguise · · ·. The relativity equations are disguises for classical laws." The character of one guilty of juggling and abstracting other peoples' ideas would appear to be anything but honorable.

The relativity theory is utterly destroyed in *The Dynamic Universe.* It is proved to be wrong in general and, what is even worse, to be deliberately deceptive, plagiaristic. "The relativity equations are everywhere disguises." It is claimed that the theory has been harmful to physical research through the widespread confusion which it has produced in the realm of physics.

This general picture of the relativity theory given in the book is unfair, to say the least. A theory which has been fully approved by the best physicists of our age, such as Planck, Laue, Michelson, etc., and whose originator has been awarded the Nobel prize, cannot be as worthless as it appears to be from the book. This argument, however, of wide recognition of the theory is not sufficient to prove its criticism to be wrong. For this the book furnishes a far stronger argument, namely, evidence of a considerable lack of knowledge of the theory on the part of the author.

The book condemns the relativity theory on the ground that it is based on definitions: "Einstein begs the question of whether time, length, etc. are relative by defining them to be so. · · · Time and length are relative by definition." Such passages reveal misunderstanding of the relativity theory. Time and length are relative in it, not by "defining them to be so," but as a consequence of the two fundamental principles or assumptions of the theory. The validity of these being granted, it can be proved conclusively that simultaneity of distant events, an interval of time, and a spatial length are different for different observers. These factors are, therefore, relative not "by definition" but by proof deduced from two principles.

That failure to understand the relativity theory underlies its criticism in the book may be seen from the following citations. The assertion is attributed to Einstein that "a gravitational field can be produced by somebody pulling on a rope" or that "pulling on a rope produces a gravitational field." This is denied with the remark that "neither pulling on ropes, nor any other means of accelerating bodies can alter the gravitational field of the earth." To put such an interpretation on Einstein's famous illustration of his equivalence principle betrays insufficient acquaintance with the relativity theory. Einstein asks us to imagine in the interstellar space a room pulled "upwards" with constant force by some being. The behavior of all objects inside the room would in nowise be distinguishable from that which they would show if the room, instead of being pulled

"upwards," were in a gravitational field acting "downwards." In both cases an observer in the room would see that objects which he lets loose approach the floor with an acceleration which is the same for all of them, the same for a small pellet of cork as for a big lump of lead. This means that acceleration in one direction through any force is equal in its observable effects to acceleration in the opposite direction through gravitational force. Only misunderstanding of the relativity theory can see in that lucid illustration the claim that "pulling on ropes can alter the gravitational field of the earth."

Further proof of miscomprehension of the relativity theory is furnished by the book through the following argument. The general relativity theory is founded upon the equality of gravitational mass and inertial mass. The book tries to destroy this foundation by proving that gravitation and inertia are dissimilar. It resorts to an imaginary experiment. The weight and inertia of a body are measured at the Equator. The body is then removed to the Pole and the measurements are repeated. They show that the weight of the body has increased but its inertia has remained the same. From this the conclusion is drawn that there is "a fundamental dissimilarity" between gravitation and inertia. Hence the basis of the general theory is destroyed. But the conclusion contains a grave error. From the experiment one may infer only the inequality of weight and inertia but not that of gravitational mass and inertial mass. That conclusion shows that gravitational mass has been confounded with weight. But the two are not identical. Gravitational mass is the quotient obtained by dividing weight by gravitational acceleration. This quotient does not vary with the place on the earth. Its value for a body weighing one kilogram in Paris can be shown to be 0.10195 everywhere on the earth. The inertial mass is expressible by the same quotient. Hence gravitational mass and inertial mass are not made unequal by that experiment. They appear to be unequal in the book because of its confounding gravitational mass with weight.

The book disparages the relativity theory for its failure to explain Nature, that is, to reveal cause and inner mechanism of all natural phenomena: "The power of the general relativity theory to explain anything is denied by some of the ablest advocates thereof. . . . The total matter theory can neither explain nor predict anything." With regard to such statements—there are many more of them in the book—the author of the relativity theory may well say, "God protect me from my friends, against my enemies I shall defend myself." The writings of some of the stanchest supporters of the theory have created the impression that it explains everything and anything in the universe; that is, discloses cause and inner workings of all natural phenomena. But the theory pretends nothing of the kind. The special relativity theory makes no claim regarding the explanation of the natural phenomena except that "two assumptions are sufficient to arrive at a simple electroydnamics of moving bodies free from contradiction." No assertion is made that these assumptions give an account of cause and workings of electromagnetism, nor even of light which plays the most important part in the theory. Only the observable optical phenomena are

treated in it, namely, the facts that light is propagated and that it has a certain velocity in a given medium through which it passes. The cause of light and its mode of propagation are left out of the question entirely.

Neither does the general theory pretend to reveal cause and inner workings of the physical phenomena of gravitation and inertia which figure most prominently in it. Nothing is alleged regarding their cause and intrinsic mode of operation except that the gravitational and inertial phenomena observable in a body may have their cause not in the body alone but also in the masses of the universe outside of it. No attempt is made to explain that cause. Censuring the relativity theory for untenable claims not made in the original monographs on it nor in related writings of its creator is unjustifiable.

The author of *The Dynamic Universe* proposes a theory of his own, called the *radiation theory*. With regard to claims it does not compare favorably with the relativity theory. The latter is distinguished through the great moderation of its claims. It does not pretend to explain any natural process, that is, to lay bare its cause and inner mechanism. The new radiation theory, on the other hand, seems to pride itself on being able to explain in this sense all physical phenomena, static electricity, light, attraction, repulsion, impenetrability, gravitation, inertia, etc.

I can say but very little about this new radiation theory because it is unintelligible to me. Whether physicists will understand it and pay attention to it remains to be seen. To me the theory appears to refute itself through the extravagance of its claims. There are a good many vague statements in the presentation of the new theory. I shall merely point out one strange assertion. Discussing "radiation displacement" the book remarks that "it has no more to do with space and time than physical phenomena in general." This can only mean that physical phenomena in general have nothing to do with space and time. Hence they must occur outside of space and time. One may well wonder how a physical phenomenon can occur in no place and at no instant. A radiation displacement having nothing to do with space is a palpable contradiction in itself. A displacement always implies space. But perhaps the "radiation displacement" of the new radiation theory is no displacement at all. What it actually is one cannot gather from the book.

Nevertheless science may indirectly benefit by the book under review. Through the work attention is called to the necessity of purging the relativity theory of dross injected into it by metaphysicians, and physicists with a bent towards mysticism. Their exaggerated claims pass for an integral part of the theory, marring its original beauty and simplicity and rendering it incomprehensible. The mystically inclined physicists would want the theory to be a panacea for all the troubles and difficulties in explaining Nature. In the end the panacea does not work, and then they advocate, as a remedy for those troubles, views akin to the exploded superstitions of the dark ages. Severe criticism is administered to such "relativists" in *The Dynamic Universe* and is fully justified. The relativity theory itself becomes involved in this criticism when their vaga-

ries pass for a part of the theory. Ask a layman what the relativity theory is, and he will probable answer that it is a doctrine asserting that a "fourth dimension" exists and that "space is curved." These ideas have been harped upon too much, and many relativists have only added to their mysteriousness. "Fourth dimension" is one of four straight lines perpendicular to each other in one point, and "curved space" has the same character as a crooked line and a bent surface. Our objection that these two things are impossible is brushed aside by the empty argument that we merely do not possess the power to visualize them. From those mystical notions the inference is drawn that right may be left and left may be right, past may come after future, and effect may precede the cause. Such preposterous conclusions are arrived at through "verbal hocus-pocus," as the book expressively characterizes the unintelligible talk of the relativists.

Einstein used those two terms in a radically different sense. Had he foreseen the confusion they have caused, he would probably have avoided them in expounding his theory. This can be done very well, as I have shown in a paper entitled "*A Simple Presentation of the Fundamentals of the Relativity Theory*" (Scientific Monthly, January, 1932). It shows that by keeping clear of those two expressions an interpretation of the theory gains in lucidity, frees the theory of all mysteriousness and renders it quite comprehensible.

There are other wrong notions about the relativity theory which constitute foreign dross to be eliminated together with the absurdities mentioned above. It is an error, for instance, that the theory has abolished the ether. Einstein himself states that "according to the general relativity theory a space without ether is inconceivable." Further, the rumor that the theory is incomprehensible and only twelve men in the whole world understand it is a myth. The statement in *The Dynamic Universe* that "few physicists pretend to understand it" will not be admitted by any competent physicist. What is true about the rumor is that even highly educated laymen have very little understanding of the theory. One reason for this is to be found in the encroachments upon it by metaphysics and mysticism. The principal reason is that the faculty for good teaching is a rare natural endowment, as rare as the gift of creating choice poetry; and there has not as yet appeared on the scene representing the relativity theory the talented actor who is both a competent physicist and an ideal teacher.

MAX TALMEY

Allgemeine Natürliche Geometrie und Liesche Transformationsgruppen. By G. Kowalewski. Berlin and Leipzig, W. de Gruyter & Co., 1931, 280 pages. RM 15.50.

In 1889, E. Cesàro published in the Rendiconti dei Lincei the first of a series of articles on the intrinsic geometry of curves. The first two chapters of the present work present this theory, for plane and space curves, as it has been developed out of Cesàro's basic ideas. Starting from the equation $\rho=\rho(s)$, which expresses the radius of curvature of a plane curve as a function of the length of arc, he shows how a Cartesian equation of the curve may be derived and how the

properties of the curve, and the loci associated with it, that are invariant under motion may be determined. Similarly, from the intrinsic equations $\kappa=\kappa(s)$, $\tau=\tau(s)$ which express the curvature and torsion of a space curve in terms of s, an analogous study is made of the invariants under motion of space curves.

The group of motions, however, is only one special type of transformation to which we may wish to subject a curve. In the next two chapters, the author generalizes to other types of Lie transformations the essentials of the ideas and methods that he has derived in the first two chapters. This discussion is more difficult to follow, not only because of its greater generality, but also because more advanced mathematical training on the part of the reader is presupposed. In the last chapter, the fundamentals of Lie's theory of transformation groups is explained and connected with the present treatment of the subject.

The author's style is a bit abrupt, but it is stimulating and suggestive. The mode of exposition is interesting, in that the theory is expounded, as far as practicable, by means of discussions of carefully chosen special cases. The reviewer believes that this form of presentation has been used very successfully. Readers who wish to learn how to use this valuable theory as a tool for their own investigations will be well pleased with the author's presentation of the subject.

C. H. SISAM

Mathematical Tables and Formulas. Compiled by Robert D. Carmichael and Edwin R. Smith. Boston, Ginn and Company, 1931. viii+270 pages. $2.00.

The principal tables in this volume are five-place tables of logarithms of numbers, of logarithms of trigonometric functions, and of natural functions. These tables are arranged in columns down the page instead of the usual arrangement across the page, making the tables rather more bulky but more convenient. There are also four-place tables, a table of important constants, a table of functions of multiples of 15°, tables of functions and logarithms of functions of angles expressed in radians, and the usual conversion tables between the circular and the sexagesimal systems. In Part II there are tables of powers, roots, and reciprocals, natural logarithms, exponential and hyperbolic functions, of multiples of $M=0.43429$ and its reciprocal, and finally a table of ten-place logarithms of prime numbers less than 1000. Part III consists of over 50 pages of formulas and theorems from algebra, geometry, trigonometry, analytic geometry and the calculus. There are ten pages of standard graphs, an integral table consisting of 360 formulas, and 50 infinite series. All this material is carefully chosen, and the book should be valuable to mathematicians.

R. A. J.

Electricity and Magnetism, The Mathematical Theory. By Vincent C. Poor. New York, John Wiley & Sons, Inc., 1931. vii+183 pages.

This monograph is substantially an abridged exposition of the classical theory of electricity and magnetism. In the author's own words, "this book should appeal to a very select class of readers, the physicists, the mathemati-

cians and the electrical engineers, who wish to orient themselves quickly in electrical theory." It is true that "... the professional electrical engineer is graduated without being able to read electrical theory. This is a deplorable situation." From my own experience, I can state that much of the future "mathematization" of electrical engineering will consist of gradually incorporating the "electromagnetic theory" as distinguished from the "circuit theory" into the thinking of every electrical engineer. Both theories represent invaluable methods of solving "everyday problems" from now on. For these reasons, we rejoice that the present book is clearly written and regret that there are serious omissions and occasional emphases on comparatively non-important aspects of the theory. An eight page introduction dealing with Vector Analysis is followed by a forty page development of electrostatics. There, the concepts of electric force, displacement, electric potential, and so on are defined but unfortunately the important idea of capacity is hardly mentioned and the fundamental electrostatic problem is not emphatically stated. The explanation of the idea of "displacement" is rather cumbersome; this is due to a somewhat unfortunate choice of experimental evidence leading to this particular concept. The magnetic field due to permanent magnets is the subject matter of the second chapter. In the next chapter, dealing with current electricity, the author develops Maxwell's equations. Although he gives the equations in a perfectly general form, no concrete example is given of the significance of these equations in the study of electric flow in conductors; like many other exponents of the classical theory, the author is interested primarily in electromagnetic waves in dielectrics. Although the choice of the subject matter is comparatively immaterial to the mathematicians, the physicists and engineers would have certainly appreciated at least some account of the "skin effect in cylindrical conductors," or some similar problem as the most elementary instance of the utility of Maxwell's equations in the solution of practical problems. Chapter IV is devoted to the dynamical theory of electric currents or—as it is commonly known among electrical engineers—the circuit theory. Starting with D'Alembert's Principle the author arrives toward the end of the chapter at the differential equation of one-mesh circuit and concludes with its solution. The electron theory and the part played by the special relativity in the theory of electricity and magnetism are the topics reserved for the last chapter. Our general impression is: First, that the book is not likely to suit the tastes of the majority of engineers because no concrete and practically significant problems are either solved or stated for solution; second, that incomplete as it is, the treatment is clear, even if old-fashioned in places, but not always the simplest; that on this account, it is suitable as a text for the classroom use in engineering colleges where it could be supplemented by the instructors.

S. A. SCHELKUNOFF

Storia delle Matematiche. Vol. II. I secoli XVI e XVII. By Gino Loria. Padua, Casa Milani ("Cedam"), 1931. 595 pages. 23 lire.

The first volume of this work appeared in 1929 and dealt chiefly with the

classical period, the middle ages, and the renaissance. In a review of that volume (this *Monthly*, vol. 36, p. 387) the late Professor Cajori paid this just tribute to the author: "Combining readability with the requirements of scientific accuracy is an undertaking in which it would be difficult to surpass the distinguished historian from Genoa." Perhaps the very fact that the Italian word for "history" is the same as that for "story" inclines the writers of that country to make their histories more readable than those of other lands. It was the Italian Libri, although writing in French, who made his history of the mathematics of his people read like a fascinating story.

Professor Loria rightly asserts that the two centuries with which he deals, the 16th and 17th, constitute a period with special characteristics of their own. As he says, the years from 1200 to 1500 were relatively barren in comparison with these two centuries. Indeed, measured by the unit of print space, his giving of only a hundred pages more to the mathematics of human history ending with 1500 (as in vol. I) is justifiable. We feel that we live at present in an era of marvellous progress, and so we do; but that the future historian will rank the 18th and 19th centuries very much higher than the 16th and 17th in the opening of new fields of great importance in the domain of mathematics is doubtful.

The chapters follow in due sequence the fifteen of the first volume, and a statement of their general contents is perhaps the best way of giving a fair idea of the ground covered and the arrangement of the material. Stated as briefly as possible the chapter titles and contents are as follows: XVI, Syncopated algebra in its apogee, Part I, Italy, with special reference to the works of Tartaglia, Cardan, Bombelli, and Benedetti; XVII, Syncopated algebra in its apogee, Part II, transalpine, with special attention to Rudolff, Stifel, Recorde, Nunes, Petri, Stevin, van Roomen, and Viète; XVIII, The influence of Humanism on mathematics, as seen in the editions of the Greek classics, and in the works of Maurolico, Commandino, Benedetti, and Guidobaldo del Monte, with an interesting mention of the dawn of the historical study of mathematics (Ramus and Bernadino Baldi); XIX, Trigonometry and cyclometry in the 16th century, at the hands of Werner, Copernicus, Rheticus, and Pitiscus, but more especially as due to Tycho Brahe, Viète, Snell, and Stevin; the sub-titles include the problem of the quadrature of the circle in the 16th century (Clavius, di Monforte, Cristoforo); XX, Aids to scientific correspondence, periodicals, scientific societies, universities; XXI, First years of a glorious century (the 17th), with special attention to Napier, Bürgi, Cataldi, Galileo, Kepler, d'Aguillon, and Bachet de Méziriac; XXII, the disciples of Galileo, particularly Cavalieri, Torricelli, Viviani, Ricci, and Borelli; XXIII, Gli algebristi della vigilia (who but an Italian could express it so poetically and at the same time so exactly?),—Girard, Harriot, Oughtred, and Hérigone; XXIV, The beginning of modern mathematics,—Descartes and Fermat; XXV, The awakening of pure geometry (The Italian is more poetical,—Risveglio, the French *réveil, réveiller*), Desargues and Pascal; XXVI, The precursors of the infinitesimal calculus,—Vincenzo and Tacquet, Roberval, Wallis, de Sluse, Stefano degli Angeli, Mengoli, N. Mercator, Barrow,

David Gregory, and Lord Brouncker; XXVII, The *intermezzo* in which the geometric field was further explored; XXVIII, the Newton-Leibniz controversy.

The work may be characterized in various ways. In the first place it gives the impression of being well-balanced. Although written by one whose interests might tend chiefly to the Italian contributions, there is no evidence of any prejudice in favor of the achievements of his countrymen. Where he has devoted more than the usual attention to Tartaglia and Cardan, he makes us his debtors by his thorough study of the history of the cubic equation, and by his bibliography of the controversy. There are numerous other cases in which his extensive knowledge of the Italian works has enabled him to call attention to matters that have not been so thoroughly considered by most historians, as for example, in the relation of Stevin to Bombelli in the matter of Decimals (p. 111).

Another distinguishing feature is the revaluation of the works of certain writers usually somewhat neglected, such as Giambattista Benedetti, Petri, Guidobaldo del Monte, Scaliger, Simon du Chesne, and d'Aguillon. On the other hand, it might have been better to have given some or more attention to writers like Juan de Ortega, the Taglientes, Vander Hoecke, Köbel, Lazesio, E. de la Roche, dal Sole, Sfortunati, de Boissière, Fernel, Scheubel, Perez de Moya, Glareanus, Digges, Gillibrand, and Henrion, some of whom, however, contributed so little as hardly to be worth mentioning in such a survey as this. The question also arises as to the desirability of neglecting generally the oriental writers of this period some of whom, like Seki Kōwa in Japan, the Jesuits and their followers in China, and Raganātha in India, made contributions worthy of brief mention at least.

A third item that will be welcomed by all readers is the judicial manner in which the Leibniz-Newton controversy is treated, a controversy which would probably never have become acute had it not been for the stupidity of Faccio di Duillier (Nicolas Fatio de Duillier).

The list of corrections to Volume I, given on pages 593–595, is formidable enough to relieve other writers when they contemplate their own shortcomings. On the other hand it raises the question as to possible errors in Volume II. That they exist will be apparent to any reader, as in *librairy* (p. 21 n), *In artem analyticen* (p. 118), *The Grounds of Arts* (p. 105), *The whestone of witte* (pp. 106, 129), *Pathway to knowledge* (good English but not the original spelling), *Teuby* (for Tenby, p. 105), *praise* (for practise, p. 106), and various others of the same nature, easily detected. It would give quite a false impression of the book to call attention to any others, limited, as generally seems to be the case, to minor typographical errors.

In general it may be said that Professor Loria has, in these two volumes, given the story of mathematics up to the year 1700 with fairness and accuracy and with sufficient completeness for the student beginning his reading in the historical field of the science. Mathematicians will look forward with even more interest to a third volume, relating to the 18th and 19th centuries if, as is doubtless the case, this is to be added to his treatment of the subject.

DAVID EUGENE SMITH

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3521. *Proposed by J. M. West, State College, Pa.*

Show that the equations of the tangents of the circle $x^2+y^2=r^2$, through the external point (a, b) are

$$(ar \pm bt)x + (br \mp at)y = r(a^2 + b^2),$$

where t is the length of the tangents from the point (a, b) to the circle.

3522. *Proposed by A. Galbraith, Ash Grove, Mo.*

To trisect any angle, approximately, with straight edge and compasses only take the vertex of the given angle as a center and with any radius describe the arc AC. Divide the chord AC into six equal parts, the middle point being B and the other two points between B and C being F and G in the order B, F, G, C. On AC produced, locate the points H and K such that $CH=\frac{1}{6}AC$ and $HK=\frac{1}{2}CH$. At F erect a perpendicular to AC intersecting the arc AC in E. With G as a center and a radius EG, describe an arc intersecting the chord AC in M. With K as a center and radius EK describe an arc intersecting AC in N. Locate point P, the mid-point of MN. With E as a center and radius PH, describe an arc intersecting AC in L. With L as a center and a radius EL describe an arc intersecting AC in O. Then CO will be the chord of the trisected arc AC.

A Note by the Editors: This is the most accurate method of trisecting any angle approximately by means of a straight edge and compasses only that has ever come under our observation.

We are publishing it for the double purpose of acquainting our readers with this rather complex though very accurate method and to have some of them determine for what angles the method is absolutely exact. We suggest that those who are interested, take a ruler and compasses and follow out the construction. An ocular proof is thus seen to be very convincing to the non-mathematical mind.

3523. *Proposed by Dewey C. Duncan, University of California.*

Solve completely the equation

$$x^7 + Ax^6 + Bx^4 + Cx^2 + D = 0,$$

A, B, C, D, being unknown constants and given that the equation has a double root and one triple root.

3524. *Proposed by W. H. Echols, University of Virginia.*

At the corners of any equilateral triangle ABC let there be hinged three equilateral triangles ALM, BNO, CPQ of any sizes or positions.

Then will the midpoints of each of the sets of three segments

$$(LQ, OP, MN), \quad (BQ, CN, OP),$$
$$(LB, OA, MN), \quad (AP, CM, LQ)$$

be the corners of an equilateral triangle.

3525. *Proposed by Henry A. Campbell, Omaha, Nebraska.*

A semicircle with center O has AC for the diameter at its base. The radius OC is prolonged to D so that $CD=OC=r$. The semi-circumference is bisected at Z, and on the arc ZC the point Y is taken so that ZY subtends at O an angle of 60°. Let the chord ZY produced cut OD in X; and with X as center and radius XD describe a semicircle on the same side of AD as that of the first one. Let B be any point on the arc AZ, and draw OB. Construct on this side of AD the angle DXM equal to the angle AOB, where M is the intersection of the side XM with the second semicircle. Let N be the projection of M on OD, and draw NB cutting again in T the semicircle at O. Prove or disprove that $NT=r$.

Editorial Note. If NT and r were equal, the angle AOB would be three times the angle ONB; and we know that no such construction will give this result for *any* position of B on the arc AZ. Determine the maximum error in this approximate method for trisecting an angle less than 90°, and show that the error is zero only for 90° and a zero angle.

3526. *Proposed by R. E. Gaines, The University of Richmond.*

If a triangle PQR be inscribed in the ellipse $x^2/a^2+y^2/b^2=1$, and if PQ and PR are tangent to the hyperbola $x^2/a^2-y^2/b^2=1$, the locus of the pole of QR with respect to the ellipse is identical with the locus of the pole of QR with respect to the hyperbola.

3527. *Proposed by R. E. Gaines, The University of Richmond.*

If a variable chord QR of a conic subtends a right angle at a fixed point P on the conic, the chord QR passes through a fixed point S. If P describes the conic, the locus of S is a conic whose equation differs from the equation of the given conic only in the constant term.

SOLUTIONS

434[1917, 328; 1918, 119]. *Proposed by E. W. Chittenden.*

Evaluate $\int_0^1 f(x)dx$ where

$$f(x) = \sum_{n=1}^{n=\infty} \frac{\operatorname{sgn}(x - x_n)}{2^n}.$$

The function sgn $x=-1, 0, +1$ according as x is negative, zero or positive. The numbers x_n form the series

$$\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{8}, \tfrac{3}{8}, \tfrac{5}{8}, \tfrac{7}{8}, \cdots$$

for which the general formula is $x_n=(2h+1)/2^k$, were k is the greatest integer such that $2^{k-1}\leqq n$ and $h=n-2^{k-1}$.

Note: This problem is reprinted because of a misprint in the original statement and also because of a lack of definiteness in the definition of the series of values x_n. An inquiry has also been received in regard to the origin of the notation "sgn x." This notation seems to be due to Kronecker; cf. Werke, vol. II, p. 500.

Solution by Herbert A. Meyer, Hanover College.

Let $f_n(x)$ be the nth term of the given series for $f(x)$. For each value of n, $f_n(x)$ assumes the value -2^{-n} in the interval $0\leqq x<x_n$ and the value $+2^{-n}$ in the interval $x_n<x\leqq 1$. When $x=x_n, f_n(x)=0$. The integral of this term from 0 to 1 is $-2^{-n}x_n+2^{-n}(1-x_n)=2^{-n}(1-2x_n)$. The term $f_n(x)$ may be represented geometrically by a broken straight line parallel to the x-axis, and its integral is the area between the x-axis and this broken line. Hence the desired integral is

$$\int_0^1 f(x)dx = \sum_{n=1}^{n=\infty}\left(\frac{1-2x_n}{2^n}\right) = 1 - \sum_{n=1}^{n=\infty}\frac{x_n}{2^{n-1}}\cdot$$

These series are absolutely convergent since x_n is not less than zero or greater than one. The error made in stopping with the rth term of the series of which the general term is $2^{1-n}x_n$ will be considerably less than 2^{1-r}.

Also solved by Ralph P. Agnew.

A Note by Otto Dunkel: The infinite series in the result may be made more rapidly convergent by summing groups of its terms. Thus, for $n=2^{k-1}+i$, $x_n=(2i+1)2^{-k}$, $i=0, 1, \cdots, 2^{k-1}-1$, $k=1, 2, 3, \cdots$. Then

$$2^{1-n}x_n = 2^{1-k-2^{k-1}}\cdot(2i+1)2^{-i}.$$

We now find the sum of the second factors for a fixed k. It is easily verified that

$$\Delta[-(2i+3)2^{1-i}] = (2i+1)2^{-i},$$

and hence

$$\sum_{i=0}^{p}(2i+1)2^{-i} = 6-(2p+5)2^{-p},\quad p = 2^{k-1}-1.$$

Hence the result may be written

$$1-4\sum_{k=1}^{\infty}[3\cdot 2^{-k-2^{k-1}} - 2^{-2^k} - 3\cdot 2^{-k-2^k}].$$

Using five terms, we find its value to 10 decimals to be .1123714457.

3140 [1925, 315]. *Proposed by C. C. MacDuffee, Ohio State University.*

Let f be any algebraic form of total degree $m>1$ in n variables, and $H(f)$ its Hessian. Let ϕ by any analytic function. Prove that

$$H[\phi(f)] = H(f)\cdot\left[\left(\frac{\partial\phi}{\partial f}\right)^n + \frac{mf}{m-1}\,\frac{\partial^2\phi}{\partial f^2}\left(\frac{\partial\phi}{\partial f}\right)^{n-1}\right].$$

In particular when $\phi(f)$ is f^r, we have Mrs. Ballantine's generalization of Problem 2908 proposed by Professor Dickson [*1921*, 326; *1923*, 41].

Again, when $\phi(f)=\log f$, we have

$$H(f) = (1-m)f^nH[\log f],$$

which is a generalization of Exercise 22 of Sir Thomas Muir's "*Budget of Exercises on Determinants*" printed in this MONTHLY [*1922*, 10].

Solution by the Proposer.

For brevity of writing set $\phi'=d\phi/df$, $\phi''=d^2\phi/df^2$, $f_i=\partial f/\partial x_i$, and $f_{ij}=\partial^2 f/\partial x_i\partial x_j$. Then

$$(1)\qquad H[\phi(f)] = \left|\frac{\partial^2\phi(f)}{\partial x_i\partial x_j}\right| = \left|\,\phi' f_{ij} + \phi'' f_i f_j\right| .$$

This can be expanded by columns into a sum of 2^n determinants. But, since all of these which involve more than one column of elements of the type $\phi'' f_i f_j$ are zero, we may write

$$(2)\qquad H[\phi(f)] = (\phi')^n H(f) + (\phi')^{n-1}\phi''\sum f_i \begin{vmatrix} f_{11} & \cdots & f_1 & \cdots & f_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{n1} & \cdots & f_n & \cdots & f_{nn} \end{vmatrix},$$

where the ith column contains the first partial derivatives of f, and the summation is for $i=1, 2, \cdots, n$. By Euler's theorem

$$f_j = \frac{1}{m-1}\sum_{k=1}^{n} x_k f_{jk},$$

and, by the use of this substitution in the ith column, the determinant in (2) expands into n determinants all of which but one is zero. The remaining one is $(m-1)^{-1}x_iH(f)$. Therefore

$$H[\phi(f)] = (\phi')^n H(f) + (\phi')^{n-1}\phi''(m-1)^{-1}H(f)\sum x_i f_i.$$

Since $\sum x_i f_i = mf$, we have the desired result

$$H[\phi(f)] = H(f)[(\phi')^n + m(m-1)^{-1}f\cdot(\phi')^{n-1}\phi''].$$

3276 [1927, 381]. *Proposed by L. L. Silverman and J. Tamarkin.*

Prove that if ν is an integer greater than or equal to 1, then

$$\int_0^\infty \frac{(1+z)^{-\nu}dz}{\log^2 z+\pi^2} = (-1)^{\nu-1}\int_0^1 \binom{t}{\nu}dt, \text{ where } \binom{t}{\nu} = \frac{t(t-1)\cdots(t-\nu+1)}{\nu!}.$$

II. *Solution by G. Szegö, Königsberg, Germany.*

Consider the function

$$F(z) = \frac{(1+z)^{-\nu}}{\log z - i\pi}(0 < \Im \log z < 2\pi;\ \nu = 1, 2, \cdots)$$

and integrate it over the contour C consisting of two circles C_r, C_R about the origin of radii r and R respectively, and of a segment of the axis of reals between $z=r$, $z=R$, described twice in two opposite directions. It is immediately seen that the contribution of the circles C_r, $C_R \to 0$ as $r\to 0$, $R\to\infty$, while the contribution of the two-fold integration along the rectilinear part of the contour C tends to

$$\int_0^\infty \frac{(1+x)^{-\nu}}{\log x - i\pi}dx - \int_0^\infty \frac{(1+x)^{-\nu}}{\log x + i\pi}dx = 2i\pi\int_0^\infty \frac{(1+x)^{-\nu}}{\log^2 x+\pi^2}dx.$$

Since the integrand $F(z)$ is single-valued and analytic in C, except for a pole at $z=-1$, the integral of the problem will be equal to the residue of $F(z)$ at $z=-1$. To compute this residue we set $z+1=\zeta$, whence

$$F(z) = \frac{\zeta^{-\nu}}{\log(\zeta-1) - i\pi} = \frac{\zeta^{-\nu}}{\log(1-\zeta)} = G(\zeta)$$

and compute the residue of $G(\zeta)$ at $\zeta=0$. But this, in view of the expansion

$$\frac{\zeta}{\log(1-\zeta)} = \sum_{\nu=0}^{\infty} a_\nu \zeta^\nu,$$

will coincide with the coefficient a_ν.

A solution of this problem by J. Tamarkin appeared in 1928, page 497.

3433. [1930, 315]. *Proposed by Thurman Andrew, Jamaica Plain, Mass.*

Consider three elastic rods of equal lengths and cross sectional areas. The cross sections have the forms of a circle, an equilateral triangle and a rectangle whose long side is twice the diameter of the circle. What will be the ratio of the torques required to turn them through a given angle about their axes? What is the shape of the solid rod which requires the greatest torque? The elastic limit is not to be exceeded.

Solution by D. L. Holl, Ames, Iowa

The solution of the torsion problem[1] subject to the condition of only a pure

[1] A. E. H. Love, *The Mathematical Theory of Elasticity*, 4th edition (1927); p. 310; *Handbuch der Physik*, Bd. VI (1928), p. 143; J. Prescott, *Applied Elasticity* (1924), p. 188.

couple over the cross section of the rod and no action on the sides, requires the solution of $\nabla^2\xi = -2\tau$ such that $\xi = 0$ on the contour. The torquing couple is then $Q = 2\mu\int\xi dA$ where the integration is carried out over the cross section of the prism, and the symbols are defined as follows: τ is twist per unit length of rod, and μ is the modulus of shear or $\mu = E/2(1+\sigma)$ with E the Young's modulus, and σ is Poisson's ratio.

For the circular rod of radius a, the solution is $\xi = \frac{1}{2}\tau(a^2 - r^2)$ and $Q_c = \frac{1}{2}\tau\pi\mu a^4$.

For the equilateral triangle,

$$\xi = \tfrac{1}{6}\tau a_1^{-1}(3xy^2 - x^3 + 4a_1^3) - \tfrac{1}{2}\tau(x^2 + y^2)$$

and

$$Q_T = (9\sqrt{3}/5)\mu\tau a_1^4,$$

where a_1 is the radius of the inscribed circle. To satisfy the condition of equal cross sections, $3\sqrt{3}a_1^2 = \pi a^2$; hence $Q_T = (\pi^2\sqrt{3}/15)\mu\tau a^4$.

For a rectangle of sides $2a$, $2b$, the solution is

$$\xi = \tau(a^2 - x^2) - \frac{32}{\pi^3}\tau a^2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cosh[(2n-1)\pi y/2a]\cos[(2n-1)\pi x/2a]}{(2n-1)^3\cosh[(2n-1)\pi b/2a]},$$

and

$$Q_R = \frac{16}{3}a^3 b\mu\tau - \left(\frac{4}{\pi}\right)^5 a^4\mu\tau \sum_{n=1}^{\infty} \frac{1}{(2n-1)^5}\tanh[(2n-1)\pi b/2a].$$

This series is rapidly convergent and, when b_1 and a_1 denote the longest and shortest dimensions, for $b_1 > 3a_1$ the following expression is sufficiently accurate:

$$Q_R = \tfrac{1}{3}a_1^3 b_1\mu\tau\left(1 - 0.630\frac{a_1}{b_1}\right).$$

To satisfy equal cross sections, $b_1 = 4a$, $a_1 = \frac{1}{4}\pi a$ in terms of radius of circle. Then

$$Q_R = \frac{\pi^3}{48}\mu\tau a^4\left(1 - 0.630\frac{\pi}{16}\right).$$

Hence

$$\begin{aligned} Q_C : Q_T : Q_R &= 1 : \frac{2\sqrt{3}}{15}\pi : \frac{\pi^2}{24}\left(1 - 0.630\frac{\pi}{16}\right) \\ &= 1 : 0.725 : 0.360. \end{aligned}$$

In each of the above cases, the origin of coördinates was taken at the centroid of the section.

To find the section which requires the greatest torque for a given twist τ per unit length, is to solve a problem of calculus of variations; the section can be shown to be a circle.

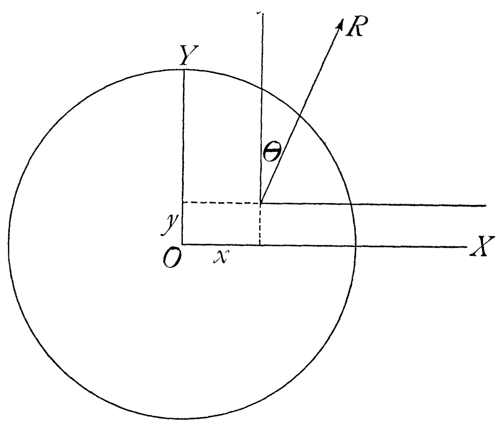
Intuitively it might be argued that since the rods have equal volume, and since the maximum stresses occur on the boundary at points nearest the center of twist, the circle, by reason of its compact area and uniformly increasing stresses towards the periphery, will absorb the most elastic energy per unit length, hence requiring the greatest torque per unit of twist τ.

3462. [1930, 508]. *Proposed by R. C. Colwell and O. R. Ford, West Virginia University.*

A circular hoop, which is free to move on a smooth horizontal plane, has sliding on it a small ring of $1/n$th its mass, the coefficient of friction between the two being μ. Initially the hoop is at rest and the ring has an angular velocity ω round the hoop. Show that the ring comes to rest relative to the hoop after a time $(1+n)/\mu\omega$.

Solution by the Proposer

Take the axes of x and y at the original position of the center of the hoop so that the coördinates x, y represent the position of the center at any time t.



The equations of motion are

(1) $$m\ddot{x} = R\sin\theta + \mu R\cos\theta,$$

(2) $$m\ddot{y} = R\cos\theta - \mu R\sin\theta,$$

(3) $$\frac{m}{n}\,\frac{d^2}{dt^2}[x + a\sin\theta] = -R\sin\theta - \mu R\cos\theta,$$

(4) $$\frac{m}{n}\,\frac{d^2}{dt^2}[y + a\cos\theta] = -R\cos\theta + \mu R\sin\theta.$$

From (1) and (3), with one integration,

$$\frac{m(n+1)}{n}\dot{x} + \frac{m}{n}(a\cos\theta\cdot\dot{\theta}) = C$$

when

$$t = 0,\ \dot{x} = 0,\ \cos\theta = 1,\ \dot{\theta} = \omega;\ \text{therefore } C = ma\omega/n$$

or

$$(n+1)\dot{x} + a\cos\theta\cdot\dot{\theta} = a\omega.$$

From this equation

$$(n+1)x + a\sin\theta = a\omega t. \tag{5}$$

From (2) and (4),

$$\frac{m}{n}\frac{d^2}{dt^2}(y + a\cos\theta) = -m\ddot{y},$$

or

$$\frac{m}{n}\ddot{y} - \frac{m}{n}\frac{d}{dt}(a\sin\theta\cdot\dot{\theta}) = -m\ddot{y},$$

or

$$\frac{m(n+1)}{n}\dot{y} - \frac{m}{n}a\sin\theta\cdot\dot{\theta} = C'.$$

When

$$t = 0,\ \dot{y} = 0,\ \sin\theta = 0;\ C' = 0;$$

hence

$$(n+1)\dot{y} - a\sin\theta\cdot\dot{\theta} = 0.$$

Integrating,

$$(n+1)y = a(1-\cos\theta). \tag{6}$$

The position of the center of gravity of the system is determined by the equations

$$\bar{x} = x + [a/(n+1)]\sin\theta,\quad \bar{y} = y + [a/(n+1)]\cos\theta.$$

Using these equations in (5) and (6), we have

$$\bar{x} = [a/(n+1)]\omega t,\quad \bar{y} = a/(n+1).$$

This equation shows that the center of gravity of the whole system travels along a straight line in the x direction with uniform velocity $a\omega/(n+1)$ and at a distance $a/(n+1)$ from that axis.

We may now write out the equations corresponding to (1), (2), (3), and (4) with regard to the center of gravity. Then

$$m\ddot{x} = m\frac{d^2}{dt^2}\left[\bar{x} - \frac{a}{n+1}\sin\theta\right] = R\sin\theta + \mu R\cos\theta, \tag{7}$$

$$m\ddot{y} = m\frac{d^2}{dt^2}\left[\bar{y} - \frac{a}{n+1}\cos\theta\right] = R\cos\theta - \mu R\sin\theta, \tag{8}$$

$$\frac{m}{n}\frac{d^2}{dt^2}\left[\bar{x}+\frac{na\sin\theta}{n+1}\right] = -R\sin\theta-\mu R\cos\theta, \tag{9}$$

$$\frac{m}{n}\frac{d^2}{dt^2}\left[\bar{y}+\frac{na\cos\theta}{n+1}\right] = -R\cos\theta+\mu R\sin\theta. \tag{10}$$

From (7) and (9),

$$\frac{m}{n}a\frac{d^2}{dt^2}\sin\theta = -R\frac{(n+1)}{n}(\sin\theta+\mu\cos\theta). \tag{11}$$

From (8) and (10),

$$\frac{m}{n}a\frac{d^2}{dt^2}\cos\theta = -R\frac{(n+1)}{n}[\cos\theta-\mu\sin\theta]. \tag{12}$$

Dividing (11) by (12), we obtain

$$\frac{\cos\theta\cdot\ddot{\theta}-\sin\theta\cdot\dot{\theta}^2}{-\sin\theta\cdot\ddot{\theta}-\cos\theta\cdot\dot{\theta}^2} = \frac{\sin\theta+\mu\cos\theta}{\cos\theta-\mu\sin\theta}.$$

This reduces to the simple form

$$\ddot{\theta}+\mu\dot{\theta}^2 = 0.$$

The integration of this equation gives

$$\mu t = \frac{1}{\dot{\theta}}-\frac{1}{\omega}. \tag{13}$$

The law of conservation of angular momentum about the center of gravity gives $I_1\,\omega=I_2\dot{\theta}$, where $I_1=ma^2/(n+1)$ is the initial moment of inertia and $I_2=m(n+2)a^2/(n+1)$ is the final moment of inertia when the hoop is set into rotation. From these equations

$$\dot{\theta} = \omega/(n+2). \tag{14}$$

Using (14) in (13) we have

$$t = (n+1)/\mu\omega.$$

Also solved by T. W. Edmondson and T. L. Smith.

3469 [1931, 50]. *Proposed by V. M. Spunar, Chicago, Illinois.*

A constant length, PQ, is measured along the tangent at any point, P, on a curve; give a geometrical construction for the center of curvature of the locus of the point Q. Also if PQ' be measured equal to PQ in the opposite direction along the tangent, prove that the point P and the centers of curvature of the loci of Q and Q' lie in a straight line.

The argument for the last part may be stated in this manner. The tangents, at Q and Q' meet in O on CP. If P_1 on (P) is very near P, the corresponding points Q_1, Q; Q_1', Q'; C_1, C are very near each other. Except for infinitesimals of higher order than the first, Q_1 may be regarded as lying upon OQ; Q_1' on OQ'; C_1, on OC. The nearer P_1 is to P, the more nearly true are these statements. Hence at the limit the three points (QC, Q_1C_1); $(Q'C, Q_1'C_1)$; (QQ', Q_1Q_1') lie in a straight line, i.e., q, q', P lie in a straight line.

A Second Proof. The geometric reasoning may be put in a form somewhat different from the above. Since the normals at P and Q meet in a point C, this point is the instantaneous center of rotation of $PC=c$. From the elementary geometry proof of this fact, it will follow that $PC=r$ is the radius of curvature at P, and that $ds/r=d\sigma/k$, where $k=QC$. Let $\angle PQC=\phi$, then from $c \tan \phi=r$, we find $c \sec^2\phi d\phi=dr$, or $k^2d\phi=cdr$. Since dr is the element of arc of the evolute (C), we have $\bar{r}d\theta=dr$, where $\bar{r}=CC'$; also $rd\theta=ds$. Combining these results we have

$$ds/d\sigma = r/k, \quad d\phi/ds = c\bar{r}/(rk^2). \tag{1}$$

Consider the two neighboring figures PQC and $P_1Q_1C_1$, and let $\Delta\psi$ denote the angle between QC and Q_1C_1. An inspection of the figure shows that $\Delta\psi=\Delta\theta-\Delta\phi$. If we set $Qq=R$, then

$$\frac{1}{R}=\frac{d\psi}{d\sigma}=\frac{ds}{d\sigma}\left(\frac{d\theta}{ds}-\frac{d\phi}{ds}\right),$$

and this reduces by (1) to

$$\frac{k}{R}=1-\frac{c\bar{r}}{k^2}, \quad \text{or} \quad \frac{R-k}{R}=\frac{c\bar{r}}{k^2}=\frac{Cq}{Qq}\cdot \tag{2}$$

To locate q draw CM perpendicular to QC cutting QP at M. From the figure we see that $c/k=k/QM$, and this with the last result in (2) gives

$$Cq/Qq = CC'/QM. \tag{3}$$

This shows how to construct q. Draw MC' cutting QC in a point which (3) shows must be q.

In the isosceles triangle QCQ', q lies on QC and q', on $Q'C$. Take $\bar{q}$ on QC so that $Q\bar{q}=Q'q'$. Then $P(Q, \bar{q}, C, q)$ is a harmonic pencil, since for q' we use $-c$ in place of c in (2). Since PC is perpendicular to PQ, the ray PC bisects the angle $\bar{q}Pq$. This shows that P, q, q' lie in a straight line.

The results in (2) may also be derived rather simply by vector analysis.

3476[1931, 111]. *Proposed by Vladimir F. Ivanoff, San Francisco, California.*

Given the cubic, $x^3+ax+b=0$, with rational roots α, β, γ; show that the equation $\alpha y^2+\beta y+\gamma=0$ also has rational roots.

Solution by Marjorie E. Halliwill, A Student at Akron University, Akron, Ohio.

Since $\alpha+\beta+\gamma=0$, the quadratic may be written $(y-1)(\alpha y-\gamma)=0$. One root is obviously rational, and the second is also rational from the given conditions.

Also solved by R. P. Agnew, H. W. Bailey, A. W. Bear, W. R. Church, O. C. Collins, A. G. Clark, Ralph Deutsch, Edward Fleisher, D. F. Gunder, J. D. Hill, J. D. Leith, Gertrude I. Mc Cain, G. I. Miller, R. E. Moritz, J. H. Neelley, A. Pelletier, E. A. Rasor, P. L. Reà, J. Rosenbaum, Wallace Smith, Mrs. B. J. Smyth, F. Underwood, B. F. Yanney, G. L. Weaver, Kamcheung Woo, and the Proposer.

3477[1931, 111]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

If two points, harmonically separated by the centers of two unequal circles (spheres), are diametrically opposite on a circle (sphere) coaxial with the given circles (spheres), these two points are the centers of similitude of the two given circles (spheres).

Note: This is the converse of the property of the circle (sphere) of similitude. (See, for instance, Nathan Altshiller-Court, *College Geometry*, p. 196.)

Solution by Edwin J. Purcell, University of Colorado.

Given two circles S and S' with radii r and r' respectively, $r \neq r'$. Without loss of generality we may take their centers O and O' as the origin and the point $(a', 0)$ respectively in rectangular Cartesian coördinates. Their equations are

$$S \equiv x^2 + y^2 - r^2 = 0, \text{ and } S' \equiv (x - a')^2 + y^2 - r'^2 = 0.$$

A third circle S'', coaxial with S and S', will have its center collinear with O and O'. Its equation is

$$S'' \equiv S + kS' \equiv (x^2 + y^2 - r^2) + k(x^2 - 2a'x + a'^2 + y^2 - r'^2) = 0, \tag{1}$$

where k is an arbitrary constant. The points of intersection, B and A, of S'' with the line of centers, $y=0$, are given by the values of x in

$$x = \frac{a'k \pm \sqrt{(-ka'^2 + kr'^2 + r^2 + k^2r'^2 + kr^2)}}{(1 + k)}. \tag{2}$$

The problem imposes the condition that the four points O, B, O', A, be harmonic. Upon substitution and simplification, this condition gives

$$(1 + k)(kr'^2 + r^2) = 0. \tag{3}$$

It is to be shown that B and A are the centers of similitude of the circles S and S'. Now if k equals -1, the equation (1) gives a straight line, which is to be excluded. Therefore, from (3),

$$k = -r^2/r'^2. \tag{4}$$

Substituting (4) in (2) and simplifying, the coördinates of B and A are

$$x = \frac{a'r}{r + r'}, \quad y = 0; \quad \text{and} \quad x = \frac{a'r}{r - r'}, \quad y = 0;$$

and these are precisely the coördinates of the centers of similitude of the two given circles, S and S' (see, for instance, G. Salmon: *Conic Sections*, 1904, page 105).

The proof of the corresponding theorm for three dimensions will be similar to that given above if we take the two given spheres as having their respective centers at (0, 0, 0) and $(a', 0, 0)$ and consider the section made by the plane $z=0$.

Editorial Note: The proof above may be put in a different form by replacing (1) by the known equivalent fact that the lengths of the tangents from B to the given circles have the same ratio as the corresponding tangents from A. Hence

$$\frac{\overline{OB}^2 - r^2}{\overline{O'B}^2 - r'^2} = \frac{\overline{OA}^2 - r^2}{\overline{O'A}^2 - r'^2}.$$

From the harmonic property we have also $OB/OA = -O'B/O'A = p$. These equations combined give $OA/O'A = -OB/O'B = r/r'$, and the desired result follows at once.

Also solved by A. Pelletier and F. Underwood.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

The preliminary official program of the International Mathematical Congress to be held in Zurich in 1932, which has just been issued, indicates that the sessions of the Congress will be on September 4–12, that all countries will be represented, and that titles of papers intended for presentation at the Congress should be sent to the Secretary of the International Mathematical Congress, École Polytechnique Fédérale, Zurich. The president of the committee on organization of the Congress is Professor R. Fueter. It is the present intention also to hold the sessions of the International Mathematical Union, which is a distinct organization, and of which Professor W. H. Young is the president, during the time of the Congress.

In commemoration of the one hundredth anniversary of the death of Laplace (in 1827) a memorial momument will be erected in the town of Beaumont, his birthplace. Dedication will take place in the summer of 1932.

Professor R. L. Moore, of the University of Texas, who was appointed by the Council of the American Mathematical Society to the Travelling Lecturership for the current academic year, gave the first series of lectures on the Pacific Coast, on the following dates: University of California at Los Angeles, November 10, 12, 13; Stanford University, November 16, 17, 18; University of Washington, November 20; University of California, Berkeley, November 23,

24, 25. The titles of his lectures were: 1. *Foundations of point set theory;* 2. *Continuous curves;* 3. *Upper semi-continuous collections.* Professor Moore also gave an address entitled *A set of axioms for plane analysis situs*, by invitation of the program committee at the meeting of the Society held at the California Institute of Technology on November 28.

Professor Willem de Sitter, director of the Leyden Observatory, has been elected an honorary member of the American Astronomical Society.

Professor R. A. Millikan, of the California Institute of Technology, has been made a Knight of the Legion of Honor by the French Government.

The following have been awarded National Research Fellowships in mathematics for the academic year 1931–32; R. P. Agnew, E. F. Beckenbach, Leonard Carlitz, Benedict Cassen, Max Coral, H. B. Curry, J. L. Dorroh, A. L. Foster, D. H. Lehmer, S. B. Littauer, E. J. McShane, G. W. Mendel, C. B. Morrey, A. L. O'Toole, A. E. Ross, Wladimir Seidel, W. J. Trijitzinsky, Hassler Whitney, S. S. Wilks, Jacob Yerushalmy, Leo Zippin. This list includes renewals.

Dr. W. S. Adams, director of the Mount Wilson Observatory, has been elected president of the American Astronomical Society.

Dr. R. P. Agnew has been appointed assistant professor of mathematics at Cornell University.

Dr. W. D. Baten has been promoted to an assistant professorship of mathematics at the University of Michigan.

Mr. William Beverly has been promoted to an assistant professorship of mathematics at Lafayette College.

Mr. W. M. Birchby has been promoted to an assistant professorship of mathematics at the California Institute of Technology.

Mr. J. C. Brixey, of the University of Oklahoma, has been promoted to an assistant professorship of mathematics.

Associate Professor Thomas Buck, of the University of California, Berkeley, has been promoted to a professorship of mathematics.

Mr. Earl Church, of Syracuse University, has been promoted from an assistant professorship of mathematics to an associate professorship of photogrammetry.

Professor Elbert H. Clarke, head of the department of mathematics and astronomy in Hiram College, Hiram, Ohio, has been granted a six months leave of absence for study at the University of Göttingen.

Dr. C. M. Cleveland has been appointed adjunct professor of applied mathematics at the University of Texas.

Assistant Professor Tobias Dantzig, of the University of Maryland, has been promoted to an associate professorship of mathematics.

Dr. M. S. Demos, of Columbia University, has been appointed assistant professor of mathematics at the University of New Hampshire.

Dr. W. L. Duren, of the College of the City of Detroit, has been appointed assistant professor of mathematics at Tulane University.

Dr. N. C. Fisk, of the University of Michigan, has been appointed dean of the Itasca Junior College, Coleraine, Minnesota.

Associate Professor M. C. Foster has been promoted to a professorship of mathematics at Wesleyan University.

Miss Marjorie L. Heckel has been appointed instructor at the University of South Dakota.

Dr. P. E. Hemke has been appointed associate professor of mechanics at the Case School of Applied Science.

Mr. A. S. Householder has been appointed to an assistant professorship of mathematics at Washburn College.

Dr. Gertrude I. McCain has been appointed professor of mathematics at Marymount College, Salina, Kansas.

Dr. G. F. McEwen has been appointed chairman of the newly created committee of the American Geophysical Union on evaporation from free water surfaces.

Miss Martha P. McGavock has been promoted to an associate professorship of mathematics at Rockford College.

Dr. L. T. Moore has been appointed assistant professor of mathematics at Brooklyn College of the City of New York.

Associate Professor D. S. Morse has been promoted to a professorship of mathematics at Union College.

Dr. C. V. Newsom has been appointed associate professor of mathematics at the University of New Mexico.

Associate Professor C. S. Porter, of the department of mathematics at Amherst College, has been appointed acting dean of the College.

Miss Mary J. Quigley, of Boston Teachers College, has been promoted to a professorship of mathematics.

Mr. H. P. Rogers, of the University of Illinois, has been appointed assistant professor of mathematics at Kent State College, Kent, Ohio.

Dr. J. A. Shohat has been promoted to an assistant professorship of mathematics at the University of Pennsylvania.

Dr. L. B. Slichter, of the California Institute of Technology, has been appointed associate professor of mathematics at the Massachusetts Institute of Technology.

Associate Professor R. G. Smith, of Kansas State Teachers College, Pittsburg, has been promoted to a professorship.

Assistant Professor Pauline Sperry, of the University of California, Berkeley, has been promoted to an associate professorship of mathematics.

Dr. Elizabeth Stafford Sokolnikoff has been apointed instructor at the University of Wisconsin.

Dr. F. G. Williams has been appointed professor of mathematics at Susquehanna University.

Dr. R. C. Yates has been appointed assistant professor of mathematics at the University of Maryland.

Richard Bolling Dunn, instructor in mathematics at Yale University, has died recently.

Dr. Stephen Marshall Hadley, who was for many years dean and professor of mathematics at Penn College, Oskaloosa, Iowa, died at Whittier, California, on November 10, 1931. He was a charter member of the Mathematical Association.

Dr. Charles Haseman, professor of mathematics and mechanics at the University of Nevada, died on July 9, 1931, after a year's illness. He was a charter member of the Association.

Mr. Alexander Knisely of Columbia City, Indiana, a charter member of the Association, died September 29, 1931.

Dr. Bessie I. Miller, of the University of Illinois, died February 4, 1931. Dr. Miller was a charter member of the Association.

Professor C. L. E. Moore, of the Massachusetts Institute of Technology, died on December 5, 1931. He was a charter member of the Association.

Professor C. W. Watts, of Virginia Military Institute, a charter member of the Association, died on July 10, 1931.

The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The prize will be awarded hereafter every three years. The last award was in December, 1929, to Professor T. H. Hildebrandt. The next award will be in December, 1932, for the period 1929-1931.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of.................................Dollars,

to be known as the...............................Fund, and to be used

for[2] { Endowment—the income only of which may be expended.
Special Projects—for which both principal and income may be expended.

Witness:...................... Signature......................

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.
[2] Indicate which one of the two purposes is desired, and omit the other.

The Association needs funds for scientific publications and for the promotion of scientific activities.

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Sixteenth Summer Meeting of the Association, Los Angeles, Calif., Aug. 29-30, 1932.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1932 and reported to the secretary.

ILLINOIS, Urbana.
INDIANA.
IOWA.
KANSAS.
KENTUCKY.
LOUISIANA-MISSISSIPPI.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May.
MICHIGAN.
MINNESOTA.
MISSOURI.
NEBRASKA.
OHIO, Columbus, Ohio, April 7.
PHILADELPHIA, Philadelphia, Pa., Nov. 26.
ROCKY MOUNTAIN, Laramie, Wyo.
SOUTHEASTERN.
SOUTHERN CALIFORNIA, San Diego, March 26.
TEXAS, Austin, Jan. 30.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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THE AMERICAN MATHEMATICAL MONTHLY

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MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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VOLUME XXXIX, 1932

NUMBER 2, FEBRUARY

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND ITHACA, N.Y.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

THE SIXTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The sixth annual meeting of the Philadelphia Section of the Mathematical Association of America was held in the Packard Laboratory of Lehigh University on Saturday, November 28, 1931. Professor Fort of Lehigh University presided at both sessions.

The attendance was thirty-three, including the following twenty-four members of the Association: S. S. Cairns, P. A. Caris, J. W. Clawson, Arnold Dresden, Tomlinson Fort, J. R. Kline, M. S. Knebelman, P. A. Knedler, P. V. Kunkel, V. V. Latshaw, F. L. Manning, A. E. Meder, Jr., G. E. Raynor, J. B. Reynolds, J. A. Roulton, C. A. Rupp, J. A. Shohat, C. A. Shook, L. L. Smail, W. M. Smith, Alexander Tartler, Willis Whited, Clement Weinstein, A. H. Wilson.

At the business meeting the following officers were chosen for next year: Chairman, Arnold Dresden, Swarthmore College; Secretary, P. A. Caris, University of Pennsylvania; Program Committee, Professors Dresden, Caris, Smith, and Morris. The next meeting of the Section will be held on Saturday, November 26, 1932, at Philadelphia.

The following papers were presented:

1. "Redundant co-ordinates" by Professor C. A. Rupp, Pennsylvania State College.

2. "On some fundamental conceptions in the theory of infinite processes" by Professor L. L. Smail, Lehigh University.

3. "Italy and geometry" by Professor W. M. Smith, Lafayette College.

4. "Different kinds of curvature" by Professor M. S. Knebelman, Princeton University.

5. "Swarthmore honors course in mathematics" by Professor Arnold Dresden, Swarthmore College.

Abstracts of the papers follow:

1. The choice, for the co-ordinates of a geometric element, of a set of quantities whose number exceeds, by more than one, the number of degrees of freedom of the element, may have the advantage of yielding superior ease in the algebraic manipulations of the co-ordinates. The price paid for the gain in manoeuverability is that the task of finding multipliers in questions of linear dependence is complicated by the fact that the superfluous co-ordinates are functions of the essential co-ordinates. In spite of such complications, it is possible, in some cases to make the machinery of linear dependence work. This paper draws some geometric consequences of the linear dependence of flat spaces in a space of n dimensions.

2. In this paper Professor Smail discussed the definitions of infinite series and of convergence, and the fundamental significance of the concept of summability of infinite processes. He criticized the usual definitions of the concept

of infinite series found in the books, and gave Knopp's definition as the only satisfactory one. He proposed that for the definition of convergence of infinite sequences the so-called general principle of convergence be substituted in place of the usual limit definition, and discussed the advantages of this form of definition. The fundamental importance of the concept of summability in the theory of infinite processes was emphasized and the earlier introduction of the notion in the systematic study of the subject was urged.

3. Professor Smith spent his last Sabbatical year working with Professor Bompiani at the University of Rome. After outlining the facilities available for graduate study in Rome, he reviewed briefly the salient characteristics of some of the more important geometries, metric and non-metric, including projective and Riemannian with their various subdivisions, and pointed out the contribution of Italian geometers to these different divisions.

4. The concept of curvature of a curve has been extended to surfaces by Gauss and Rodrigues. About fifty years later Riemann defined the curvature of a space of more than two dimensions for a given orientation. It so happens that this curvature is not adequate for the study of either the extrinsic or topological properties of the space, and it is only within the last two or three years that the question of curvature has undergone a closer scrutiny. In this paper some of these newer results are discussed.

5. Referring to the 1931 catalogue of Swarthmore College, pages 42 and following, to R. C. Brook's "Reading for Honors at Swarthmore," and to President Aydelotte's contribution to the volumes entitled "Five College Plans" for information as to the general features of the honors work at Swarthmore, this paper gives an account of the way in which this plan is realized, particularly in mathematics and the natural sciences.

P. A. Caris, *Secretary*

SUMMABILITY OF SERIES[1]

By C. N. MOORE, University of Cincinnati

1. *Introduction.* In the study of the various functions that occur in mathematical theory and its applications to science and engineering one of the most potent methods of investigation is expansion in series. The chief value of this method lies in the fact that it enables us to express new and more complicated functions in terms of simpler and better known functions. Examples of such expansions are met early in our study of mathematics. For instance the binomial expansion of $(1+x)^n$ for the case of negative or fractional exponents leads to an infinite series in powers of x. This expansion is a special case of the expansions due to Taylor and Maclaurin, which are treated in the usual first course in calculus. The functions that admit of such expansions form a wide class and they possess the fundamental property that when they are generalized

[1] A lecture delivered by invitation at the S.P.E.E. Summer Session for Teachers of Mathematics to Engineering Students at Minneapolis, Sept. 4, 1931.

to complex values they admit of similar expansions in the complex plane. This beautiful generalization of the theorems of Taylor and Maclaurin was one of the important discoveries due to Cauchy.

In order for a function to have a power series development in the neighborhood of a point it must possess derivatives of all orders at that point. While the class of functions that has this property is a very important one, it is nevertheless a highly restricted group when compared to functions in general. More general expansions are obviously necessary for the study of wider classes of functions. Moreover, even in the case of functions that possess a power series development, expansions in terms of other simple functions may be more useful for certain types of investigation.

Of coordinate importance with power series and next to them in simplicity we have Fourier series or developments in sines and cosines of integral multiples of x. These developments were first studied in connection with certain problems of mathematical physics and they still rank as the expansions of greatest utility in the application of mathematics to other sciences. This arises from the frequent occurrence of periodic phenomena and the importance of wave motion as a hypothesis in explaining other phenomena.

Among the important problems of mathematical physics whose solution frequently involves Fourier series may be mentioned those that arise in the study of the flow of heat and electricity. Such problems also lead to the consideration of other related expansions, such as developments in Bessel's functions, Laplace's functions, Sturm-Liouville functions, and so on. These various functions have in common with the sine and cosine the properties of being oscillatory in character and satisfying differential equations of the second order. The expansions in terms of such functions have many properties analogous to those of Fourier series, but their study naturally presents greater difficulty.

Outside of power series, Fourier series and related developments, the type of series that has been most extensively studied is that known as Dirichlet's series. Such series have the form $\Sigma a_n e^{-\lambda_n s}$, where the λ's are a set of real constants such that λ_n becomes infinite with n, the a's are a set of constants real or complex, and s is a complex variable. So far the applications of such series have been primarily in the field of pure mathematics, and more particularly in the analytic study of the Theory of Numbers. However, like other conceptions[1] that first arose in the study of pure mathematics, they may prove later to have important applications in other sciences.

In order to apply mathematical theory to concrete situations we need always a method of obtaining numerical results. Thus in the case of an infinite series of constant terms we must have some convention as to a numerical value to be attached to the series. The simplest and most natural one is based on the conception known as convergence. Given a series

$$u_0 + u_1 + u_2 + \cdots + u_n + \cdots, \tag{1}$$

[1] Such as conic sections, non-Euclidean geometry, complex numbers, continued fractions.

we form

$$s_n = u_0 + u_1 + \cdots + u_n \tag{2}$$

and consider the behavior of s_n with increasing n. If s_n approaches a definite limit s as n becomes infinite, we say that the series converges to the value s. If no limit exists we say that the series diverges. A series may diverge because the related s_n becomes positively or negatively infinite, or because it oscillates between finite or infinite limits. The latter type of divergent series is sometimes referred to as oscillatory.

2. *Use of divergent series in the eighteenth century.* The mathematicians of the eighteenth century found that the formal expansions of many simple functions failed to converge for values of the argument for which the function corresponding to the series was perfectly well defined. They found on the other hand that in many of these cases the divergent expansions could be used in various ways to obtain valid results just as well as if they were convergent. They therefore felt the need of attaching a numerical value to a divergent series that would be in agreement with the value of the function whose expansion gave rise to the series.

The procedure of two of the leading mathematicians of the period can be adequately illustrated in terms of the simple divergent series

$$1 - 1 + 1 - 1 + \cdots .$$

We note that the successive values of s_n are alternately 1 and 0. Since these two values occur with equal frequency, Leibnitz contended that on the basis of the Theory of Probability the value $\frac{1}{2}$ should be ascribed to the series. Suppose, on the other hand, that we introduce successive powers of x in the terms of the series, thus obtaining the power series

$$1 - x + x^2 - x^3 + \cdots .$$

For values of x numerically less than unity the series converges and has the value $1/(1+x)$. Now allow x to approach 1 from below. The function defined by the series approaches $\frac{1}{2}$; the series itself, if we take the limit term by term, approaches the series with which we started. For this reason Euler contended that the value $\frac{1}{2}$ should be ascribed to the original series. The surprising feature here is that these two entirely different methods of approach lead to the same value for the series. This coincidence is not an accident, however, but a special case of a general theorem proved by Frobenius in 1880, to which we will refer later.

It is apparent that the procedure of Leibnitz and Euler in the case of the simple series $1-1+1-1+\cdots$ is entirely out of harmony with present day notions of rigor in mathematical analysis. These notions are the result of a gradual development from ideas that first came into prominence in the mathematical world early in the nineteenth century as a consequence of the pronouncements of such outstanding mathematicians as Cauchy and Abel. The

steadily rising standards of rigor, coupled with the fact that the use of divergent series occasionally led to serious contradictions, resulted in a gradually diminishing interest in such series during the first three fourths of the nineteenth century. At the same time the rigorous theory of convergent series was being actively developed by students of this field of mathematics, and thus the way was being paved for a rigorous treatment of divergent series.

3. *Beginnings of the modern theory*. The first definite advance in the direction of a rigorous theory to appear in a mathematical journal is found in a paper by Frobenius which was published in Crelle's Journal in 1880. The essential contribution of Frobenius was the proof of the following theorem.

If for the series (1) *the* s_n *defined by* (2) *are such that the limit*, $\lim_{n\to\infty}(s_0+s_1+\cdots+s_n)/(n+1)$, *exists and is equal to* s, *then the series* $u_0+u_1x+\cdots+u_nx^n+\cdots$ *will converge for* $|x|<1$ *to a function* $f(x)$ *such that* $\lim_{x\to 1-0}f(x)=s$. It is apparent that the agreement between the procedures of Leibnitz and Euler in the case of the series $1-1+1-1+\cdots$ is a special case of Frobenius's theorem.

Two years after the publication of Frobenius's paper, his result was considerably generalized by Hölder. Let us represent the successive arithmetic means of the s's by $s_0^{(1)}, s_1^{(1)}, \cdots s_n^{(1)}, \cdots$. Then, if $s_n^{(1)}$ fails to approach a limit as n becomes infinite, it seems natural to repeat the process originally used by forming successive arithmetic means of the $s^{(1)}$'s, labeling them $s_0^{(2)}, s_1^{(2)}, \cdots s_n^{(2)}, \cdots$. If $s_n^{(2)}$ approaches a limit s we define that limit to be the generalized sum of the series. If $s_n^{(2)}$ does not approach a limit, we repeat the process, forming $s^{(3)}$'s and studying the behavior of $s_n^{(3)}$ as n becomes infinite. If for any integer r, $s_n^{(r)}$ approaches a limit s as n becomes infinite, we define that limit to be the generalized sum of the original series. In modern terminology we say that the series is summable (Hr) to the value s. We are now in a position to state Hölder's generalization of Frobenius's theorem. *If the series* (1) *is summable* (Hr) *to the value* s *for any integer* r, *then the series* Σu_nx^n *will converge for values of* x *numerically less than* 1 *to a function* $f(x)$ *such that* $\lim_{x\to 1-0}f(x)=s$.

In 1890, ten years after the publication of Frobenius's paper, Cesàro attacked a problem in series that is naturally suggested by the behavior of the Cauchy product of two convergent series. Given two infinite series, Σu_n and Σv_n, their Cauchy product is defined as the series $\Sigma(u_nv_0+u_{n-1}v_1+\cdots+u_0v_n)$. If both the original series are convergent and at least one of them is absolutely convergent, it can be shown that the product series will always converge to a value that is the product of the values of the two given series. If, however, neither of the original series is absolutely convergent, it may happen that the product series diverges. If then there can be found methods of summing divergent series that will give to them values consistent with their mode of formation, it seems reasonable to expect that some one of these methods will serve to sum the product series of two convergent series to the proper value. Cesàro found that the simple method used by Frobenius had this interesting and important property. However, the problem suggested by the multiplication of

convergent series was not completely elucidated by this discovery, nor did Cesàro's investigations stop at that point. Suppose that we are considering the product of three series. The product of two of them may diverge, and we are thus led to consider the behavior of the product of a convergent series and a series that can be summed by the method of Frobenius. If we start with four series, their products, taking two at a time, may each diverge, and we thus have the problem of multiplying two divergent series of a certain type, and so on. It was Cesàro's aim to answer all questions of this order, and thus to furnish the means for discussing the behavior of the product of any finite number of series. He succeeded brilliantly in his purpose, obtaining a complete solution of the problem in a six page paper that involved only analysis of a simple type.

Cesàro arrived at his goal by generalizing the process of forming an arithmetic mean of the first n partial sums of the series in a different manner from that adopted by Hölder. Instead of forming successive means from the means previously obtained, he conceived the idea of forming successive weighted means, the weights changing in a regular manner. Let us set

$$(4)\quad S_n^{(r)} = s_n + rs_{n-1} + \frac{r(r+1)}{2!}s_{n-2} + \cdots + \frac{r(r+1)\cdots(r+n-1)}{n!}s_0,$$

$$(5)\quad A_n^{(r)} = \frac{(r+1)(r+2)\cdots(r+n)}{n!}$$

$$(6)\quad \sigma_n^{(r)} = \frac{S_n^{(r)}}{A_n^{(r)}}.$$

The $A_n^{(r)}$ can be readily shown to be the sum of the coefficients of the s's in the expression for $S_n^{(r)}$, and therefore $\sigma_n^{(r)}$ is a weighted mean of $s_0, s_1, \cdots, s_n$. For $r=0$, $\sigma_n^0=s_n$, and consequently convergence is included in our general scheme of summation for this value of the parameter. For $r=1$, $\sigma_n^{(1)}=\sum_{i=0}^{i=n}s_i/(n+1)$, and therefore summation by the ordinary arithmetic mean is included for the value unity of the parameter. For values of $r>1$, it is apparent from the above formula that the early s's are weighted more than the later ones, and therefore it seems probable that series which oscillate more strongly can be summed by using larger values of r. Since the general effect of forming the Cauchy product of two or more oscillating series is to increase the degree of oscillation, one would expect the general multiplication theorem to depend on the value of r in formula (6), and this expectation is realized. In modern terminology a series for which $\sigma_n^{(r)}$ approaches a limit σ as n becomes infinite is said to be summable (Cr). Using this definition, Cesàro's general theorem may be stated as follows. *The Cauchy product of a series summable* (Cr) *to value* A *by a series summable* (Cs) *to value* B *is always summable* $(C, r+s+1)$ *to value* AB. This includes the result regarding convergent series as the special case where $r=0=s$.

It is apparent at first sight that there is a rather close relationship between Hölder's generalization of Frobenius's method and that of Cesàro. Furthermore

we readily find that a necessary condition for either summability (Hr) or (Cr) is $\lim_{n\to\infty} u_n/n^r = 0$. This suggests that the two methods are of the same general scope, and mathematicians working with them felt fairly certain that they were either equivalent or approximately so. Nevertheless, the discussion of the question of equivalence seemed at first to involve algebraic difficulties of a serious character. Finally, in 1907, Knopp proved that summability (Hr) implies summability (Cr) to the same value, and somewhat later Schur and W. B. Ford proved independently that summability (Cr) implies summability (Hr) to the same value. These early proofs involved rather complicated analysis, but it has since been found that the equivalence theorem can be proved in a quite elementary manner. The first simple proof was given by Schur in 1913.

4. *Work of Borel and others on analytic extension.* We spoke at the beginning of the importance of developments in power series in the case of functions possessing such developments. For some functions, such as the sine and cosine, the corresponding power series converges for all values of the independent variable. In the case of other quite simple functions, however, the range of convergence of the power series may be quite limited. For example the power series development of the function $1/(1-z)$ is

$$1 + z + z^2 + \cdots + z^n + \cdots, \tag{7}$$

and this series converges only within the unit circle. The Theory of Functions of a Complex Variable furnishes the reason for this restricted region of convergence. The function $1/(1-z)$, simple as it is, possesses an infinite discontinuity at the point $z=1$, and such a discontinuity on the circumference of a circle whose center is the point about which we develop the function always prevents convergence outside the circle. In spite of the failure of the convergence of the series for values of $z \geqq 1$, there is obviously such a close relationship between the series and the function that we should expect to find the value of the function for such values of z by using a suitable method of summing the divergent series. Cesàro's success in summing the product of any finite number of convergent series suggested to Borel the possibility of summing power series outside their circle of convergence by using some similar scheme. It was apparent at once that neither the method of Cesàro nor that of Hölder would suffice, since the necessary condition for their applicability is not satisfied by a power series beyond the circle of convergence.

Borel found it useful, however, to take Cesàro's scheme of weighted means as a point of departure, but he introduced an important new element in this scheme. Instead of considering a weighted mean of a finite number of sums and then allowing the number of sums to become infinite, he considered a weighted mean of the infinite set of sums and allowed the weights to vary in a definite manner. Thus he formed

$$F(x) = \frac{\displaystyle\sum_{n=0}^{\infty} s_n \frac{x^n}{n!}}{e^x} \tag{8}$$

and considered the limit of $F(x)$ as x becomes infinite. This definition is of interest on account of being a logical development from Cesàro's scheme, but a closely related definition which is obtained by a simple transformation of this one, is more useful in the analytic applications. We set

$$u(x) = u_0 + u_1 x + u_2 \frac{x^2}{2!} + \cdots + u_n \frac{x^n}{n!} + \cdots \tag{9}$$

and form

$$\int_0^\lambda e^{-x} u(x) dx. \tag{10}$$

If the limit of (10) as λ becomes infinite exists, we call this limit the generalized sum of the series. This definition has substantially the same scope as the earlier one.

The application of Borel's scheme to the power series development of $1/(1-z)$ is extremely simple and serves to show the scope of the method. We set

$$u(x, z) = 1 + zx + \frac{z^2 x^2}{2!} + \cdots = e^{zx}. \tag{11}$$

Then we form

$$\int_0^\lambda e^{-x} u(x, z) dx = \int_0^\lambda e^{(z-1)x} dx = \frac{e^{(z-1)\lambda}}{(z-1)} - \frac{1}{z-1} \cdot \tag{12}$$

It is apparent that for all values of z in the complex plane such that the real part of z is less than 1, the first term on the right hand side of (12) approaches zero as λ becomes infinite. Therefore for all such values of z Borel's method serves to sum the series (7) to the appropriate value $1/(1-z)$. We have thus vastly extended the region in which the power series development can be utilized.

Shortly after Borel's investigations Leroy devised a more powerful method of summing a divergent series, which may be defined as follows. Given any series, $\Sigma u_n(z)$, we set

$$F(z, t) = \sum_{n=0}^{\infty} \frac{\Gamma(nt + 1)}{\Gamma(n + 1)} u_n(z) \qquad (0 < t < 1).$$

and define the value of the series as $\lim_{t \to 1-0} F(z, t)$, provided that this limit exists. It can readily be shown that Leroy's method will sum the series (7) to the value $1/(1-z)$ for all values of z in the complex plane except real values greater than one. In 1920, some twenty years after Leroy's work, Mittag-Leffler devised a method which will sum the series (7) to the value $1/(1-z)$ for all values of z for which the latter expression is defined, that is for all values of z except $z=1$.

We have seen how the various methods of summation serve to extend the region in which the series (7) can be summed to the value $1/(1-z)$. These methods are equally effective in the case of power series in general, provided that the function to which the power series corresponds is analytic in certain regions beyond the circle of convergence. They thus furnish an analytic extension of the function. To show the scope of the methods we shall state the facts in the case where the function is analytic throughout the complex plane, except for a finite number of singular points, none of which is at the origin. Suppose that we use Cauchy's generalization of Maclaurin's theorem to obtain the development $a_0+a_1z+a_2z^2+\cdots$. This series will converge within a circle whose center is at the origin and whose radius is the distance from the origin to the nearest singular point. Draw lines through the singular points perpendicular to the lines joining these points with the origin. These lines form a polygon, known as the Borel polygon, within which the series is summable to the value of the function by Borel's method. Draw lines from the singular points to infinity that are the prolongations of the lines from the origin to these singular points. The entire plane, except the lines extending out from the singular points, forms a region known as the Mittag-Leffler star. Leroy's method will sum the series to the value of the function everywhere in the Mittag-Leffler star. Finally, Mittag-Leffler's method will sum the series to the function everywhere, except at the singular points themselves.

5. *Summation of Fourier series and related developments.* We have seen what could be done with power series by the use of general methods of summation. Let us turn now to some other types of series and consider first of all the important class known as Fourier series. The Fourier series associated with a function $f(x)$, defined and integrable in the interval $(-\pi, \pi)$, may be written in the form

$$\tag{13} \tfrac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + \cdots + (a_n \cos nx + b_n \sin nx) + \cdots,$$

where

$$\tag{14} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

If the function satisfies certain fairly general conditions, the series will converge to the value of the function at all points of continuity of the function. However, there exist continuous functions for which the series fails to converge at points that are densely distributed in every sub-interval of the interval $(-\pi, \pi)$. Shortly after Borel's work on power series, it occurred to Fejér that some of the methods for summing divergent series might be applied to advantage in the case of Fourier series. An investigation of the matter showed that this was indeed the case and led to the following noteworthy result. *If the function of which the Fourier series is a development is finite except at a finite number of points and is integrable in the sense of Riemann, throughout the interval* $(-\pi, \pi)$, *then the Fourier series is summable* $(C1)$ *to the value of the function at all points of continu-*

ity and to the value $\frac{1}{2}[f(x+0)+f(x-0)]$ *at all points of discontinuity where this expression has a meaning.*

We have spoken earlier of developments analogous to Fourier series, such as developments in Laplace's functions or Bessel's functions, which likewise have important applications in mathematical physics. Shortly after his investigation of Fourier series Fejér made an analogous study of developments in Laplace's functions. A little later C. N. Moore investigated the developments in Bessel's functions, and Haar studied the developments in Sturm-Liouville functions. In all these cases it was the method of summation due to Cesàro that proved fruitful. In general the results were analogous to those already obtained for Fourier series, but in the case of developments in Laplace's functions and Bessel's functions certain important differences were found in the neighborhood of the point where the differential equation defining these functions has a singular point.

The investigations just described were merely the first applications of general methods of summation to the developments in question. Many subsequent studies have been made and the field of research opened up by the initial investigations is still far from exhausted. It is not possible to give any detailed account of the various results that have been obtained in a general lecture on the whole subject of summability of series. It may be stated, however, that the papers in which these results are found form an important part of the mathematical literature of the present century.

6. *Summation of Dirichlet's series.* There is one further type of series to which we have referred in the introduction, namely series of the form $\Sigma a_n e^{-\lambda_n s}$, known as Dirichlet's series. A brief account of some of the investigations of these series seems desirable, since it will show the importance of modifying various methods of summability in order to adapt them to the specific needs of certain types of series. The first investigations of the summability of Dirichlet's series were applications of Cesàro's method to certain special types of Dirichlet's series by Bohr and M. Riesz. In passing to the case of the general Dirichlet's series, Riesz found it desirable to replace Cesàro's method by a more general one, which he termed summation by typical means. For the type of series initially studied, summation by Riesz's typical mean is equivalent to summation by a corresponding Cesàro mean, but even in this case the method due to Riesz is a more effective weapon in studying the behavior of Dirichlet's series.

7. *Conclusion.* The preceding account of what has been accomplished by the use of various methods of summation in the investigation of divergent series is necessarily somewhat sketchy. I trust that it will serve to give some idea of the very notable success that has attended the application of these methods. It is easy to see that among series in general a convergent series is a very exceptional case. For convergence it is necessary first of all that the general term of the series approach zero as a limit, and many important series where the general term does approach zero fail to converge. It is apparent that if we are to achieve any comprehensive knowledge of series we must make use of methods of evaluat-

ing divergent series. The methods which we have described have amply proved their worth in the studies that have been made. We have every reason to expect that further application of these methods and others yet to be devised will vastly extend our knowledge of the important field of mathematics known as the theory of infinite series.

ON THE GEOMETRY OF THE TRIANGLE[1]

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Introduction. The purpose of Part A of this discussion is to develop an analytic treatment, using conjugate coordinates, of the extensive metric figure associated with an arbitrary triangle and to present some inter-relations in this figure which are believed to be new. Comprehensive bibliographies covering known relations can be found in *Mathesis*.[2] Part B is concerned with certain projective properties.

Part A

I. *Conjugate Coordinates.*

1. The conjugate coordinates[3] x and y of a point in a plane are defined by the equations, $x=X+iY$, $y=X-iY$, where X, Y are rectangular Cartesian coordinates and i is the imaginary unit. A complex number whose absolute value is unity is called a turn and will be denoted by such symbols as t, T, t_j, etc. The operations of multiplication, division and extraction of roots applied to turns give results which are turns.

2. A real line is represented by a linear equation $tx+y=ct$, where c is any complex number, $\bar{c}$ is the conjugate of c, and t a turn such that $\bar{c}=ct$. The point c is the reflexion of the origin in the line, and the turn t gives the inclination of the line and will be called the clinant. Two lines with clinants t_1 and t_2 are parallel if $t_1=t_2$ and perpendicular if $t_1=-t_2$. The equation of the line whose clinant is t and which passes through the point a is $tx+y=ta+\bar{a}$. The equation of the line through the points a and b is

$$\begin{vmatrix} x & y & 1 \\ a & \bar{a} & 1 \\ b & \bar{b} & 1 \end{vmatrix} = 0.$$

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[1] Accepted for publication during the editorship of W. H. Bussey.

[2] *Mathesis Supplements*, vol. 6 (1886), vol. 10 (1890), vol. 15 (1895), vol. 27 (1907), and Year 1922 beginning page 50.

[3] W. B. Carver, *The failure of the Clifford chain*, American Journal Mathematics vol. 42 (1920), p. 137; Frank Morley, *On the metric geometry of the plane N-line*, Transactions of the American Mathematical Society, vol. 1 (1900), p. 97; R. M. Winger, *An Introduction to Projective Geometry*, D. C. Heath and Co. (1923), p. 324.

3. If the point a is the center of a circle and $|r|$ its radius, its map equation is $x=a+rt$. The conjugate equation of this circle is $(x-a)(y-\bar{a})=r\bar{r}$, where the product $r\bar{r}$ is real and greater than 0 and is the square of the radius.

II. *The Incenter and Excenters.*

1. The origin of the system of coordinates is taken at O, the circumcenter of the triangle ABC. No generality is lost by taking the circumcircle as the unit circle and assigning to the vertices A, B and C the turn values t_1, t_2 and t_3 respectively. The orthocenter of such a triangle is given by the sum of the coordinates of its vertices, that is $t_1+t_2+t_3$.

2. The points A and B divide the unit circle into two arcs AB and the midpoints of these arcs are given by $\pm(t_1t_2)^{1/2}$. We may define T_3 as that one of the square roots of t_1t_2 which represents the midpoint of that arc AB which contains the point C. A similar agreement defines T_1 and T_2. It is evident at once that $T_1T_2\div T_3=\pm t_3$, and it can be shown that under the above definition $T_1T_2\div T_3=+t_3$; similarly that $T_2T_3\div T_1=t_1$ and $T_3T_1\div T_2=t_2$.

3. It is readily seen that the orthocenter of the triangle $(-T_1)(-T_2)(-T_3)$ is the incenter I_0 of ABC and that the orthocenter of the triangle $(-T_1)(+T_2)(+T_3)$ is the excenter I_1 opposite A of ABC. The coordinates of the points $I_i(i=0\cdots 3)$ then follow.

$$\begin{aligned} I_0 &= -T_1-T_2-T_3=-\Sigma_1,\\ I_1 &= -T_1+T_2+T_3=\Sigma_1-2T_1,\\ I_2 &= T_1-T_2+T_3=\Sigma_1-2T_2,\\ I_3 &= T_1+T_2-T_3=\Sigma_1-2T_3,\end{aligned}$$

where[1] $\Sigma_1=T_1+T_2+T_3$.

If A', B', C', I_i' are the reflexions of A, B, C, $I_i(i=0\cdots 3)$ respectively through O, then their coordinates are the negative of the coordinates of the latter. Agronomof[2] has pointed out that the circumcenters of the triangles formed from the set I_i' taken three at a time form the orthocentric set of points I_i and visa versa.

4. If the triangle ABC is isosceles, the relation among the turns T_i is $T_1^2=T_2T_3$, $T_2^2=T_1T_3$ or $T_3^2=T_1T_2$ depending whether the equal sides meet in A, B, or C. Should any two of these conditions hold at the same time, the third necessarily does and the triangle is equilateral. Throughout the rest of this paper (except §XIII) it will be assumed that t_1, t_2, t_3 and T_1, T_2, T_3 do not satisfy any of these conditions, i.e., that the triangle ABC is neither isosceles or equilateral.

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[1] $\Sigma_i(i=1, 2, 3)$ will be used to denote the usual symmetric functions of the T's; i.e., the T's are the roots of the equation $T^3-\Sigma_1T^2+\Sigma_2T-\Sigma_3=0$.

[2] Mathesis, vol. 27 (1907), pp. 14–15.

III. *A set of Transformations.*

1. To facilitate our procedure we will define a set of transformations $J_i(i=1, 2, 3)$ which will have the meaning that the application of J_i to an expression in $T_j(j=1, 2, 3)$ will change the signs of every T except T_i. For instance, $J_1(I_0)=I_1$, $J_2(I_0)=I_2$, $J_3[J_2(I_0)]=I_1$, etc.

IV. *The Points of Contact* A_i, B_i, C_i *of the In- and Ex-circles* (*Centers at* $I_i(i=0 \cdots 3)$) *with the Sides* BC, CA, AB *Respectively.*

1. By eliminating y between $(1/T_1^2)x+y=(T_2^2+T_3^2)/\Sigma_3$ and $(-1/T_1^2)x+y=\Sigma_1/T_1^2-\Sigma_1$, which are respectively the equations of BC and the line through I_0 perpendicular to BC, we find

$$A_0 = [T_1T_2^2 + T_1^2T_2 + T_1T_3^2 + T_1^2T_3 - T_2^2T_3 - T_2T_3^2] \div [2T_2T_3].$$

Similarly

$$B_0 = [T_1T_2^2 + T_1^2T_2 + T_2T_3^2 + T_2^2T_3 - T_1^2T_3 - T_1T_3^2] \div [2T_1T_3],$$

$$C_0 = [T_2T_3^2 + T_2^2T_3 + T_1T_3^2 + T_1^2T_3 - T_1^2T_2 - T_1T_2^2] \div [2T_1T_2].$$

The application of $J_i(i=1, 2, 3)$ to A_0, B_0, C_0 will give A_i, B_i, C_i. The triangle $A_iB_iC_i$ can not be a right triangle, as this would make two sides of the triangle ABC parallel.

V. *The Radii of the In- and Ex-circles.*

1. The radius r_0 (positive) of the incircle, center at I_0, is $[(A_0-I_0)(\bar{A}_0-\bar{I}_0)]^{1/2}$, or $(\Sigma_3-\Sigma_1\Sigma_2)/2\Sigma_3$. The application of $J_i(i=1, 2, 3)$ to $-r_0$, that is, to $(\Sigma_1\Sigma_2-\Sigma_3)/2\Sigma_3$, will give r_i (also positive).

VI. *The Symmedian Points of the Triangles* $A_iB_iC_i(i=0 \cdots 3)$.

1. In an arbitrary triangle XYZ whose vertices are the turn values u_1, u_2, u_3 respectively, the symmedian point is

$$\text{(a)} \qquad [6F_1F_3 - 2F_2^2]/[9F_3 - F_1F_2],$$

where $F_i(i=1, 2, 3)$ denote the ordinary symmetric functions of u_1, u_2, u_3. It can be shown that if u_1, u_2, u_3 are turns, $9F_3-F_1F_2$ can vanish only if $u_1=u_2=u_3$, hence the expression (a) always has a meaning.

2. To find the symmedian point K_0 of the triangle $A_0B_0C_0$ the coordinate system may be shifted by means of the transformation $x'=(x-I_0)/r_0$. This transformation shifts the origin to I_0 and causes a stretching (or shrinking) so that the incircle whose radius is r_0 becomes the unit circle, thereby making A_0, B_0, C_0 have turn values. If the new coordinates of the points A_0, B_0, C_0 are denoted by A_0'', B_0'', C_0'' respectively, then

$$A_0'' = 2\Sigma_3(A_0 - I_0)/(\Sigma_3 - \Sigma_1\Sigma_2) = - T_1.$$

Simiarly $B_0''=-T_2$ and $C_0''=-T_3$. $\Sigma_1\Sigma_2-\Sigma_3=(T_2+T_3)(T_3+T_1)(T_1+T_2)$. If $T_2+T_3=0$, $\sqrt{t_3t_1}=\pm\sqrt{t_1t_2}$, or $t_3=t_2$. Hence $\Sigma_1\Sigma_2-\Sigma_3$ can not vanish if the vertices of ABC are distinct. Making use of (a),

$$K_0'' = (2\Sigma_2^2 - 6\Sigma_1\Sigma_3)/(9\Sigma_3 - \Sigma_1\Sigma_2).$$

The inverse transformation $x = r_0x' + I_0$ applied to this point will give its coordinate with respect to the original system of coordinates. By this means the symmedian point K_0 of $A_0B_0C_0$ is found to be

$$[4\Sigma_1^2\Sigma_2\Sigma_3 - \Sigma_1\Sigma_2^3 + \Sigma_2^2\Sigma_3 - 12\Sigma_1\Sigma_3^2]/\Sigma_3[9\Sigma_3 - \Sigma_1\Sigma_2].$$

The application of $J_i(i=1, 2, 3)$ to this expression will give K_i, the symmedian point of the triangle $A_iB_iC_i$.

VII. *A Collinear Relation of Three Points.*

1. Theorem A: *The nine point center N and the symmedian point K of a triangle XYZ (which is not a right triangle) are collinear with the orthocenter H of the tangential triangle $\alpha\beta\gamma$ of XYZ.*

2. Again taking for the vertices XYZ the turn values u_1, u_2, u_3 respectively we have the following tabulation:

$$\alpha = 2u_2u_3/(u_2 + u_3), \quad \beta = 2u_1u_3/(u_1 + u_3), \quad \gamma = 2u_1u_2/(u_1 + u_2),$$
$$H = (2F_2^2 - 2F_1F_3)/(F_1F_2 - F_3).$$

($F_1F_2 - F_3 \neq 0$ if XYZ is not a right triangle). Since the nine point center is midway between the orthocenter and the circumcenter we have $N = F_1/2$; and, from §VI-1, $K = (6F_1F_3 - 2F_2^2)/(9F_3 - F_1F_2)$. Since

$$\begin{vmatrix} H & \overline{H} & 1 \\ N & \overline{N} & 1 \\ K & \overline{K} & 1 \end{vmatrix}$$

is found to vanish, it follows that H, N, and K are collinear.

3. In the general figure the triangle ABC is the common tangential triangle of the triangles $A_iB_iC_i(i=0, 1, 2, 3)$. If the orthocenter of ABC is Q, then as a consequence of Theorem A there exists on Q a pencil of four lines N_iK_iQ, where N_i is the nine point center of the triangle $A_iB_iC_i$.

VIII. *The Hessian Lines of the Figure.*

1. Theorem B: *It is known that in any triangle XYZ*[1] *the circumcenter O and the symmedian point K lie on the Hessian line*[2] *of the triangle. The orthocenter P of the pedal triangle and the reflexion H' of the orthocenter of the tangential triangle through its circumcenter are also on the Hessian line and satisfy the harmonic relation* $[OPKH']$.[3]

[1] To meet later needs of the theorem, XYZ must not be a right triangle.

[2] This line is so named because the Hessian (equated to 0) of the cubic whose roots are the turns representing the vertices of XYZ defines two so called Hessian points which with O and K satisfy a harmonic relation upon the line. The linear relation OKP was known to Tucker; see p. 175 of Milne's *Companion to the Weekly Problem Papers* (Macmillan and Co., 1888).

[3] Throughout this discussion $[ABCD]$ will represent a harmonic range of points where C and D are harmonic conjugates with respect to A and B.

2. Using u_1, u_2, u_3 as the coordinates of the vertices of XYZ, the coordinates of the points concerned are:
$O=0$, $K=(6F_1F_3-2F_2^2)/(9F_3-F_1F_2)$, $P=(3F_1F_3-F_2^2)/2F_3$, $H'=(6F_1F_3-2F_2^2)/(F_1F_2-F_3)$. The equation of OK is

$$(F_1^2-3F_2)x+(3F_1F_3-F_2^2)y=0.$$

It may be easily verified that P and H' satisfy this equation, thereby showing that O, K, P and H' are collinear points. In order to show that these points form a harmonic set $[OPKH']$ it is necessary that

$$[(O-K)(P-H')]/[(O-H')(P-K)]=-1,$$

or, since O is at the origin,

$$P=2KH'/(K+H'),$$

and a direct substitution from above will substantiate this harmonic property.

3. Now consider any one of the four triangles (not right triangles) $A_iB_iC_i (i=0,1,2,3)$ (or any one of the four triangles $A_i'B_i'C_i'$). Its circumcenter is at $I_i(I_i')$, the orthocenter of its pedal triangle is $P_i(P_i')$, and the orthocenter of its tangential triangle reflected through O is $Q'(Q)$. Therefore there are four Hessian lines concurrent at Q' and four at Q, each bearing a harmonic set of points $[I_iP_iK_iQ']([I_i'P_i'K_i'Q])$. The former come from the triangles $A_iB_iC_i$ and the latter from the triangles $A_i'B_i'C_i'$.

4. Each of the eight Hessian lines of the triangles $A_iB_iC_i$ and $A_i'B_i'C_i'$ is also the Hessian line of another triangle. The Hessian line $I_iQ'(I_i'Q)$ of the triangle $A_iB_iC_i(A_i'B_i'C_i')$ is the Hessian line of the triangle whose vertices are those points of the set $I'(I)$ which bear subscripts different from i.

IX. *A Relation Between Certain Symmedian Points.*

1. Theorem C: *The symmedian point L_i' $(i=0,1,2,3)$ (L_i) of a triangle whose vertices are those points of the set $I'(I)$ which bear subscripts different from i is mid-way between the orthocenter Q' of $A'B'C'$ (Q of ABC) and the symmedian point $K_i(K_i')$ of the triangle $A_iB_iC_i(A_i'B_i'C_i')$.*

2. It will be shown first that L_0', the symmedian point of the triangle $I_1'I_2'I_3'$ (circumcenter at I_0) is mid-way between Q' and K_0. From §II-3,

$$I_0=-\Sigma_1,\quad I_1'=-\Sigma_1+2T_1,\quad I_2'=-\Sigma_1+2T_2,\quad\text{and}\quad I_3'=-\Sigma_1+2T_3.$$

Then by means of the transformation $x'=(x+\Sigma_1)/2$ and its inverse we are able to find

$$L_0'=[3\Sigma_1\Sigma_3-4\Sigma_2^2+\Sigma_1^2\Sigma_2]/[9\Sigma_3-\Sigma_1\Sigma_2].$$

The points Q' and K_0 are respectively

$[2\Sigma_1\Sigma_3-\Sigma_2^2]/\Sigma_3$ and $[4\Sigma_1^2\Sigma_2\Sigma_3-\Sigma_1\Sigma_2^3+\Sigma_2^2\Sigma_3-12\Sigma_1\Sigma_3^2]/\Sigma_3[9\Sigma_3-\Sigma_1\Sigma_2]$(§$VI$-2).

Since $Q'+K_0=2L_0'$, the fact that L_0' is mid-way between Q' and K_0 is established.

3. The application of the transformations $J_i(i=1, 2, 3)$ to the numbers throughout the above work will establish this property for the other Hessian lines concurrent at Q'. From the symmetry of the figure it is clear that whatever is true of the pencil of Hessian lines on Q' is also true of the pencil on Q.

X. *The Four Feuerbach Points.*[1]

1. If $\phi_i(i=0, 1, 2, 3)$ represents the point of contact of the in- or ex-circle (center at I_i) and the nine point circle of ABC, we have

$$\phi_0 = [\Sigma_2^2 - 2\Sigma_1\Sigma_3]/2\Sigma_3 - \Sigma_2/2\Sigma_1.$$

$\Sigma_1 \neq 0$, for if $\Sigma_1=0$ the triangle ABC must be isosceles. The application of $J_i(i=1, 2, 3)$ to ϕ_0 gives ϕ_i. These points $\phi_i(i=1, 2, 3)$ always exist, since the denominators involved can never vanish.

XI. *The Four Points* $\lambda_i(i=0, 1, 2, 3)$ *of Lemoine.*

1. The point λ_i is defined[2] as the harmonic conjugate of ϕ_i in the set $[\phi_i\lambda_i I_i R]$, where R is the center of the nine point circle containing the ϕ's. The point λ_i will exist except in the case where ϕ_i is the mid-point of I_iR. It is clear that ϕ_i, I_i and R are collinear from their definitions. λ_0 can be found by substituting I_0 (§II-3), ϕ_0 (§X-1) and $R=[\Sigma_2^2-2\Sigma_1\Sigma_3]/2\Sigma_3$ into the expression

$$[(\phi_0 - I_0)(\lambda_0 - R)]/[(\phi_0 - R)(\lambda_0 - I_0)] = -1,$$

thereby giving

$$\lambda_0 = [\Sigma_2^2 - 2\Sigma_1\Sigma_3]/2\Sigma_3 + \Sigma_2^2/2[\Sigma_1\Sigma_2 - 2\Sigma_3].$$

The denominator

$$\Sigma_1\Sigma_2 - 2\Sigma_3 = \Sigma_3(\Sigma_1\Sigma_2/\Sigma_3 - 2) \neq 0$$

in as much as $\Sigma_1\Sigma_2/\Sigma_3$ (the square of the distance OI_0) is always less than unity. In other words, ϕ_0 is never the mid-point of the segment I_0R. The application of $J_i(i=1, 2, 3)$ to λ_0 gives λ_i. The application of J_1 to $(\Sigma_1\Sigma_2-2\Sigma_3)/2\Sigma_3$ gives

$$\left[\frac{(T_1 - T_2 - T_3)(-T_1T_2 - T_1T_3 + T_2T_3) - \Sigma_3}{2T_1T_2T_3}\right] - \tfrac{1}{2}.$$

However, the expression in brackets is r_1 (§V-1) and, hence, the denominator of λ_1 vanishes if $r_1=1/2$. In general, then, if the radius r_i of the circumcircle of $A_iB_iC_i(i=1, 2, 3)$ equals $1/2$, ϕ_i bisects the segment I_iR and the point λ_i does not exist.

[1] Mathesis, vol. 36, pp. 50–51.

[2] This is not Lemoine's definition of these points. He defines them by the linear properties in XI–2. See Nouvelles Annales de Mathématiques, 3 série, tome 5 (1886), p. 122.

2. Historically (see last footnote), these λ's are not defined from the above harmonic property, but rather, λ_0, λ_1, λ_2, λ_3 are defined as the respective points of concurrence of the following groups of lines:

$$\begin{array}{cccc} A\phi_1 & A\phi_0 & A\phi_3 & A\phi_2 \\ B\phi_2, & B\phi_3, & B\phi_0, & A\phi_1. \\ C\phi_3 & C\phi_2 & C\phi_1 & C\phi_0 \end{array}$$

XII. *A Four Line* $p_i (i=0, 1, 2, 3)$.

1. It is obvious that the line joining I_i and I_j, where $i \neq j$, and $i, j=0, 1, 2, 3$, passes through the corresponding[1] vertex of the triangle ABC. The point P_{ij} will be defined as the harmonic conjugate of this vertex with respect to I_i and I_j. It is well known[2] that these six points are the vertices of a four-line having the same diagonal triangle ABC as the four-point $I_i (i=0, 1, 2, 3)$. The lines of the four-line will be represented by p_i $(i=0, 1, 2, 3)$, where P_{ij} is not the intersection of p_i and p_j, but the intersection of the other two lines of the four-line. It will be shown that the four lines are the axes of perspectivity of the pairs of triangles with centers of perspectivity at the ϕ's and λ's (§XI-2); and also that each of the six points P_{ij} lies on the corresponding line of each of the complete four-points made up of the ϕ's, the I's and the λ's; i.e. P_{ij} is the point of concurrence of the lines $\phi_i\phi_j$, I_iI_j and $\lambda_i\lambda_j$.

2. From the definitions $[I_0I_1P_{01}A]$, $[I_0I_2P_{02}B]$, $[I_0I_3P_{03}C]$ we obtain, respectively,

$$\begin{aligned} P_{01} &= T_1(T_2^2 + T_3^2 - T_1^2 + T_2T_3)/(T_1^2 + T_2T_3), \\ P_{02} &= T_2(T_1^2 + T_3^2 - T_2^2 + T_1T_3)/(T_2^2 + T_1T_3), \\ P_{03} &= T_3(T_2^2 + T_1^2 - T_3^2 + T_1T_2)/(T_3^2 + T_1T_2). \end{aligned}$$

The points P_{12}, P_{23}, P_{31} could be obtained likewise from their definitions $[I_1I_2P_{12}C]$, $[I_3I_2P_{23}A]$, $[I_1I_3P_{31}B]$ respectively, but we note in a simpler manner that $P_{12}=J_1(P_{03})=J_2(P_{03})$, $P_{23}=J_2(P_{01})=J_3(P_{01})$, $P_{31}=J_1(P_{02})=J_3(P_{02})$. The expressions for P_{01}, P_{02} and P_{03} always have a meaning and those for P_{12}, P_{23} and P_{31} do, provided the original triangle is not isosceles (§II-4).

3. By direct substitution it may now be shown that each of these points P_{ij} lies on three lines, as follows:

$$\begin{array}{ll} P_{01} \text{ on } BC,\ \phi_0\phi_1,\ \lambda_0\lambda_1, & P_{12} \text{ on } AB,\ \phi_1\phi_2,\ \lambda_1\lambda_2, \\ P_{02} \text{ on } AC,\ \phi_0\phi_2,\ \lambda_0\lambda_2, & P_{13} \text{ on } AC,\ \phi_1\phi_3,\ \lambda_1\lambda_3, \\ P_{03} \text{ on } AB,\ \phi_0\phi_3,\ \lambda_0\lambda_3, & P_{23} \text{ on } BC,\ \phi_2\phi_3,\ \lambda_2\lambda_3. \end{array}$$

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[1] This correspondence is between the letters ABC and the three ways of separating 0123 into two pairs, thus, A with 01–23, B with 02–13, etc.

[2] A. Clebsch, *Leçons sur la Géométrie*, tome 1, p. 71.

These linear relations when considered in connection with the pencils of lines on the ϕ's and λ's (XI-2) establish the fact that the points P_{ij} are arranged by threes upon the four lines $p_i (i=0, 1, 2, 3)$ which are the axes of perspectivity of the triangles perspective from the ϕ's and from the λ's. The significance of the subscripts is that p_i is that line of the four-line which contains the three of the six points P_{ij} which do not bear the subscript i.

4. In going back to the definition (XI-1) of the points λ_i it will be recalled that there exists on R a pencil of four lines of the type $RI_i\phi_i\lambda_i (i=0, 1, 2, 3)$. If all the possible pairs of triangles perspective from R are selected (where each triangle will have vertices consisting entirely of I's, ϕ's or λ's), then in the light of the theorem of Desargues there will be certain axes of perspectivity. Since the anharmonic ratio of the points on each line is constant for the whole pencil, there are not twelve axes of perspectivity, but only four, and these constitute the four-line p_i.

5. By selecting proper four-points or by direct substitution it can be shown that these points P_{ij} enter into other harmonic relations: $[CAP_{02}P_{13}]$, $[BCP_{01}P_{23}]$ and $[ABP_{03}P_{12}]$.

XIII. *Some Special Cases.*

1. Thus far in this discussion there have been three types of triangles ABC which have caused vanishing denominators: (1) isosceles triangles, (2) equilateral triangles, and (3) triangles having a radius of an escribed circle equal to 1/2. Type (3) will receive no further treatment than that already given it (XI). Types (1) and (2), however, have been ruled out of the past discussion, and more consideration will be given them now.

2. *The isosceles triangle.* The triangle ABC is isosceles under any one of three conditions: $T_1^2=T_2T_3$, $T_2^2=T_1T_3$, or $T_3^2=T_1T_2$, depending on whether its equal sides meet in A, B, or C. It will suit the present purpose to investigate the figure when ABC is isosceles due to the condition $T_1^2=T_2T_3$, with $T_2^2\neq T_1T_3$ and $T_3^2\neq T_1T_2$. For such a triangle all the expressions representing points for the general case will apply to this case when these expressions do not contain vanishing denominators. Thus, all the points I_i, ϕ_i, λ_i, P_{ij} (except P_{23}), K and R exist and are represented by the same expressions which represent them in the general case. The nine point circle of ABC and the two in- and ex-circles with centers at I_0 and I_1 are tangent to each other at the mid-point of BC; therefore, $\phi_0=\phi_1=(t_2+t_3)/2$. In satisfying the harmonic relation $[I_0I_1P_{01}A]$, P_{01} becomes $(t_2+t_3)/2$ also. The segment I_2I_3 is bisected by A, therefore there is no point P_{23}. The remaining P_{ij} points exist and the four-line p_i is special only in the sense that p_0 and p_1 are parallel. The points λ_2 and λ_3 are on BC. The perpendicular bisector of BC contains the points A, O, I_0, λ_0, R, K, $\phi_0=\phi_1=P_{01}$ and I_1 and this line is both the Euler and the Hessian line of ABC.

3. *The equilateral triangle.* The conditions for an equilateral triangle ABC are $T_1^2=T_2T_3$, $T_2^2=T_1T_3$ and $T_3^2=T_1T_2$. In order to simplify the discussion let the axis of reals be located so that $T_1=t_1=1$. Then $T_2T_3=1$, $T_2^2=T_3$, $T_3^2=T_2$,

from which it follows that $T_2=\omega$, $T_3=\omega^2$, or $T_2=\omega^2$, $T_3=\omega$, where $1, \omega, \omega^2$ represent the cube roots of 1. The expressions for the points I_i obtained in §II evidently hold for the equilateral triangle and we have $I_0=-(1+\omega+\omega^2)=0$, $I_1=-2$, $I_2=-2\omega$, and $I_3=-2\omega^2$. Since the nine point circle of ABC is the same as the incircle with center at I_0, the Feuerbach point ϕ_0 is indeterminate. However, $\phi_1=-1/2$, $\phi_2=-\omega/2$ and $\phi_3=-\omega^2/2$, which are the same as the results obtained by substituting in the expressions for the ϕ-points in the general case. Similarly for the λ's; λ_0 does not exist and $\lambda_1=1=A$, $\lambda_2=\omega=B$, and $\lambda_3=\omega^2=C$. The points P_{12}, P_{13}, P_{23} and the line p_0 do not exist, in as much as C, B and A bisect the line segments I_1I_2, I_1I_3 and I_2I_3 respectively. We also find that $P_{01}=-1/2=\phi_1$, $P_{02}=-\omega/2=\phi_2$, and $P_{03}=-\omega^2/2=\phi_3$. The Euler and Hessian lines are both indeterminate.

Part B

XIV. *Projective Properties.*

1. Except for certain classes of triangles which have been specially treated (XIII), the points and lines associated with the triangle have been the points and lines of ordinary geometry and not the points and lines at infinity of projective geometry. From now on no distinction will be made and throughout this section the words point and line will be used in the broader sense of projective geometry.

XV. *The Set of Points Q_{ij}.*

1. There are six points determined by the six pairs of lines $(\phi_i\phi_j)$ and $(\lambda_k\lambda_m)$, and these points constitute the set Q_{ij}, where i, j, k, m are distinct and equal 0, 1, 2, 3. For example, Q_{01} is the intersection of the lines $\phi_0\phi_1$ and $\lambda_2\lambda_3$.

2. From XI-2 there exists a pencil of four lines on each of the points A, B, C. The pencil on A is

$$\begin{matrix} \lambda_2\phi_3 \\ \lambda_1\phi_0 \\ \phi_1\lambda_0 \\ \phi_2\lambda_3 \end{matrix} \quad \text{or} \quad \begin{matrix} \lambda_1\phi_0 \\ \lambda_3\phi_2 \\ \phi_1\lambda_0 \\ \phi_3\lambda_2 \end{matrix}.$$

With A as center of perspectivity there are four pairs of perspective triangles considering all the possible triangles from each column as arranged above. The corresponding sides define the harmonic sets of points $[RCQ_{03}Q_{12}]$ and $[RBQ_{02}Q_{13}]$. Similarly, using B and C as centers we get respectively $[RAQ_{01}Q_{23}]$, $[RCQ_{03}Q_{12}]$ and $[RAQ_{01}Q_{23}]$, $[RBQ_{02}Q_{13}]$.

3. There exist twelve lines of the type $Q_{ij}Q_{jk}P_{ik}$, where i, j, k are distinct and equal 0, 1, 2, 3. Consider the triangles $\phi_0\phi_2\phi_1$ and $\lambda_2\lambda_0\lambda_3$ perspective from B. The axis of perspectivity is $Q_{01}Q_{12}P_{02}$. The complete truth of the statement may be verified by selecting other perspective triangles.

4. Then there exist twelve harmonic sets of points, six of the type

$[P_{ij}Q_{ij}\phi_i\phi_j]$ and six of the type $[P_{ij}Q_{km}\lambda_i\lambda_j]$, where i, j, k, m are distinct and equal to 0, 1, 2, 3. These may be arrived at by the proper selections of complete four-points.

XVI. *The Complete Pencil of Lines on R.*

1. On the seven lines through R there exist the following seven harmonic sets, the last three being repeated with the Q's interchanged (XI-1, XV-2):

$$RI_0\phi_0\lambda_0,\quad RI_2\phi_2\lambda_2,\quad RAQ_{01}Q_{23},\quad RBQ_{02}Q_{13},\quad RCQ_{03}Q_{12},$$
$$RI_1\phi_1\lambda_1,\quad RI_3\phi_3\lambda_3,\quad RAQ_{23}Q_{01},\quad RBQ_{13}Q_{02},\quad RCQ_{12}Q_{03}.$$

If these ten sets were arranged in a single column and if all the possible pairs of perspective triangles were selected by using as vertices of one triangle three points selected from one column within the tabulation and for the other triangle the corresponding three points selected from either other column, the axis of perspectivity would be, in each case, one of the four lines p_i (XII). The point of intersection of any corresponding sides will be either a point P_{ij} or one of twelve new points which will be denoted by T_{i-jk}, where i, j, k are distinct and equal to 0, 1, 2, 3. In designating the T's as the points of concurrence of certain sets of lines, the following notation will be clear:

$$\begin{aligned}
T_{0-12} &= (\phi_3Q_{12})(\lambda_3Q_{03})(I_3C), & T_{1-02} &= (\phi_3Q_{02})(\lambda_3Q_{13})(I_3B),\\
T_{0-23} &= (\phi_1Q_{23})(\lambda_1Q_{01})(I_1A), & T_{1-03} &= (\phi_2Q_{03})(\lambda_2Q_{12})(I_2C),\\
T_{0-13} &= (\phi_2Q_{13})(\lambda_2Q_{02})(I_2B), & T_{1-23} &= (\phi_0Q_{23})(\lambda_0Q_{01})(I_0A),\\
T_{2-01} &= (\phi_3Q_{01})(\lambda_3Q_{23})(I_3A), & T_{3-01} &= (\phi_2Q_{01})(\lambda_2Q_{23})(I_2A),\\
T_{2-03} &= (\phi_1Q_{03})(\lambda_1Q_{12})(I_1C), & T_{3-02} &= (\phi_1Q_{02})(\lambda_1Q_{13})(I_1B),\\
T_{2-13} &= (\phi_0Q_{13})(\lambda_0Q_{02})(I_0B), & T_{3-12} &= (\phi_0Q_{12})(\lambda_0Q_{03})(I_0C).
\end{aligned}$$

2. It may be easily shown that p_i contains three points of the type T_{i-jk} in addition to the three points P_{jk}. For instance, p_0 contains P_{23}, P_{13}, P_{12}, T_{0-23}, T_{0-13} and T_{0-12}. Moreover, if P_{jk}, P_{kl} and P_{jl} are on p_i, then T_{i-jk} will be the conjugate of P_{jk} in the harmonic set $[P_{jk}T_{i-jk}P_{kl}P_{jl}]$, similarly, T_{i-kl} will be the conjugate of P_{kl}, etc. This means that if a homogeneous one-dimensional coordinate system were established on the line p_i and if the three points P_{jk}, P_{kl}, P_{jl} were given by a binary cubic form, then the three points T_{i-jk}, T_{i-kl}, T_{i-jl} would be given by the cubic covariant form.[1]

3. The T's also enter into harmonic sets of points upon the sides of the orthocentric quadrilateral (vertices at $I_i(i=0, 1, 2, 3)$). Consider the following six relations:

$$\begin{aligned}
&T_{1-23}T_{0-23}P_{01}AI_0I_1, && T_{2-03}T_{1-03}P_{12}CI_1I_2,\\
&T_{2-13}T_{0-13}P_{02}BI_0I_2, && T_{3-02}T_{1-02}P_{13}BI_1I_3,\\
&T_{3-12}T_{0-12}P_{03}CI_0I_3, && T_{3-01}T_{2-01}P_{23}AI_2I_3.
\end{aligned}$$

[1] A. Clebsch, *Leçons sur la Géométrie*, tome I, page 272.

On the line I_0I_1, for example, there exist the harmonic sets $[AP_{01}T_{0-23}T_{1-23}]$, $[AI_0P_{01}T_{1-23}]$, $[AI_1T_{0-23}P_{01}]$, and $[AP_{01}I_0I_1]$. (See XII-2). Similar relations can be found on the remaining lines.

XVII. *Projective Coordinates.*

1. A homogeneous coordinate system will be entirely determined if the points A, B, C and I_0 are assigned the following coordinates: $A=(1:0:0)$, $B=(0:1:0)$, $C=(0:0:1)$, and $I_0=(1:1:1)$. The P's and the remaining I's follow readily from their properties in XII. We have

$$P_{01} = (0:1:1),\quad P_{12} = (1:-1:0),\quad I_1 = (-1:1:1),$$
$$P_{02} = (1:0:1),\quad P_{13} = (1:0:-1),\quad I_2 = (1:-1:1), \text{ and}$$
$$P_{03} = (1:1:0),\quad P_{23} = (0:1:-1),\quad I_3 = (1:1:-1).$$

Let $(a:b:c)$ be the projective coordinates of R. The point ϕ_0 lying on the line joining R and I_0 will have the projective coordinates $(va+1:vb+1:vc+1)$. We may assume that this constant v has been incorporated into the a, b, and c (which are, therefore, no longer homogeneous), and the coordinates of ϕ_0 then become $(a+1:b+1:c+1)$. Then the conditions (XV) on ϕ_i and λ_i determine their coordinates as follows:

$$\phi_0 = (a+1:b+1:c+1),\qquad \lambda_0 = (a-1:b-1:c-1),$$
$$\phi_1 = (a+1:b-1:c-1),\qquad \lambda_1 = (a-1:b+1:c+1),$$
$$\phi_2 = (a-1:b+1:c-1),\qquad \lambda_2 = (a+1:b-1:c+1),$$
$$\phi_3 = (a-1:b-1:c+1).\qquad \lambda_3 = (a+1:b+1:c-1).$$

The Q's and the T's follow directly from their definitions (XV and XVI).

$$Q_{01} = (a+1:b:c),\quad T_{0-12} = (1:1:-2),\quad T_{2-01} = (2:1:-1),$$
$$Q_{02} = (a:b+1:c),\quad T_{0-23} = (-2:1:1),\quad T_{2-03} = (-1:1:2),$$
$$Q_{03} = (a:b:c+1),\quad T_{0-13} = (1:-2:1),\quad T_{2-13} = (1:2:1),$$
$$Q_{12} = (a:b:c-1),\quad T_{1-02} = (1:2:-1),\quad T_{3-01} = (2:-1:1),$$
$$Q_{13} = (a:b-1:c),\quad T_{1-03} = (1:-1:2),\quad T_{3-02} = (-1:2:1),$$
$$Q_{23} = (a-1:b:c),\quad T_{1-23} = (2:1:1),\quad T_{3-12} = (1:1:2).$$

XVIII. *The Transformation from Conjugate to Projective Coordinates.*

1. The transformation

$$\begin{aligned} \rho x_1 &= A_1X + B_1Y + C_1,\\ \rho x_2 &= A_2X + B_2Y + C_2, \qquad (1)\\ \rho x_3 &= A_3X + B_3Y + C_3, \end{aligned}$$

where the coefficients are real and the determinant of the transformation is different from zero, connects Cartesian coordinates (X, Y) with projective co-

ordinates $(x_1:x_2:x_3)$. Conjugate coordinates (x, y) are related to the Cartesian coordinates by the equations $x=X+iY$, $y=X-iY$, or $X=(x+y)/2$, $Y=(x-iy)/2i$, and these equations substituted in (1) give a transformation of the same non-singular type.

2. In conjugate coordinates the equations of the sides BC, AC, and AB of the original triangle are respectively

$$x + T_1^2 y - T_1^2(T_2^2 + T_3^2)/\Sigma_3 = 0,$$
$$x + T_2^2 y - T_2^2(T_1^2 + T_3^2)/\Sigma_3 = 0,$$
$$x + T_3^2 y - T_3^2(T_2^2 + T_1^2)/\Sigma_3 = 0.$$

If A, B, C and I_0 have the coordinates as in XVII, the equations of BC, AC and AB are respectively $x_1=0$, $x_2=0$ and $x_3=0$. Hence, the transformation connecting the two coordinate systems will be

$$\begin{aligned} x_1 &= k_1[x + T_1^2 y - T_1^2(T_2^2 + T_3^2)/\Sigma_3], \\ x_2 &= k_2[x + T_2^2 y - T_2^2(T_1^2 + T_3^2)/\Sigma_3], \\ x_3 &= k_3[x + T_3^2 y - T_3^2(T_1^2 + T_2^2)/\Sigma_3], \end{aligned} \tag{2}$$

where k_1, k_2 and k_3 are still to be determined. This may be done by means of the point I_0 which has the projective coordinates $(1:1:1)$ and conjugate coordinates $(-\Sigma_1, -\overline{\Sigma}_1)$. Substituting in transformation (2) we get $k_1:k_2:k_3 = 1/T_1:1/T_2:1/T_3$. Therefore the required transformation is

$$\begin{aligned} x_1 &= T_2T_3x + T_1\Sigma_3 y - T_1(T_2^2 + T_3^2), \\ x_2 &= T_1T_3x + T_2\Sigma_3 y - T_2(T_1^2 + T_3^2), \\ x_3 &= T_1T_2x + T_3\Sigma_3 y - T_3(T_1^2 + T_2^2). \end{aligned} \tag{3}$$

3. If this transformation (3) is applied to the conjugate coordinates of R (see XI), its projective coordinates are found to be

$$(a:b:c) = [(T_1^4 + T_2^2T_3^2)/T_1:(T_2^4 + T_1^2T_3^2)/T_2:(T_3^4 + T_1^2T_2^2)/T_3].$$

Therefore,

$$a = k(T_1^4 + T_2^2T_3^2)/T_1, \quad b = k(T_2^4 + T_1^2T_3^2)/T_2, \text{ and } c = k(T_3^4 + T_1^2T_2^2)/T_3,$$

where k is a factor of proportionality. Using the point ϕ_0, it may be shown that $k = -1/2\Sigma_3$. Whence,

$$\begin{aligned} &T_1^4 + T_2^2T_3^2 + 2T_1^2T_2T_3a = 0, && (\mathrm{I}) \\ (4)\qquad &T_2^4 + T_1^2T_3^2 + 2T_1T_2^2T_3b = 0, && (\mathrm{II}) \\ &T_3^4 + T_1^2T_2^2 + 2T_1T_2T_3^2c = 0. && (\mathrm{III}) \end{aligned}$$

If a, b, and c are given arbitrary values, we would have a projective figure which would possess all the projective properties of the metric figure. Moreover, for an arbitrary selection of a, b, and c the figure obtained would not in general be a projection of our metric figure. In order that a figure with projective co-

ordinates as given in XVII may be a projection of our metric figure, there are certain conditions imposed on a, b, and c.

4. These restriction are implied in the set of equations (4) above. If equation ($^{\mathrm{II}}$) is considered as a quadratic equation in T_1, the roots are

$$[-b \pm \sqrt{(b^2-1)}]T_2^2/T_3.$$

Substituting these values in ($^{\mathrm{I}}$) and ($^{\mathrm{III}}$) gives

$$(\alpha)\quad (-b \pm \sqrt{[b^2-1]})^4(T_2^6/T_3^6) + 2a(-b \pm \sqrt{[b^2-1]})^2(T_2^3/T_3^3) = -1,$$

$$(\beta)\quad (-b \pm \sqrt{[b^2-1]})^2(T_2^6/T_3^6) + 2c(-b \pm \sqrt{[b^2-1]})(T_2^3/T_3^3) = -1,$$

where the sign before the radical is taken the same in each equation. Either of these equations will define the ratio T_2/T_3, yet T_2/T_3 must be a turn and a restriction on b can be obtained by using that fact. Take an equation as $Ax^2+Bx+C=0$, where A, B, and C may be real or complex and x a turn. Then $\bar{C}x^2+\bar{B}x+\bar{A}=0$. Therefore, $A=k\bar{C}$, $B=k\bar{B}$ and $C=k\bar{A}$, where k is a factor of proportionality. The application of these conditions to equation (β), bearing in mind that a, b, and c are real, makes it necessary that $|b|\leqq 1$. Similarly it is necessary that $|a|\leqq 1$ and $|c|\leqq 1$. Whenever such an expression occurs as $\pm\sqrt{[a^2-1]}$, where $|a|\leqq 1$, it will be written $\pm i\sqrt{[1-a^2]}$, where $\sqrt{[1-a^2]}$ means the positive (or zero) root.

5. There are, however, further conditions on a, b, c. Let equations (α) and (β) be solved for T_2^3/T_3^3. We get, respectively,

$$\frac{-a \pm i\sqrt{[1-a^2]}}{[-b \pm i\sqrt{(1-b^2)}]^2} \text{ and } \frac{-c \pm i\sqrt{[1-c^2]}}{-b \pm i\sqrt{(1-b^2)}},$$

where the signs in the two denominators are to be the same and the signs of the numerators are independent of those in the denominators and of each other. By equating equals and simplifying, we obtain the symmetrical relation

$$(A)\qquad [-a \pm i\sqrt{(1-a^2)}][-b \pm i\sqrt{(1-b^2)}][-c \pm i\sqrt{(1-e^2)}] = 1,$$

where the three double signs are quite independent. Hence, if a, b, and c are real numbers which are each of absolute value less than or equal to 1, a necessary condition that T's can be found satisfying condition (4) is that (A) should hold for some choice of signs. It is clear that if the condition holds for some choice of signs it will hold for the opposite choice.

It may now be shown that this is a sufficient condition on a, b, and c. Given then, that $|a|\leqq 1$, $|b|\leqq 1$, and $|c|\leqq 1$, and $(-a+iA)(-b+iB)(-c+iC)=1$, where A, B, and C denote respectively $\pm\sqrt{[1-a^2]}$, $\pm\sqrt{[1-b^2]}$, and $\pm\sqrt{[1-c^2]}$ for some fixed choice of signs, values may be found for T_1, T_2, and T_3 satisfying conditions (4) as follows: Take

$$T_1^3:T_2^3:T_3^3 = 1/(-a+iA):1/(-b+iB):1/(-c+iC),$$

or

$$T_1^3 = K^3/(-a+iA),\quad T_2^3 = K^3/(-b+iB),\quad T_3^3 = K^3/(-c+iC),$$

where K is an arbitrary turn. Then $T_1=K/\sqrt[3]{[-a+iA]}$, $T_2=K/\sqrt[3]{[-b+iB]}$, and $T_3=K/\sqrt[3]{[-c+iC]}$. Let the cube roots in the denominators be so chosen that their product is 1. Their product for any choice will be 1, ω, or ω^2, but after any two of them have been fixed arbitrarily the third can be so chosen that their product is 1. We then have $T_1T_2T_3=K^3$. That the set of T's so chosen satisfy conditions (4) may be seen by direct substitution. Therefore there are nine sets of T's which satisfy the conditions. If $T_1T_2T_3$ are one such set, the nine sets are

$$\begin{array}{ccccccccc} T_1 & T_2 & T_3 & T_1 & \omega\, T_2 & \omega^2 T_3 & T_1 & \omega^2 T_2 & \omega\, T_3 \\ \omega\, T_1 & \omega\, T_2 & \omega\, T_3 & \omega\, T_1 & \omega^2 T_2 & T_3 & \omega\, T_1 & T_2 & \omega^2 T_3 \\ \omega^2 T_1 & \omega^2 T_2 & \omega^2 T_3 & \omega^2 T_1 & T_2 & \omega\, T_3 & \omega^2 T_1 & \omega\, T_2 & T_3, \end{array}$$

and they group themselves (by threes) as indicated. Each group does not contain three different types of triangles, but the same triangle affected by rotation through an angle of 120° or 240°. Also the arbitrary turn K causes an arbitrary rotation of the triangle. Therefore, of the nine sets of T's there can be formed only three different kinds of triangles. Hence, if a, b, and c are chosen as real numbers of absolute values less than or equal to 1 to satisfy condition (A), then there are three sets of T's which will give three and only three essentially different metric figures. Any one of these triangles can be obtained from any other by holding one vertex fixed and rotating the second and third vertices around the circumcircle through 120° and 240° respectively.

6. The amount of arbitrariness of a, b, and c will next be considered. Condition (A) represents, in fact, eight conditions, some one of which must be satisfied for a fixed choice of signs. If given $|a|\leqq 1$, $|b|\leqq 1$ (both real) and if the signs of $\pm\sqrt{[1-a^2]}$ and $\pm\sqrt{[1-b^2]}$ are arbitrarily chosen, then (A) may be written

$$(B) \qquad (-a+iA)(-b+iB)(-c\pm i\sqrt{[1-c^2]})=1,$$

where, as earlier, A and B each includes the sign of the radical. Regrouping (B), squaring and solving for c, we obtain $c=AB-ab$, which is a real number. Checking this value of c in (B) it is found that the equation is satisfied only if the + sign is chosen in the last parenthesis. Hence, a value of c has been determined which will satisfy some one of the eight conditions in (A). If A and B had signs which are opposite to those which were assigned them, c would still be $AB-ab$, and this value will satisfy one of the eight conditions. Again suppose the square roots are chosen as A and $-B$ (or $-A$ and B); then $c=-AB-ab$. This value will likewise satisfy one of the eight conditions. It can be shown that both of these values of c are of absolute value less than or equal to 1. Hence, given a and b, both real and of absolute value less than or equal to 1, there can be found two values of c, namely $c_1=AB-ab$ and $c_2=-AB-ab$, such that the two sets abc_1 and abc_2 each satisfy two of the eight conditions in (A).

If a and b are not distinct but both equal to u, then $c=-1$, $1-2u^2$. If $a=b=c=u$, then $u=-1$, $1/2$. When $u=-1$, either there is no triangle or the

triangle is equilateral. If $u=1/2$, then condition (A) requires that the signs of the radicals be the same. Whence, from §5 we get the same deductions as when $u=-1$.

Summary. If in equations (4) the three turns T_1, T_2, and T_3 are given, then the three real numbers a, b, and c will be uniquely determined. If, however, three arbitrary real values are given to a, b, and c, it will not in general be possible to find three turns satisfying equations (4). But if the given a, b, and c satisfy the relations

$$|a| \leqq 1, \ |b| \leqq 1, \ |c| \leqq 1;$$

$$(C) \qquad [-a \pm i\sqrt{(1-a^2)}][-b \pm i\sqrt{(1-b^2)}][-c \pm i\sqrt{(1-c^2)}] = 1,$$

then and only then can values of the T's be found, and in this case three homogeneously distinct sets of turn values can be found.

Geometrically, this means if we start with a metric figure, such a metric figure will correspond to a fixed set of T's and for the projective coordinates of the points of that figure, or the points of any figure projective with it, we could use the coordinates of §XVII in which a, b, and c would have their values uniquely determined. Since these values must necessarily satisfy conditions (C), they give rise to three sets of T's, the original set and two others, and the metric figures determined by these three sets are projective with each other. On the other hand, if one gives to a, b, and c arbitrary values, the figure whose points have the coordinates of §XVII will not in general be projective with the metric figure. But, if a, b, and c are given values satisfying conditions (C), then the figure whose points have the coordinates of §XVII will be projective with the metric figure, in fact with three metric figures which are essentially different.

ON MATHEMATICAL LIFE IN HUNGARY[1]

By TIBOR RADÓ, Ohio State University

Hungary has turned out, in the course of the last fifty years, a surprisingly large number of fine mathematicians. The time allotted for this talk would certainly not allow me to give any adequate idea of their contributions to our science; so I have thought I had better not try the impossible. Instead of attempting to sketch what Hungarian mathematicians have done in the way of research, I shall restrict myself to speak of what they are doing in the way of taking care of the next mathematical generation—certainly a business of major importance. In this respect, one thing which is likely to interest you is the *Eötvös-prize.*

This prize was founded in 1894 by the Hungarian Mathematical Society, to commemorate the elevation to the Ministry of Education of the President

[1] A paper read at the Cleveland Meeting of the Mathematical Association of America, on January 1, 1931, by invitation of the program committee.

of the Society, the famous Hungarian physicist, Baron Eötvös. The prize, or more exactly the first and the second prize and one or more honorable mentions, are awarded to the winners of the national mathematical competition for students, held in October every year, and open to the young folks who graduated from the secondary schools in June of the same year. The good mathematicians among these young folks are in October, generally speaking, either at some University, studying mathematics and physics in order to obtain, after four years of study, a secondary teachers diploma, or they are at the Polytechnical School in Budapest. As a matter of fact, several prominent Hungarian mathematicians started studying engineering, and the winner of the Eötvös-prize has many times been a student of the Polytechnical School.

So every October, since 1894, the national competition is entered by ambitious and hopeful baby-mathematicians, of an average age of eighteen years. The competition is sponsored by the Hungarian Mathematical Society. The competitors get three problems and four hours to solve them; they are allowed to use books and everything they want to; the problems are selected, however, in such a way that practically nothing, save one's own brains, can be of any help. As a matter of fact, it does not even help one to have been a real good boy or girl and to have learned all one's formulas and logarithms. The prize is not intended for the good boy; it is intended for the future creative mathematician.

Is it possible to test by problems creative mathematical ability? Certainly it would be possible to prove the impossibility of such an attempt on the basis of psychological theories. First of all, a boy can be really able, and still be unable to solve anything in four hours, in strange surroundings. There is the great danger of discouraging this type. There are many other objections; it might be interesting, but certainly futile, to discuss them, simply because the experience of almost forty years shows conclusively that this mathematical competition *could* be made a success.

Recently, Professor Kürschák, the most enthusiastic promoter of the idea, published in one volume all the problems of the past competitions, with the solutions and a record of the first and second Eötvös prize winners. This book is exceedingly delightful reading; and besides it makes it possible to analyze the working of the Eötvös prize. In order to give you an idea of its working, I pick out the year 1897, and I shall tell you what happened then.

In 1897, one of the three problems was the following.

Prove that the angles α, β, γ of any triangle satisfy the inequality

$$\sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma < \tfrac{1}{4}.$$

A first noteworthy feature of the problem is that it certainly can be clearly understood by anybody who has graduated from any secondary school. The notions involved are absolutely familiar. The type of the problem is unusual; indeed, in the secondary schools mathematics consists mostly of identities and computations, whilst here we have got to prove an inequality. Of course, secondary teaching may vary with time and country; I suppose, however, that if

inequalities of the above type had been drilled in Hungarian secondary schools around the year 1897, the above problem certainly would not have been selected for the competition.

According to the book of Professor Kürschák, this problem has been attacked by the first and the second prize winners in the following ways.

Solution by the boy b_1. (I am sorry to say, no girl g ever won the Eötvös-prize.) We have

$$\tfrac{1}{2}\alpha = 90^\circ - \tfrac{1}{2}(\beta + \gamma) < (90^\circ - \tfrac{1}{2}\beta) < 90^\circ.$$

For angles less than 90°, the sine increases with the angle; so it follows that

$$\sin \tfrac{1}{2}\alpha < \sin (90^\circ - \tfrac{1}{2}\beta) = \cos \tfrac{1}{2}\beta$$

and

$$(1) \qquad \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta < \cos \tfrac{1}{2}\beta \sin \tfrac{1}{2}\beta = \tfrac{1}{2} \sin \beta \leqq \tfrac{1}{2}.$$

Suppose the notation has been chosen in such a way that γ is the *smallest* angle of the triangle; then $\gamma \leqq 60^\circ$ and

$$(2) \qquad \sin \tfrac{1}{2}\gamma \leqq \sin 30^\circ = \tfrac{1}{2}.$$

So it follows from (1) and (2) that

$$\sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma < \tfrac{1}{4}.$$

Solution by the boy b_2. This fellow remembered, first of all, quite a lot of his plane geometry. He started with the formula

$$(3) \qquad \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma = \rho/4r$$

where ρ is the radius of the inscribed and r the radius of the circumscribed circle. As evidently $\rho < r$, the boy b_2 obtained, without any computations, the desired inequality

$$\sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma < \tfrac{1}{4}.$$

The boy b_2 then did a very curious thing. Instead of enjoying the nice solution he had found, he started wondering if the result were as good as possible; and he asked the question if the bound, $\tfrac{1}{4}$, could not be lowered. And, indeed, he completed his solution by showing that

$$\sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma \leqq \tfrac{1}{8},$$

and that $\tfrac{1}{8}$ cannot be replaced by any smaller number. To show this, he refers to the formula of Euler

$$(4) \qquad d^2 = r^2 - 2\rho r,$$

where ρ, r have the previous meaning, and d is the distance of the centers of the inscribed and of the circumscribed circles. From (4) it follows that $r^2 - 2\rho r \geqq 0$, and therefore that $\rho/r \leqq \tfrac{1}{2}$. Substituting into (3), we get

$$\sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma \leqq \tfrac{1}{8};$$

and, as for the equilateral triangle ($\alpha=\beta=\gamma=60°$) the left-hand side is exactly equal to $\frac{1}{8}$, the constant $\frac{1}{8}$ cannot be improved any more.

You may be impressed also by the fact that the boy b_2 remembered the formula (4) of Euler. I suppose that back in 1897 the geometry of the triangle had been much more attended to in secondary schools than in these days, when the young people get calculus in the upper grades. So it may not be astonishing at all that the boy b_2 remembered that formula which in these days the students possibly never hear of. Really impressive, however, is the fact that the boy b_2 showed what we might call *the pioneer spirit.* He arrived at the proposed goal all right, but he kept on going until he satisfied himself that it was certainly impossible to go any further. This pioneer spirit makes the scientist; it makes, as a matter of fact, the leaders in all fields of human activity.

You will not be very much surprised if I tell you that the boy b_2 has developed into a mathematician of international fame. However, before discussing the 1897 situation any further, I want to give a third solution of the above problem, due to boy B. He is denoted by B, because he was some grown-up mathematician; and indeed, the picture would be very incomplete if I would not tell something concerning the general interest which is given to this competition by the adult Hungarian mathematicians. They try to solve the problems as nicely as possible, and mainly as psychologically as possible. That is to say, they try to solve the problems as a boy who is gifted but who did not waste any too much time on learning things would handle the question. In this respect, the brilliant solution of the boy b_2 is open to the psychological objection that a gifted, but very lazy boy would not remember all the formulas the boy b_2 was using. So some grown-up mathematician proposed the following solution of the problem. We have

$$\cos(\tfrac{1}{2}\alpha + \tfrac{1}{2}\beta) = \cos\tfrac{1}{2}\alpha \cos\tfrac{1}{2}\beta - \sin\tfrac{1}{2}\alpha \sin\tfrac{1}{2}\beta,$$

$$\cos(\tfrac{1}{2}\alpha - \tfrac{1}{2}\beta) = \cos\tfrac{1}{2}\alpha \cos\tfrac{1}{2}\beta + \sin\tfrac{1}{2}\alpha \sin\tfrac{1}{2}\beta.$$

Subtraction gives

$$2\sin\tfrac{1}{2}\alpha \sin\tfrac{1}{2}\beta = \cos(\tfrac{1}{2}\alpha - \tfrac{1}{2}\beta) - \cos(\tfrac{1}{2}\alpha + \tfrac{1}{2}\beta). \tag{5}$$

Now, on account of the relation $\alpha+\beta+\gamma=\pi$,

$$\cos(\tfrac{1}{2}\alpha + \tfrac{1}{2}\beta) = \sin\tfrac{1}{2}\gamma.$$

Besides,

$$\cos(\tfrac{1}{2}\alpha - \tfrac{1}{2}\beta) \leqq 1. \tag{6}$$

So it follows from (5) that

$$2\sin\tfrac{1}{2}\alpha \sin\tfrac{1}{2}\beta \leqq 1 - \sin\tfrac{1}{2}\gamma.$$

Multiplying by $\sin\frac{1}{2}\gamma$ and using the inequality between the geometric and the arithmetic means, we obtain

$$2 \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma \leqq (1 - \sin \tfrac{1}{2}\gamma) \sin \tfrac{1}{2}\gamma \leqq \left(\frac{(1 - \sin \frac{1}{2}\gamma) + \sin \frac{1}{2}\gamma}{2}\right)^2 = \tfrac{1}{4},$$

that is to say

$$\sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma \leqq \tfrac{1}{8}. \tag{7}$$

The sign of equality can only hold here if it holds in (6), that is to say if $\alpha = \beta$. As the notation was arbitrary, this means that *any* two angles of the triangle must be equal, that is to say $\alpha = \beta = \gamma = 60°$. In this case, we have in (7) obviously the sign of equality.

So far we have described one of the 1897 problems. The point is, of course, did the boys b_1 and b_2 fulfill the expectations which the promoters of the Eötvös-prize hoped the winners ought to fulfill? The year 1897 is in a certain way representative in this respect. I mentioned already that the boy b_2, with the pioneer spirit, developed into one of the prides of Hungarian mathematics. As to the boy b_1, I really do not know what became of him—certainly he did not ever do any creative work in mathematics. Viewed as a whole, this situation is characteristic for the almost forty years of the history of the Eötvös-prize; several of the winners are distinguished mathematicians today, and many of them disappeared from the scene completely.

As a matter of fact, no matter what kind of human ability you try to test, it is certain that a test, that is to say an artificial emergency, never can conclusively show what would happen in a real emergency. Making this necessary allowance, it may be fairly said that the Eötvös-prize is a real success. In the long series of the winners since 1894 we certainly see names which never occured again in connection with mathematical research. On the other hand, many of the winners developed into distinguished mathematicians. The important point, however, is that the Eötvös-prize *is* considered as a success in Hungary. I mean that the bright boys are real anxious to win it, and the winner is then helped in all ways to develop his abilities. To show the present-day attitude, the following little anecdote may be helpful. Every year, when the grown-up mathematicians discuss the problems proposed to and the solutions devised by the young competitors for the Eötvös-prize, one of the prides of Hungarian mathematics tells to anybody who might listen the story of why he did *not* get the prize when it was his year. He points out that in October of that year he was in Zürich, studying engineering at the Polytechnical School there, and he also points out that when his friends wrote him about the problems *he solved them in just one half of an hour.*

The great thing about the Eötvös-prize is exactly that it is considered highly desirable by the young folks leaving the secondary schools, and that the grown-up mathematicians feel that a fair percentage of the winners is made up of potential creative mathematicians, and so the winners are taken care of accordingly. This is certainly a very gratifying situation, and the credit for it

goes to people like Professor Kürschák who helped selecting the problems and who kept on watching and stimulating the winners—the crucial points of the whole scheme.

As a newcomer to this country, I could hardly venture suggestions for something analogous here in America. One thing I feel quite sure about is that the story of the Eötvös-prize, as it is presented in the book of Professor Kürschák, would make a highly interesting reading for those in this country who are anxious to maintain and to increase the enthusiasm for our beautiful science.

SOME MATHEMATICAL ASPECTS OF THE NEW PHYSICS[1]

By J. H. VAN VLECK, University of Wisconsin

The most fundamental and far-reaching formulation of the new quantum mechanics is probably that in terms of matrices, along the lines of the Dirac transformation theory. The term "wave mechanics" is a bit misleading, as the wave idea is only a part of the whole story. Nevertheless, for simplicity, attention is confined in the present paper to phenomena capable of description by means of the Schrödinger wave equation. For a system with one degree of freedom, this equation, which is to be regarded as a fundamental postulate, is

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}(W - V(x))\psi = 0. \tag{1}$$

Here h is Planck's constant 6.55×10^{-27} ergs $\times$ sec., m is the mass of the particle, $V(x)$ is its potential energy, a function of the position x, and W is its total (i.e. kinetic+potential) energy, a constant.

One immediately wonders how quantized phenomena can be extracted from (1), as a differential equation suggests things continuous rather than discrete. The answer is that one must confine one's attention to what may be called "civilized solutions" of Eq. (1), and this requirement restricts the constant W to certain particular values. I purposely use the term "civilized" instead of some more exact word such as "analytic" because the conditions which it is necessary to impose on ψ are rather less severe than that ψ be analytic, and furthermore vary somewhat with the physical conditions of the problem. It is usually sufficient that ψ be single-valued, continuous together with its first derivatives, and finite except perhaps at a few points at which ψ becomes infinite in such a way that $\int |\psi|^2 dx$ exists. Now although Eq. (1) possesses two arbitrary constants of integration and hence an infinity of solutions, it often turns out that with an arbitrary value of W not one of these solutions is civilized. Instead it often happens that a civilized solution is obtained only if W is given certain particu-

[1] A paper given before the Mathematical Association of America at its meeting at the University of Minnesota, September 8, 1931.

lar, discrete values, usually called the "characteristic values." (Eigenwerte). The corresponding civilized solutions ψ are called the "characteristic functions" (Eigenfunktionen). The characteristic values of W, when substituted in the relation

$$h\nu = W' - W'' \tag{2}$$

enable one to calculate the spectral frequencies ν of an atom or molecule. Eq. (2) is called the Bohr frequency condition, and is merely an expression of the conservation of energy. When an atom passes from a state of energy W' to another W'', it must dispose of an amount of mechanical energy $W' - W''$ which is converted into radiant energy; according to the light-quantum hypothesis the energy of a radiation of frequency ν is $h\nu$, yielding immediately Eq. (2). One can immediately understand from (2) why the spectral lines of an atom have definite frequencies, since by (2) the spectral frequencies have only certain particular values whenever this is true of W. However, usually there are certain regions in which the characteristic values of W form a continuous rather than discrete manifold, and this is nicely reflected physically in thė fact that atoms possess continuous as well as discrete spectra.

Great success has attended the calculation of the characteristic colors of the various chemical elements, especially the lighter or simpler ones, by means of (2); but instead of pursuing this further I shall outline briefly some of the different mathematical methods which can be used to find the characteristic values and characteristic functions of Eq. (1). These methods may be roughly divided into four classes:

(a) Infantile methods.

(b) Polynomial methods.

(c) Numerical integration.

(d) Power series in a parameter (perturbation theory).

By (a) I mean cases where the appropriate solutions of the differential equation are obvious by inspection. This situation presents no particular mathematical interest, but in order to have a simple concrete illustration of characteristic

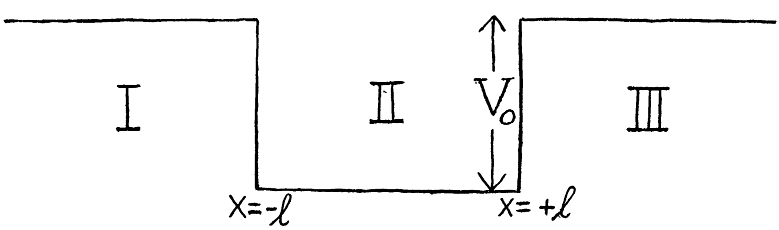

FIG. 1.

functions and values, it is perhaps well to work out an easy problem by this method. Let us therefore consider the problem of a particle in a box extending from $x = -l$ to $x = +l$ (Fig. 1). If we treat the walls as discontinuities in potential energy of amount V_0, then we may take

$$\text{I}\quad V = V_0 \quad (x < -l), \quad \text{II}\quad V = 0 \quad (-l < x \leqslant +l), \quad \text{III}\quad V = V_0 \quad (x > l). \tag{3}$$

Here region II represents the interior of the box, and I, III the space outside it. It is convenient to consider separately the differential equation (1) for the regions I, II, III, as there is then great simplification arising from the fact that V has one of the constant values (3) rather than varying with x as in most problems.

It is necessary to consider separately the cases $W < V_0$, and $W > V_0$, which correspond respectively to the particle being unable and able to climb out of the box in classical mechanics. If we introduce (3) and assume $W < V_0$, the solutions of (1) for the regions I–II–III are

$$\text{I}\quad \psi = Ae^{-\alpha x} + Be^{\alpha x}, \qquad \text{II}\quad \psi = C\cos(\beta x - \epsilon), \qquad \text{III}\quad \psi = De^{-\alpha x} + Ee^{\alpha x}$$

where $\alpha^2 = 8\pi^2 m(V_0 - W)/h^2$, $\beta^2 = 8\pi^2 mW/h^2$. To avoid singularities at $x = -\infty$ and $x = +\infty$ we can immediately take $A = E = 0$. In order for ψ to be a civilized solution, it is necessary that both ψ and $d\psi/dx$, and so also $d \log \psi/dx$, be continuous at the boundaries I–II and II–III. From the requirement that this logarithmic derivative be continuous at the boundary I–II one finds

$$\beta \tan(\beta l + \epsilon) = \alpha. \tag{4}$$

The analogous equation for the boundary II–III differs only in that $+\epsilon$ is replaced by $-\epsilon$. This analogous equation is consistent with (4) only if $\epsilon = 0$ or $\epsilon = \frac{1}{2}\pi$, (mod. π). When ϵ is specialized to 0 or $\frac{1}{2}\pi$, and explicit values are substituted for α and β, Eq. (4) becomes a transcendental equation for W. It has a discrete succession of roots W, which are the desired characteristic values. (We have made only $d \log \psi/dx$ continuous, but ψ as well is easily made continuous by proper choice of the amplitudes B, C, D.)

When $W > V_0$, the solutions in regions I and III become trigonometric instead of exponential, and it is no longer necessary to sacrifice some of the arbitrary constants to avoid singularities at $x = \pm\infty$. In consequence one readily finds that ψ, $d\psi/dx$ can be made continuous at the boundaries for all values of W in the interval $W > V_0$. Thus this case furnishes an example of a continuous distribution of characteristic values, to be contrasted with the discrete one in the interval $W < V_0$. If we adopt the physicists' "energy-level diagram," whereby the allowed values of W are represented by the heights of vertical lines, the situation is as represented in (a), Fig. 2, except that in our box problem the valley of potential energy is, of course, rectangular instead of round.

In region II, V equals zero, so that the particle is there entirely free. By comparison of the solution for the region II with the standard expression $\cos(2\pi x/\lambda - \epsilon)$ for a periodic disturbance in terms of wave-length, one sees that a free particle may be regarded as having a wave-length $\lambda = 2\pi/\beta = (h^2/2mW)^{\frac{1}{2}}$. As $W = \frac{1}{2}mv^2$, this may be written $\lambda = h/mv$, which is the well-known formula of the Davisson-Germer effect (see page 95).

(b) As illustrative of the polynomial method we may consider the harmonic

oscillator, i.e. $V=\frac{1}{2}ax^2$. On introducing the following abbreviations and changes of variable

$$A = \frac{4\pi W}{h}\left(\frac{m}{a}\right)^{1/2}, \quad y = \frac{m^{1/4}a^{1/4}(2\pi)^{1/2}}{h^{1/2}}x, \quad \psi = e^{-y^2/2}f,$$

the wave equation (1) takes the form

$$\frac{d^2f}{dy^2} - 2y\frac{df}{dy} + (A - 1)f = 0.$$

If we assume a series solution $f=y^k(c_0+c_2y^2+c_4y^4 \cdots)$, then from the usual indicial equation it is found that either $k=0$ or $k=1$. The recursion formula for determining the coefficients is

$$\frac{c_{j+2}}{c_j} = \frac{2j + 2k + 1 - A}{(j + k + 1)(j + k + 2)} .$$

The series will terminate at say $j+k=n$ if $A=2n+1$. In virtue of the definition of A, this demands that

$$W = (n + \tfrac{1}{2})h\nu_0, \quad \text{where} \quad \nu_0 = \frac{1}{2\pi}\left(\frac{a}{m}\right)^{1/2}. \tag{5}$$

Eq. (5) gives the characteristic values for the harmonic oscillator, as the corresponding solutions are highly civilized, viz. an exponential times a polynomial. These are the only characteristic values, for if the series does not terminate it is found that ψ becomes infinite at $x=\pm\infty$.

To use the polynomial method successfully it is necessary that the recursion formula involve only two coefficients. Unfortunately this condition is fulfilled in only a limited number of problems, apparently in every case precisely those problems which are integrable in terms of circular functions in classical mechanics. In most problems it is necessary to resort to an approximate method of solving (1).

(c) Numerical integration is occasionally useful. It sometimes happens that the differential equation possesses two singular points, such that with arbitrary W one can develop an analytic solution about one of these points, and another such solution about the other point. The necessary and sufficient condition that the solution be civilized is that these two developments join on to each other continuously; i.e., represent one and the same solution. It may be possible to meet this condition only if W be given certain particular values, which are the desired characteristic values.

An indirect but powerful method of performing the numerical integration is to minimize the integral whose Euler equation is (1). This method is known as the Ritz method.

(d) Development of the solution as a power series in a parameter is a device

very commonly used. To employ this method it is essential that one term in the potential energy be small, so that the latter can be written:

$$V(x) = V'(x) + \lambda V''(x),$$

where $|V''| \ll |V'|$, and where λ is some constant. It is further necessary that the "unperturbed problem," which is obtained by setting $\lambda = 0$, be capable of exact solution. When these conditions are met, one may develop ψ and W as power series in λ, so that $\psi = \psi_0 + \lambda\psi_1 + \cdots$, $W = W_0 + \lambda W_1 + \cdots$. These developments are substituted in (1) and successive powers of λ equated to zero. For instance, from the coefficients of λ^0 and λ^1 one has respectively

$$\frac{d^2\psi_0}{dx^2} + \frac{8\pi^2 m}{h^2}(W_0 - V'(x))\psi_0 = 0, \quad \frac{d^2\psi_1}{dx^2} + \frac{8\pi^2 m}{h^2}[(W_0 - V')\psi_1 + (W_1 - V'')\psi_0] = 0.$$

The first of these equations may be used to determine ψ_0, W_0, then the second to determine ψ_1, W_1, and so on.

An illustration of perturbation theory is furnished by consideration of the Schrödinger equation for a simple pendulum confined to a plane. Here the moment of inertia I and angle ϕ replace the mass m and linear displacement x in (1), while the potential energy V is $-mgl \cos \phi$. Thus this equation is of the form

$$\frac{d^2\psi}{d\phi^2} + (a + b \cos \phi)\psi = 0. \tag{6}$$

As ψ is a cyclic coordinate, the requirement of single-valuedness demands that $\psi(\phi + 2\pi) = \psi(\phi)$. The determination of the characteristic values and functions of (6) subject to this boundary condition constitutes a well known mathematical problem, the so-called "Mathieu" problem, unfortunately only capable of approximate solution. If $|b| \ll |a|$ one may use a series in $\lambda = b/a$. This amounts to figuring perturbations starting with the free rotator, as there is no potential energy when $b = 0$. If $|b| \gg |a|$, one may expand $\cos \phi$ as a Taylor's series, and if one keeps only the first two terms one has merely the wave equation of the harmonic oscillator, which may now be used as the "unperturbed system" or point of departure for calculating the effect of the higher order terms.

The Mathieu problem described above is perhaps rather hackneyed, but an interesting variant is encountered in connection with the transmission of electrons through a crystal. For our purposes it will suffice to regard the crystal as a fixed lattice of heavy positive ions through which the light, mobile electrons migrate. The potential energy of an electron in the field due to the positive ions is clearly a periodic function of x with period d, where d is the "grating spacing," i.e., the distance between two neighboring positive ions. With V an arbitrary Fourier series in $2\pi x/d$, Eq. (1) is of the form known as Hill's equation, and is too general to be useful. If for simplicity the periodic potential function is taken to be a cosine wave (not a particularly good physical approximation), then $V = A \cos (2\pi x/d)$, and the wave equation (1) now again takes the Mathieu

form (6), if we let ϕ denote the ratio $2\pi x/d$. However, ϕ as thus defined is no longer a cyclic coordinate, since it gives the position along what may be dubbed an "infinite roller-coaster" (cf. (b), Fig. 2) instead of the position in the arc of a pendulum as in the preceding paragraph. Hence one no longer has the boundary condition $\psi(\phi+2\pi)=\psi(\phi)$, and this fact makes the determination of the characteristic values, which has been done by P. Morse and by Brillouin, different than for the ordinary Mathieu problem described agove. The general solution of (6) or of any Hill equation can be shown by group theory or its equivalent to be of the form $e^{a\phi}f(\phi)+e^{-a\phi}f(-\phi)$ where $f(\phi+2\pi)=f(\phi)$; one need no longer demand that $a=2\pi ni$ but instead only that a be a pure imaginary. With this more lenient boundary condition, the characteristic values turn out to fall in

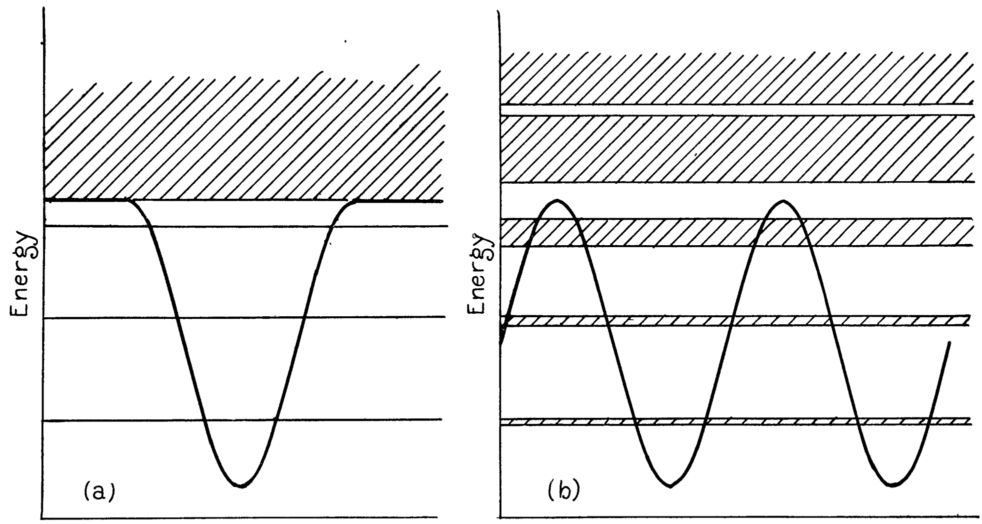

FIG. 2.

continuous manifolds, separated by gaps, as shown in (b), Fig. 2, kindly reproduced from Morse's paper. The situation is to be contrasted with that for a single atom ((a), Fig. 2), where there is only one continuous region (cf. our analysis for the "box" problem) and is confirmed experimentally in that continuous x-ray spectra from crystals are found to contain gaps not observed in similar spectra from monatomic gases.

The results described in the preceding paragraph have an important application to the Davisson-Germer effect, in which crystals are found to reflect electrons of certain particular velocities. Usually this effect is explained by a rather loose optical analogy which appeals to the fact that an electron may be regarded as having a wave length h/mv (cf. p. 92) and that in optics a crystal transmits light of certain wave-lengths and reflects others. Fig. 2, however, shows that a more fundamental explanation can be given in terms of the theory of characteristic values. An electron is capable of transmission through the

crystal ("can ride the roller-coaster") only if its energy falls inside one of the characteristic intervals indicated by dark bands in (b), Fig. 2. If the electron has any other energy or velocity it cannot be transmitted and so has no alternative but to be reflected.

Note that in (b) Fig. 2 some of the bands representing transmission through the crystal fall below the top of the potential wave. This is a manifestation of the fact that in quantum mechanics particles are able to pass through regions of negative kinetic energy and so surmount hills which they could never climb in classical mechanics (a sort of rough analog to the penetration of the geometric shadow in the wave theory of light,—not found in geometrical optics). Conduction in a solid can thus be regarded as electrons playing leap-frog from one valley of potential energy to another. The leaping of classically unsurmountable hills furnishes the long-sought mechanism of radioactive decay. If an alpha particle embedded in a nucleus be assumed to have a crater-shaped potential energy curve, then in quantum mechanics there is always a finite probability that a particle inside the crater may pass over (or perhaps better, through) the rim of the crater and coast out to freedom. In the hands of Gurney, Condon, and Gamow, this model has been extraordinarily fruitful in explaining the empirical relations between the decay periods of the radioactive elements and the velocities of their escaping alpha particles.

References: for an introductory survey of the mathematical problems connected with the Schrödinger wave equation, Condon and Morse's "Quantum Mechanics" (McGraw-Hill) is recommended; for the more advanced matrix transformation theory, Dirac's "Principles of Quantum Mechanics" (Oxford University Press).

ON THE EXPANSION OF AN INTEGRAL OF STIELTJES

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1. In his celebrated memoir on continued fractions,[1] Stieltjes considered the function

$$F(z, \lambda) = \int_0^\infty \frac{(1-\lambda)e^{-zu}du}{e^{(1-\lambda)u} - \lambda},$$

where λ is real and positive, and z is a complex variable. He showed that if $\lambda < 1$:

$$F(z, \lambda) = \sum_1^\infty \frac{(1-\lambda)\lambda^{n-1}}{z + n(1-\lambda)}, \tag{1}$$

[1] Stieltjes: *Recherches sur les fractions continues,* Annales de Toulouse, vol. 8, J, pp. 1–122, and vol. 9, A, pp. 1–47, 1894–95. Also published in vol. 32 of the Mémoires présentés à l'Académie des Sciences de l'Institut National de France, and in Stieltjes's collected works.

and when $\lambda > 1$:

$$F(z, \lambda) = \sum_1^\infty \frac{(\lambda - 1)\lambda^{-n}}{z + (n-1)(\lambda - 1)}, \tag{2}$$

while if $\lambda = 1$:

$$F(z, \lambda) = \int_0^\infty e^{-u} du/(z + u). \tag{3}$$

Hence if

$$\phi(\omega, \mu) = 1 + \frac{\omega}{1+\mu} + \frac{\omega^2}{1+2\mu} + \cdots, \tag{4}$$

we have, in the first case:

$$\phi(\omega, \mu) = 1 + \omega\mu^{-1} F(z, \lambda), \lambda < 1, \omega = \lambda, \mu = (1 - \lambda)/z, \tag{5}$$

and in the second case:

$$\phi(\omega, \mu) = \omega^{-1}\mu^{-1} F(z, \lambda), \lambda > 1, \omega = \lambda^{-1}, \mu = (\lambda - 1)/z. \tag{6}$$

The series (4) was used by Poincaré in *Les Méthodes nouvelles de la mécanique céleste*, tome II, page 3.

In the following article we will discuss certain arithmetic properties of the coefficients in the asymptotic expansions, $P(1/z)$, of $F(z, \lambda)$ and other related functions.

2. *The asymptotic series for* $F(z, \lambda)$. If we set

$$X = (1 - \lambda)/(e^{(1-\lambda)u} - \lambda),$$

and differentiate X, k times in succession, with respect to u we will obtain:

$$d^k X/du^k = (-1)^k X^{k+1} c_k(\lambda, t), \; t = e^{(1-\lambda)u}, \tag{7}$$

where

$$c_k(\lambda, t) = a_{k,0}\, t^k + a_{k,1} t^{k-1}\lambda + \cdots + a_{k,k-1}\, t\lambda^{k-1}.$$

Hence if

$$c_k \equiv c_k(\lambda) = a_{k,0} + a_{k,1}\lambda + \cdots + a_{k,k-1}\lambda^{k-1},$$

we have:

$$X = c_0 - c_1 u + \frac{1}{2!} c_2 u^2 - \cdots,$$

and

$$F(z, \lambda) \sim \frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \cdots.$$

On differentiating both members of (7) with respect to u, equating the value of $d^{k+1}X/du^{k+1}$ so found to that obtained by replacing k by $k+1$ in (7), and then setting $u=0$, one may equate coefficients of like powers of λ and obtain the fundamental recursion formula[1] for the $a_{k,i}$:

$$(8)\qquad a_{k+1,i} = (i+1)a_{k,i} + (k-i+1)a_{k,i-1},\ a_{k,0} = 1,\ a_{i,j} = 0 \quad \text{if } j > i-1 \quad \text{or} \quad \text{if } j < 0.$$

Using (8), the following relation between the polynomials $c_k(\lambda)$ readily results:

$$(9)\qquad c_k = (1-\lambda)\lambda\frac{dc_{k-1}}{d\lambda} + (1+[k-1]\lambda)c_{k-1},\ c_0 = 1.$$

The relation (8) or (9) may be used to compute the c_k. The latter may be used to study the location and character of the roots of these polynomials. Since what follows does not depend on these considerations we will merely state the results. *The roots of c_k are all real, negative, simple, and are separated by those of c_{k-1}.*[2]

Explicit formulas for the $a_{k,i}$ may be obtained as follows. By (1):

$$c_k = \sum_1^\infty i^k(1-\lambda)^{k+1}\lambda^{i-1},\ (\lambda < 1).$$

But

$$(1-\lambda)^{k+1} = 1 - \binom{k+1}{1}\lambda + \binom{k+1}{2}\lambda^2 - \binom{k+1}{3}\lambda^3 + \cdots + \binom{k+1}{k+1}\lambda^{k+1},$$

and consequently:

$$\begin{aligned} c_k &= a_{k,0} + a_{k,1}\lambda + a_{k,2}\lambda^2 + \cdots + a_{k,k-1}\lambda^{k-1}, \\ &= 1^k \\ &\quad + \left[2^k - \binom{k+1}{1}1^k\right]\lambda \\ &\quad + \left[3^k - \binom{k+1}{1}2^k + \binom{k+1}{2}1^k\right]\lambda^2 + \cdots, \end{aligned}$$

and therefore:

$$(10)\quad a_{k,i} = (i+1)^k - \binom{k+1}{1}i^k + \binom{k+1}{2}(i-1)^k - \cdots + (-1)^i\binom{k+1}{i}1^k.$$

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[1] These numbers occur elsewhere in the literature. See, for example, Chr. Zeller: Bulletin des Sciences Mathématiques, ser. 2, vol. 5, p. 195; E. Locchi: Monatshefte für Mathematik, vol. 4 (1893), p. 85.

[2] By equation (14) §4, the c_k are reciprocal polynomials.

The integral (3) is that of Laguerre and, as is well known, has the formal power series development

$$\frac{1}{z} - \frac{1!}{z^2} + \frac{2!}{z^3} - \frac{3!}{z^4} + \cdots . \tag{11}$$

From (11) it follows that

$$c_n(1) = a_{n,0} + a_{n,1} + \cdots + a_{n,n-1} = n!. \tag{12}$$

Combining (10) and (12) we obtain the curious formula

$$k! = \sum_{i=0}^{k-1} (-1)^i \binom{k+1}{i} S_{k-i}^k, \qquad \text{where} \qquad S_{k-i}^k = 1^k + 2^k + \cdots + (k-i)^k .$$

3. Since the symmetric group $G_{n!}$ contains $n!$ elements, equation (12) suggests that we try to separate the elements of $G_{n!}$ into n sets, $K_{n,p}$, $p=0, 1, 2, 3, \cdots, n-1$, containing $a_{n,p}$ elements, respectively, and such that $K_{n,p}$ shall have certain distinguishing properties. To do that we write out the possible linear permutations of the natural numbers $1, 2, \cdots, n$ for the cases $n=2, 3, 4$, and note that in these three cases the following theorem is true.

THEOREM 1: *The number of distinct linear permutations:*

$$i_1, i_2, i_3, \cdots, i_n,$$

of the first n natural numbers, having the property that just p inequalities:

$$i_k < i_{k-1} \tag{13}$$

hold, is equal to $a_{n,p}$.

Assuming the theorem true for $n=N$, we will prove its validity for $n=N+1$, and since it is obviously true for $n=2$, its truth for all $n>2$ will follow. When (13) holds we will say that there is an adjacent inversion in the permutation. Then, according to our assumption, the numbers of permutations of $1, 2, 3, \cdots, N$ having just p and $p-1$ adjacent inversions are $a_{N,p}$ and $a_{N,p-1}$, respectively. When the number $N+1$ is adjoined to the set, the total number of permutations is multiplied by $N+1$. We may think of $N+1$ new permutations of $N+1$ numbers being made from any permutation P of N numbers by inserting $N+1$ successively between adjacent numbers of P, and preceeding and following the numbers of P. Let Q denote the number of permutations of $N+1$ numbers having just p adjacent inversions. Then clearly nothing is contributed to the number Q by permutations P having less than $p-1$ or more than p adjacent inversions. If P contains just p adjacent inversions, then on inserting $N+1$ between two successive numbers $r>s$, the number of adjacent inversions remains the same. There are p such insertions possible. One more desired permutation results if $N+1$ follows the numbers of P. This gives, in all, $(p+1)a_{N,p}$ as part of the number Q. Finally if P contains exactly $p-1$ adjacent inversions

then it must contain $N-p$ adjacent permanences. Thus if $N+1$ is inserted between successive numbers $r<s$ an adjacent inversion results. One more is obtained if $N+1$ precedes the numbers of P. In all we then have $(N-p+1)a_{N,p-1}$ contributed in this case. There are no others. Hence

$$Q = (p + 1)a_{N,p} + (N - p + 1)a_{N,p-1}.$$

Hence by (8):

$$Q = a_{N+1,p}.$$

We now define the sets $K_{n,p}$ of elements of $G_{n!}$.

DEFINITION: *The set $K_{n,p}$ of elements of $G_{n!}$ contains all substitutions:*

$$t = \begin{pmatrix} 1, & 2, & 3, & \cdots, & n \\ i_1, & i_2, & i_3, & \cdots, & i_n \end{pmatrix},$$

in which the permutation: $i_1, i_2, i_3, \cdots, i_n$ contains just p adjacent inversions.

4. *The sets $K_{n,p}$.* Let

$$T = \begin{pmatrix} 1, & 2, & 3, & \cdots, & n \\ n, & n-1, & n-2, & \cdots, & 1 \end{pmatrix},$$

Then we will prove the following theorem.

THEOREM 2. *The transform of any element in $K_{n,p}$ by T is an element of $K_{n,p}$, that is:*

$$T^{-1}K_{n,p}T = K_{n,p}.$$

Considering $T^{-1}tT$ one observes that k goes into $n-k+1$, then $n-k+1$ into i_{n-k+1}, and then if we suppose i_{n-k+1} replaced by j_{n-k+1}, we have:

$$T^{-1}tT = \begin{pmatrix} 1, & 2, & 3, & \cdots, & n \\ j_n, & j_{n-1}, & j_{n-2}, & \cdots, & j_1 \end{pmatrix}.$$

Now one easily verifies that $i_{n-k} \lessgtr i_{n-k+1}$ implies $j_{n-k+1} \lessgtr j_{n-k}$, and consequently $j_n, j_{n-1}, j_{n-2}, \cdots, j_1$ contains p adjacent inversions if $i_1, i_2, i_3, \cdots, i_n$ does.

Let

$$T^{-1}t_pT = \bar{t}_p$$

be conjugate to t_p in $K_{n,p}$. Then we separate the elements of $K_{n,p}$ into two classes: $\{e_p\}$, $\{g_p\}$. The elements e_p are such that $\bar{e}_p \neq e_p$, while the $g_p = \bar{g}_p$ are self-conjugate elements with respect to T.

Let

$$t_p = \begin{pmatrix} 1, & 2, & 3, & \cdots, & n \\ i_1 & i_2, & i_3, & \cdots, & i_n \end{pmatrix},$$

$$s \equiv \begin{pmatrix} i_1, & i_2, & i_3, & \cdots, & i_n \\ i_n, & i_{n-1}, & i_{n-2}, & \cdots, & i_1 \end{pmatrix}.$$

Then clearly $t_{n-p-1}=t_ps$ belongs to $K_{n,n-p-1}$. It follows that

(14) $$a_{n,p} = a_{n,n-p-1}.$$

Let r_p denote the number of elements g_p in $K_{n,p}$. Then we will show that:

(15) $$r_p = r_{n-p-1}.$$

We have:

$$s = (i_1, i_n)(i_2, i_{n-1})(i_3, i_{n-2}) \cdots$$
$$= t_p^{-1}Tt_p,$$

since the operation $t_p^{-1}Tt_p$ may be obtained by performing the substitution t_p on the cycles of T. We then have:

(16) $$t_ps = Tt_p = t_{n-p-1},$$

and accordingly:

(17) $$T^{-1}(Tt_p)T = \bar{t}_{n-p-1} = t_pT.$$

Let now $g_p^{(i)}$, $i=1, 2, 3, \cdots, r_p$ be the distinct elements g_p in $K_{n,p}$. Then in $K_{n,n-p-1}$ there will be elements $Tg_p^{(i)}$, $i=1, 2, 3, \cdots, r_p$, and these will be elements g_{n-p-1}, $i=1, 2, 3, \cdots, r_p$, since $Tg_p^{(i)}=g_p^{(i)}T$, $i=1, 2, 3, \cdots, r_p$. These are all distinct, for $Tg_p^{(i)}=Tg_p^{(j)}$, $(i\neq j)$, implies that $g_p^{(i)}=g_p^{(j)}$, contrary to hypothesis. Furthermore, $K_{n,n-p-1}$ cannot contain more elements g than $K_{n,p}$, for on reversing the above argument one would find more than r_p elements g in $K_{n,p}$ Hence we have proved (15).

Obviously the totality of elements g in $G_{n!}$, forms a group, G, a subgroup of $G_{n!}$.

5. *Multiplication relations*. As an immediate consequence of (16), (17) we may state the following theorem.

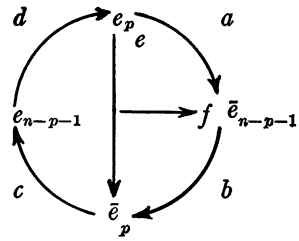
THEOREM 3. (*i*) *If g_p and g_q are any elements g in $K_{n,p}$, $K_{n,q}$, respectively, then*

$$g_{n-p-1}g_q = g_pg_{n-q-1}.$$

(*ii*) *Let e_p, $\bar{e}_p$ be any pair of conjugate elements in $K_{n,p}$. Then $e_{n-p-1}=Te_p$, $\bar{e}_{n-p-1}=T\bar{e}_p=e_pT$ is a pair of conjugate elements in $K_{n,n-p-1}$, and the following equations obtain*:

$(a)\ e_p\bar{e}_{n-p-1} = e_p^2T,$	$(a_{-1})\ \bar{e}_{n-p-1}e_p = Te_{n-p-1}^2,$
$(b)\ \bar{e}_{n-p-1}\bar{e}_p = e_p^2T,$	$(b_{-1})\ \bar{e}_p\bar{e}_{n-p-1} = e_{n-p-1}^2T,$
$(c)\ \bar{e}_pe_{n-p-1} = Te_p^2,$	$(c_{-1})\ e_{n-p-1}\bar{e}_p = e_{n-p-1}^2T,$
$(d)\ e_{n-p-1}e_p = Te_p^2,$	$(d_{-1})\ e_pe_{n-p-1} = Te_{n-p-1}^2,$
$(e)\ e_p\bar{e}_p = \bar{e}_{n-p-1}^2,$	$(e_{-1})\ \bar{e}_pe_p = e_{n-p-1}^2,$
$(f)\ e_{n-p-1}\bar{e}_{n-p-1} = \bar{e}_p^2,$	$(f_{-1})\ \bar{e}_{n-p-1}e_{n-p-1} = e_p^2.$

These may be exhibited by the following diagram:



in which the multiplications are carried out in the directions of the arrows in accordance with $a, b, c, \cdots$, *and in the opposite directions in accordance with* $a_{-1}, b_{-1}, c_{-1}, \cdots$.

When $n=4$ we find that the substitutions of G are as follows: $g_0=1$, (in $K_{4,0}$); $g_1=(13)(24)$, $g_1'=(23)$, $g_1''=(1243)$, (in $K_{4,1}$); $g_2=(12)(34)$, $g_2'=(14)$, $g_2''=(1342)$, (in $K_{4,2}$); $T=g_3=(14)(23)$, (in $K_{4,3}$). The elements e are: $e_1=(123)$, $\bar{e}_1=(243)$, $e_2=(142)$, $\bar{e}_2=(134)$; $e_1'=(132)$, $\bar{e}_1'=(234)$, $e_2'=(143)$, $\bar{e}_2'=(124)$; $e_1''=(12)$, $\bar{e}_1''=(34)$, $e_2''=(1423)$, $\bar{e}_2''=(1324)$; $e_1'''=(1432)$, $\bar{e}_1'''=(1234)$, $e_2'''=(13)$, $\bar{e}_2'''=(24)$.

When $n=5$, we find that G is of order 8, just as when $n=4$. The following are the results: $g_0=1$, (in $K_{4,0}$); $g_2=(24)$, $g_2'=(12)(45)$, $g_2''=(1254)$, $g_2'''=(1452)$, $g_2^{(\mathrm{IV})}=(14)(25)$, $g_2^{(\mathrm{V})}=(15)$, (in $K_{5,2}$); $T=g_4=(15)(24)$, (in $K_{5,4}$). The sets $K_{5,1}$, $K_{5,3}$ contain no elements of G.

6. *The group G.* We will now prove the following theorem concerning the group G defined in §4.

THEOREM 4. *When $n=2k$ or $2k+1$, G contains $2^k k!$ elements. Denote by $b_{n,p}$ the number of elements of G contained in $K_{n,p}$. Then*

$$\begin{aligned}
&(a)\ b_{2k+1,2r+1} = 0,\\
&(b)\ b_{2k+1,2r} = (2k-2r+1)b_{2k-1,2r-2} + (2r+1)b_{2k-1,2r},\\
(18)\qquad &(c)\ b_{2k,2r} = (2k-2r)b_{2k-2,2r-2} + (2r+1)b_{2k-2,2r} + b_{2k-2,2r-1},\\
&(d)\ b_{2k+1,2r} = b_{2k,2r-1} + b_{2k,2r},\\
&(e)\ b_{n,p} = b_{n,n-p-1}.
\end{aligned}$$

Note: (a), (b), (c) may be combined to give the single relation

$$(a\,b\,c):\quad b_{n,p} = (n-p)b_{n-2,p-2} + (p+1)b_{n-2,p} + b_{n-2,p-1},$$

where $p=2r$ if n is odd.

Let $g=(1, i_1)(2, i_2)(3, i_3), \cdots$, be an element of G. Then $T^{-1}gT=g$ implies that $T=(i_1, i_n)(i_2, i_{n-1})(i_3\ i_{n-2})\cdots$, and hence

$$(19)\qquad i_p + i_{n-p+1} = n+1, \qquad (p = 1, 2, 3, \cdots).$$

Now, whether $n=2k$ or $2k+1$, there are just $2^k k!$ permutations $i_1, i_2 \cdots i_n$ satisfying (19). This proves the first part of the theorem.

We now prove the relations (18). Consider a permutation of $2k-1$ numbers:

$$P : i_1,\ i_2,\ i_3,\ \cdots,\ i_{k-1},\ i_k,\ i_{k+1},\ \cdots,\ i_{2k-1},$$

for which (19) holds. Obviously $i_k = k$. Suppose $i_p \gtrless i_{p+1}$, $(p<k)$. Then by (19), $i_{2k-p-1} \lessgtr i_{2k-p}$. It follows that P must contain an even number of adjacent inversions, thus proving (18) (a).

Now suppose the numbers of permutations P having $2r-2$ and $2r$ adjacent inversions are $b_{2k-1,2r-2}$, $b_{2k-1,2r}$, respectively. Then if 0, $2k$ are inserted in P at equal distances from i_k, a permutation of 0, 1, 2, $\cdots$, $2k$ is obtained satisfying (19). If P has $2r-2$ adjacent inversions, it will have $2k-2r$ adjacent permanences. Hence, since if 0 or $2k$ is inserted between $r<s$ an adjacent inversion results, we may obtain $2k-2r$ permutations of $2k+1$ numbers having $2r$ adjacent inversions, from each permutation P. Another such permutation results if $2k$ precedes and 0 follows the numbers of P. Hence there are $(2k-2r+1)b_{2k-1,2r-2}$ obtainable in this manner. If P contains $2r$ adjacent inversions, $(2r+1)b_{2k-1,2r}$ are obtainable. There are no more. This proves (18) (b). The proof of (18) (c) may be carried out in essentially the same way.

To prove (18) (d) we start with the numbers:

$$1,\ 2,\ 3,\ \cdots,\ k,\ k+2,\ \cdots,\ 2k+1.$$

If we arrange them in some order:

$$i_1,\ i_2,\ \cdots,\ i_k,\ i_{k+1},\ \cdots,\ i_{2k},$$

satisfying (19), and such that there are just $2r-1$ adjacent inversions, then $i_k > i_{k+1}$, and if $k+1$ is inserted between i_k and i_{k+1}, the resulting permutation must contain $2r$ adjacent inversions. If P contains $2r$ adjacent inversions, then $i_k < i_{k+1}$ and when $k+1$ is inserted between these two numbers, no adjacent inversions are added. Hence we have (18) (d).

In §4 we showed that $K_{n,p}$ and $K_{n,n-p-1}$ contain equal numbers of elements of G. Equation (18) (e) is a restatement of that fact.

7. *The group G and the functions $F_1(z,\lambda)$, $F_2(z,\lambda)$.* If we set:

$$B_{2k}(\lambda) = b_{2k,0} + b_{2k,1}\lambda + \cdots + b_{2k,2k-1}\lambda^{2k-1}, \tag{20}$$

then with the aid of (18) (c) (e), we may verify that:

$$B_{2k} = \lambda(1-\lambda^2)\frac{dB_{2k-2}}{d\lambda} + (1+\lambda+2\lambda^2[k-1])B_{2k-2}. \tag{21}$$

Then if:

$$B(x,\lambda) = B_0 + B_2 x + B_4 x^2/2! + \cdots,\ (B_0 = 1), \tag{22}$$

we will readily see that if $|\lambda|<1$, then $|B_{2k}|<2^k k!$, and consequently the series (22) converges if $|\lambda|<1$, $|x|<\frac{1}{2}$. We may verify by means of (21) that the function $B(x,\lambda)$ satisfies the partial differenial equation:

$$(1 - 2\lambda^2 x)\frac{\partial \Phi}{\partial x} - \lambda(1 - \lambda^2)\frac{\partial \Phi}{\partial \lambda} = (1 + \lambda)\Phi,$$

whence we obtain:

$$\Phi(x, \lambda) = \frac{\lambda - 1}{\lambda}\Psi[x(\lambda^2 - 1) - \log \lambda],$$

where Ψ is an arbitrary function. Let Φ_0 denote that determination of Φ which reduces to 1 when $x = 0$. Then

$$B(x, \lambda) = \Phi_0 = (1 - \lambda)/(e^{(\lambda^2-1)x} - \lambda),$$

and hence if:

$$(23) \qquad F_1(z, \lambda) = \int_0^\infty \frac{(1 - \lambda)e^{-zu}du}{e^{(1-\lambda^2)u} - \lambda},$$

we have the following theorem.

THEOREM 5. *The function* $F_1(z, \lambda)$ *defined by* (23) *has the asymptotic development*:

$$F_1(z, \lambda) \sim \frac{B_0}{z} - \frac{B_2}{z^2} + \frac{B_4}{z^3} - \cdots,$$

in which the $k+1$*th coefficient*, B_{2k}, *is a polynomial of degree* $2k-1$ *in* λ *in which the* $p+1$*th coefficient*, $b_{2k,p}$, *is the number of those elements of* G, *the group of all substitutions on* $2k$ *numbers which are commutative with* $T = (2k, 1)(2k-1, 2)(2k-2, 3) \cdots$, *which are contained in the set* $K_{2k,p}$.

We may employ Theorem 5 to obtain explicit formulae for the $b_{2k,p}$. In fact if $\lambda > 1$,

$$F_1(z, \lambda) = \sum_{n=1}^{\infty} \frac{(1 - \lambda)\lambda^{n-1}}{z + n(1 - \lambda^2)},$$

and consequently:

$$B_{2k}(\lambda) = (1 - \lambda)\sum_{n=1}^{\infty}\lambda^{n-1}n^k(1 - \lambda^2)^k,$$
$$= 1^k + [2^k - 1^k]\lambda + \left[(3^k - 2^k) - \binom{k}{1}1^k\right]\lambda^2 + \cdots.$$

Hence:

$$(24)\ b_{2k,p} = \sum_{i=0}^{r}[(p + 1 - 2i)^k - (p - 2i)^k](-1)^i\binom{k}{i}, \textit{ if } p = 2r \text{ or } 2r + 1.$$

Now by (18) (d), (24):

$$b_{2k+1,2r} = \sum_{i=0}^{r}(2r+1-2i)^k(-1)^i\binom{k+1}{i}, \tag{25}$$

and if we set:

$$B_{2k+1}(\lambda) = b_{2k+1,0} + b_{2k+1,2}\lambda + \cdots + b_{2k+1,2k}\lambda^k,$$

it may be verified that the following theorem is true:

THEOREM 6. *Set*

$$F_2(z, \lambda) = \int_0^\infty \frac{(1-\lambda)e^{(1-\lambda-z)u}}{e^{2(1-\lambda)u} - \lambda}\, du.$$

Then $F_2(z, \lambda)$ has the asymptotic development:

$$F_2(z, \lambda) \sim \frac{B_1}{z} - \frac{B_3}{z^2} + \frac{B_5}{z^3} - \cdots,$$

in which B_{2k+1} is a polynomial in λ of degree k in which the $p+1$th coefficient is $b_{2k+1,2p}$, the number of elements of G contained in $K_{2k+1,\,2p}$.

In fact,[1] if $\lambda < 1$:

$$F_2(z, \lambda) = \sum_{n=1}^{\infty} \frac{(1-\lambda)\lambda^{n-1}}{z + (2n-1)(1-\lambda)},$$

$$B_{2k+1} = \sum_{n=1}^{\infty}(2n-1)^k\lambda^{n-1}(1-\lambda)^{k+1},$$

and we will find that the coefficient of λ^p in this expression is $b_{2k+1,2p}$, given by (25).

8. *Connection with* tan x, sec x. Let E_n, Q_n, denote the number of permutations of $1, 2, 3, \cdots, n$ having an even, and an odd number of adjacent inversions respectively. Then clearly:

$$E_n = [c_n(1) + c_n(-1)]/2, \qquad Q_n = [c_n(1) - c_n(-1)]/2,$$

and,

$$E_n - Q_n = c_n(-1)\begin{cases} = 0 \text{ if } n \text{ is even}, \\ \neq 0 \text{ if } n \text{ is odd}. \end{cases}$$

Now from the equation:

[1] If $\phi_2(\omega, z)$ is the series obtained from (4) by dropping out the 1st, 3rd, 5th, $\cdots$ terms, then ϕ_2 is expressible in terms of $F_2(z, \lambda)$ in a manner analogous to (5), (6).

$$\frac{(1-\lambda)}{e^{(1-\lambda)u}-\lambda} = c_0 - c_1 u + c_2 u^2/2! - \cdots,$$

we will obtain, after certain reductions:

$$\frac{(1-\lambda)\sinh(1-\lambda)u}{1-2\lambda\cosh(1-\lambda)u+\lambda^2} = c_1 u + c_3 u^3/3! + c_5 u^5/5! + \cdots. \tag{26}$$

Setting $\lambda = -1$ in (26), gives:

$$\sinh 2u/(1+\cosh 2u) = c_1(-1)u + c_3(-1)u^3/3! + c_5(-1)u^5/5! + \cdots,$$

from which we find, using the identities $\cosh x = \cos ix$, $\sinh x = -i \sin ix$,

$$\tan x = c_1(-1)x - c_3(-1)x^3/3! + c_5(-1)x^5/5! - \cdots.$$

We then have the following theorem.

THEOREM 7. *Let E_n, Q_n denote the number of permutations of* 1, 2, 3, $\cdots$, n *having an even, respectively, an odd number of adjacent inversions. Then if we set:*

$$A_{2n-1} = (-1)^{n+1}(E_{2n-1} - Q_{2n-1}),$$

we have:

$$\tan x = A_1 x + A_3 x^3/3! + A_5 x^5/5! + \cdots.$$

Let us set

$$\sec x = A_0 + A_2 x^2/2! + A_4 x^4/4! + \cdots.$$

Then it is known that

$$A_{2n} = \sum_{i=0}^{n-1} \binom{2n-1}{2i+1} (-1)^i A_{2n-2i-1},$$

and, consequently, the A_{2n} are expressible in terms of the $E_k - Q_k$.

Let $i_1, i_2, i_3 \cdots i_n$ be a permutation of 1, 2, 3, $\cdots$, n in which the differences in $i_k - i_{k+1}$, $k = 1, 2, 3, \cdots$, are alternately positive and negative. Such a permutation is called an alternating permutation. André[1] has shown that the number $2A_n$ is the number of alternating permutations of 1, 2, 3, $\cdots$, n. The connection of these with the Bernoulli numbers is well known.

From Theorem 6 one will readily find that

$$\sec x = B_1(-1) - B_5(-1)x^2/2^2.2! + B_9(-1)x^4/2^4.4! - \cdots,$$

and consequently

$$A_{2n} = (-\tfrac{1}{4})^n B_{4n+1}(-1).$$

We may therefore state the following theorem.

[1] Desire André: *Sur les permutations alternées*, Journal de Mathématiques (3), 7, 167–184.

THEOREM 8. *Let the numbers of permutations:* $i_1, i_2, i_3, \cdots, i_{4n+1}$, *of* 1, 2, 3, $\cdots$, $4n+1$, *such that*

$$i_p + i_{4n-p+2} = 4n + 2, \qquad p = 1,2,3, \cdots,$$

(*i.e. corresponding to the subgroup* G *in* $G_{(4n+1)!}$), *and having a number of adjacent inversions divisible by 4, and not divisible by 4, respectively, be* E'_{4n+1} *and* Q'_{4n+1}. *Set*

$$A_{2n} = (-1/4)^n(E'_{4n+1} - Q'_{4n+1}).$$

Then

$$\sec x = A_0 + A_2x^2/2! + A_4x^4/4! + \cdots, \quad (A_0 = 1).$$

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

NOTE ON SYSTEMS OF ORTHOGONAL CONICS

By B. H. BROWN, Dartmouth College

In a paper entitled "*Sur les courbes orthogonales composées de coniques,*" Appell[1] gives the following list of all such systems:

(1) confocal conics
  (a) ellipse-hyperbola
  (b) both families parabolas;
(2) coaxial circles—(4 well-known types, including degenerate cases);
(3) equilateral hyperbolas

$$x^2 - y^2 = \lambda, \quad xy = \mu;$$

(4) ellipses of eccentricity $\frac{1}{2}\sqrt{2}$, and parabolas:

$$2x^2 + y^2 = \lambda, \quad y^2 = \mu x.$$

It is then somewhat disconcerting to find Niewenglowski[2] giving the following orthogonal system of conics

$$(5) \qquad (x + 2y)^2 = \lambda(x + y), \quad y = -3x + \mu x^2,$$

a system obviously not included in Appell's list.

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[1] Appell, Grunert, vol. LXIII, 1879, p. 50.

[2] Niewenglowski, *Cours de Géométrie Analytique*, Paris, 1911, vol. 2, p. 318, ex. 5.

The explanation of this apparent discrepancy is simple. Appell assumes "orthogonal" to mean "orthogonal at every finite point of intersection." With this assumption his enumeration is complete. The families given by Niewenglowski have slopes

$$-\frac{x}{3x+2y} \text{ and } \frac{3x+2y}{x},$$

and hence there is orthogonality except when $x=y=0$. In general the parabolas of the first family are tangent to the line $x+y=0$ at the origin, while the parabolas of the second family are tangent to the line $3x+y=0$ at the origin. Thus at the origin, in general, each curve of one family cuts every curve of the other family at $\tan^{-1}\frac{1}{2}$.

The point to be emphasized is that Appell's assumption is a somewhat arbitrary restriction on a problem of some interest, namely: "In a given region, what orthogonal systems of conics exist?" As far as I know this problem is unsolved; it is, apparently, one of considerable difficulty. I have been unable to find any orthogonal systems other than those listed.

The researches of Von der Mühll[1] show that the restriction to *isometric* orthogonal systems is even heavier than that of Appell. His enumeration of all isometric systems of conics does not include either (4) or (5), and consists, with proper choice of parameters, of all the varieties and only those listed under (1), (2), and (3).

A Non-uniformly Convergent Series

By Margaret Gurney, Brown University

It is of some importance, for pedagogical purposes, to have non-artificial examples illustrating the possible difficulties attendant upon a too sanguine application to infinite series of operations which are known to be legitimate for polynomials. Such a one is the following convergent but non-uniformly convergent series

$$\cot z = g'(z) + \frac{1}{z} + \sum_{\nu=1}^{\infty} \frac{2z}{z^2-\nu^2\pi^2} \tag{1}$$

which arises when we try to construct $\sin z$ from the knowledge that its zeros are the points $z=k\pi$ ($k=0, \pm 1, \pm 2, \cdots$). The series (1) is obtained by taking the logarithmic derivative of

$$\sin z = e^{g(z)}\, z \prod_{\nu=1}^{\infty} (1-z^2/(\nu^2\pi^2)), \tag{2}$$

[1] Von der Mühll, *Über die Abbildung von Ebenen auf Ebenen*, Crelle, vol. 69. The actual problem solved by Von der Mühll is somewhat more general: namely, to find all the isometric systems at least one family of which are conics.

in which $g(z)$ is an entire function which is to be determined. It is readily proved that $g'(z)$ is a constant, and the whole question reduces to the determination of this constant.

In (1), if we let z become infinite through pure imaginary values of the variable (i.e. take $z=iy$, where y is real and becomes positively infinite), we have $\cot z=i(e^{-y}+e^{y})/(e^{-y}-e^{y})$, and $\lim_{z\to\infty}\cot z=\lim_{y\to\infty} i(e^{-y}+e^{y})/(e^{-y}-e^{y}) = -i$. On the assumption that $\sum_{\nu=1}^{\infty} 2z/(z^2-\nu^2\pi^2)$ is uniformly convergent in the infinite region, we would have

$$\lim_{z\to\infty}\sum_{\nu=1}^{\infty} 2z/(z^2-\nu^2\pi^2) = \sum_{\nu=1}^{\infty} \lim_{z\to\infty} 2z/(z^2-\nu^2\pi^2) = 0.$$

This requires that $g'(z)$ be $-i$. But for z real we have $\cot z$ real, and $1/z+\sum_{\nu=1}^{\infty} 2z/(z^2-\nu^2\pi^2)$ real, and we are led to the statement that the difference of two real numbers is an imaginary number. The fallacy comes in the assumption that $\sum_{\nu=1}^{\infty} 2z/(z^2-\nu^2\pi^2)$ is uniformly convergent in the infinite region. That this is not the case can be readily seen by a consideration of the remainder, $R_n=\sum_{\nu=n}^{\infty} 2z/(z^2-\nu^2\pi^2)$. For $z=iy$ we have

$$\frac{i}{2}R_n = \sum_{\nu=n}^{\infty}\frac{y}{y^2+\nu^2\pi^2} \sim \int_n^{\infty}\frac{y\,dt}{y^2+t^2\pi^2} = \left[\frac{1}{\pi}\tan^{-1}\frac{t\pi}{y}\right]_n^{\infty}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\frac{n\pi}{y}\right].$$

By the proper choice of y (>0), we may make this difference, and therefore $iR_n/2$, anything we like between 0 and $\frac{1}{2}$; for example, for $y=n\pi$, we have $(\pi/2-\tan^{-1} 1)/\pi=\frac{1}{4}$. As a result, for a pre-assigned ϵ it is not in general possible to find an M such that, for every $n\geqq M$, we have $|R_n|>\epsilon$, for every z, and the condition for the uniform convergence of $\cot z$ in the infinite region is not fulfilled. The condition is fulfilled, however, for z in the finite region of the plane; and if we allow z to approach 0, we readily obtain $g'(z)=0$, which, in view of (1), leads to the known result $g(z)\equiv 0$.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

College Algebra. By L. J. Rouse. New York, John Wiley and Sons, 1931. x+346 pages. $2.25.

A Treatise on Algebraic Plane Curves. By Julian L. Coolidge. Oxford, The Clarendon Press, 1931. xxiv+514 pages. $10.00.

Philosophy and Modern Science. By Harold T. Davis. Bloomington, Indiana, The Principia Press, 1931. xvi+336 pages. $3.50.

The Constitution of Atomic Nuclei and Radioactivity. By G. Gamow. International Series of Monographs in Physics. Oxford, The Clarendon Press, 1931. 114 pages. $3.50.

Vector and Tensor Analysis. By A. P. Wills. New York, Prentice-Hall, Inc., 1931. xxxii+286 pages. $5.00.

The Foundations of Mathematics. By F. P. Ramsey. New York, Harcourt, Brace, & Co., 1931. xviii+292 pages.

Mathematics. A Textbook for Technical Students. By B. B. Low. New York, Longmans Green, 1931. vi+448 pages.

The Taylor Series. An Introduction to the Theory of Functions of a Complex Variable by P. Dienes. Oxford, The Clarendon Press, 1931. With Bibliography. x+532 pages. $7.50.

Elements of the Theory of Integers. By Joseph Bowden. Garden City, N. Y., Published by the Author, 1931. x+268 pages. $2.50.

The Queen of the Sciences. By E. T. Bell. Century of Progress Series. Baltimore, Williams and Wilkins, 1931. 138 pages. $1.00.

Vectorial Mechanics. By Louis Brand. New York, John Wiley and Sons, 1930. xviii+544 pages. $5.00.

Foundations of the Theory of Algebraic Numbers. By Harris Hancock. Volume I, Introduction to the General Theory. New York, The Macmillan Company, 1931. xx+604 pages. $8.00.

The Theory of Spherical and Ellipsoidal Harmonics. By E. W. Hobson. Cambridge University Press, 1931. xii+500 pages. $11.50.

The Emergence of Life, being a treatise on Mathematical Philosophy and Symbolic Logic by which a new theory of space and time is evolved. By J. B. Burke. Oxford University Press, 1931. 396 pages. $7.50.

Brief College Algebra. By W. L. Hart. New York, D. C. Heath, 1932. viii+334 pages; answers 30 pages. $1.96.

REVIEWS

Introduction to Vector Analysis. By L. R. Shorter. London, Macmillan & Co. 1930. xiv+356 pages.

This is an interesting and useful book and constitutes a unique addition to the literature of this important subject. Its appearance will be welcomed particularly by those who, not having had the advantage of systematic instruction in Vector Analysis, are forced to acquire the desired knowledge through their individual efforts. Of such there are many. Among them will be found those who were in the mind of Professor Kelland when he wrote in his preface to Introduction to Quaternions: "Hamilton and Tait write for mathematicians and they do well, but the time has come when it behooves someone to write for those who desire to become mathematicians."

The plan of Mr. Shorter's book is admirably adapted to the needs of such persons. The proofs of theorems and the solutions of problems are printed in such detail that one reads them as one would a story.

The author has shown even further consideration for the reader, for, whenever reference is made to a formula previously deduced, that formula is repeated in full in the immediate text, thus obviating the necessity of frequent references to preceding pages.

The text includes a large variety of examples of which about a hundred are taken with Dr. Coffin's permission from his Vector Analysis. All of the examples are fully worked out. In this respect we note a departure from the common practice of including a list of unworked problems, but the innovation is well advised in a textbook of this sort.

Sir William Rowan Hamilton at the age of seventeen described his method of studying geometry in these words: "I glance at the title of a proposition and then work it, resolved not to assist myself by text or figure until I conquer the difficulty by my own resources. In general I find this very easy—sometimes not. Still I have observed my rule."

With this process obviously in mind Mr. Shorter has wisely collected the statements of his examples in groups by themselves while their solutions or demonstrations appear in later pages, thus making it easy for a student to try his hand at a solution before consulting that offered by the author. The reader of this book should not miss the opportunity to use the material in this way, for as the author himself suggests, "he will get a more thorough knowledge of the subject and he may also have the satisfaction of hitting on methods which are more simple and concise than those in the text."

The use of scalar and vector products as well as the operator "del" often yields results with such startling ease and swiftness that one may be tempted to question the validity of the process. The author has taken pains to set the mind of the reader at rest on this point by showing that all the important formulae can also be established by the processes of the calculus.

In his development of the subject the author carries the reader along by a succession of easy and natural steps until together they have constructed an instrument of analytical research which is both elegant and powerful.

One may doubt the wisdom of introducing (see page 205) the expression $\nabla \boldsymbol{a}$, the juxtaposition of ∇ and a vector, unless the reader is to become better acquainted with dyads and dyadics later in the book. But this notation is soon discarded, though perhaps not before the reader has experienced a disturbing confusion.

The scope of the work may be judged when one knows that the subjects treated lie chiefly in the field of geometry, kinematics and dynamics, and that the last chapter includes a deduction of Euler's dynamical equations, the divergence theorem and the theorem of Stokes.

As might be expected in the first edition of a book of this character there are typographical errors. The reviewer has preserved a list of those noted in the

hope that it may be useful to the author in the preparation of a second edition a demand for which is predicted.

T. C. ESTY

Elementargeometrie der Ebene und des Raumes. (Göschen's Lehrbücherei, Band 16.) By Professor Max Zacharias. Berlin, DeGruyter, 1930. 252 pages. Rm. 13.50.

The author follows the example of Euclid, the earliest writer on elementary geometry, in that he has prepared a book which is not intended as an instruction book on the geometry of the plane and of space for young students but is a scientific development of this theory based on Hilbert's axioms (with occasional slight modifications) and consequently is of interest to the mature student who has some acquaintance with the philosophical notions which have been presented by Hilbert in *Grundlagen der Geometrie*, by Klein in *Elementarmathematik vom höheren Standpunkte aus*, by Enriques in *Questioni riguardanti le matematiche elementari*, and others.

The first section treats of the notions of straight line and of plane, the axioms of order, union, congruence, continuity, and parallelism and their consequences. The second section gives the theory of similarity and its applications which beginning with Euclid's theorems on similar triangles include the circle, the continuous division of a line segment, the theorems of Menelaus, Ceva, Pascal, perspectivities, pole and polar relation, and inversion with respect to a circle. In the third section the topic is plane area, a topic not touched upon by Euclid in his thirteen books. The fourth section, Körperlehre, includes the many topics (exceptions noted below) which in the middle ages and in modern times have been added to the original nucleus, viz., stereometry, Euler's polyhedron theorem, regular polygons, volumes, motion, symmetry, geometry on a sphere, and stereographic projection.

At every stage Prof. Zacharias has given valuable and interesting historical notes, in fact has depicted the historical development of the theory with a completeness which makes the work a contribution of considerable importance.

The modern geometry of the triangle and of the tetrahedron and the theory of conic sections are not discussed because of limitation of space.

M. I. LOGSDON

The Elementary Theory of Tensors with Applications to Geometry. By T. Y. Thomas. New York, McGraw-Hill Book Co., 1931. ix+122 pages. $2.50.

This attractive introduction into the theory of tensors contains selections from a course on mechanics given during recent years to an undergraduate class at Princeton University. For this reason the stress is laid not so much on the algebraic and analytical side of the theory of tensors as on the application of the tensor theory to simple problems in kinematics and dynamics. The author shows how vectors and tensors of the second order enter into those questions and how

elegantly they can be solved in tensor notation. The book is therefore particularly adapted to the needs of physics or engineering students who first like to know what you can do with tensors in their field before they start out on a general study.

In many respects the book looks like the common textbook on vectors and dyads translated into the new language of tensor calculus. This has the particular advantage that the student knows from the beginning that vector calculus of his physics and mechanics is only a special case of the tensor calculus of relativity, elasticity, and advanced geometry. It follows the main trend of the times, which seems to be a gradual usurpation of different fields by the symbolism of tensor calculus, as the algebra of invariants (Weitzenböck), the differential geometry of curves and surfaces in ordinary space (Duschek), and elasticity (Nadai).

As particular features of the book we may mention the proof that an arbitrary point transformation which leaves a scalar product of a vector with itself invariant is an orthogonal transformation (p. 29), the subsequent proof that in such case measurement of distance is possible (p. 34), the discussion of simple principles connected with the theory of relativity (p. 42, p. 89), the extensive discussion of the motion of a rigid body, with the screw motion and the theorem of Coriolis (Ch. IV). Chapter V, on Newtonian dynamics, gives Mach's definition of mass, and leads to Lagrange's equations and a number of illustrative examples.

The author has to meet certain difficulties by using tensor notations instead of old vector notations. For the scalar product it is easy enough, it is v_iw_i with the Einstein summation convention. But the vector product offers difficulties. The author meets them by simply writing a new symbol: $C^1=A^2B^3-A^3B^2$ (p. 48); this, however, takes away a good deal of the usefulness of the old vector analysis, which is built upon rules connecting the scalar and vector products. Another way is that used by Duschek, who writes $e_{ijk}a_ic_j=c_k$, understanding by e_{ijk} the skew symmetric tensor of value ± 1, 0 (Differentialgeometrie I, p. 31). Thomas introduces this symbol (p. 50), but does scarcely anything with it.

The result is that at the place where the vector product has to be used, namely, in the motion of a rigid body, we get an alternating tensor instead (p. 75); which perhaps is not incorrect, but seems rather inconsistent.

This inconsistency is probably due to the fact that the classification of tensors according to their groups and the relation of vector analysis to tensor analysis is not clearly indicated. On p. 40 the author correctly points out that certain quantities, as the vector product, only have a meaning under the rotational group (Thomas calls it the orthogonal group, but it seems customary to use the word orthogonal for the group of rotations and reflexions; see, e.g., Pascal's Repertorium I, 1, p. 131). But he does not mention clearly that under this group there is no longer any difference between covariant and contravariant quantities. As a result he works with the general affine tensors in the problem of the rotation of a body, which is a problem belonging to the rotational group,

and as a result gets the vector of rotation in the form of a mixed tensor of second order obeying the curious condition (p. 75).

$$\overset{i}{w_k} + \overset{k}{w_i} = 0,$$

which indeed has an invariant meaning under the rotational group, but only here. But there it should better be written with lower indices, while w_{ik} should be from the beginning identified with a vector. This is done only occasionally at the end (p. 76). We therefore think that the relation of ordinary vector analysis to tensor analysis might have been more clearly indicated.

This criticism, however, does not affect the general usefulness of this little book, which is probably the best text we have at present for beginners, especially for students in the applied sciences.

D. J. STRUIK

A Critical Study of the Teaching of Elementary College Mathematics. By Joseph Seidlin. New York, Teachers College Bureau of Publications, 1931. x+108 pages. $1.75.

This book is announced as "a pioneer investigation of actual classroom procedures employed by teachers of college mathematics in twenty Eastern colleges and universities. An attempt is made to summarize, criticize, and evaluate present-day methods of teaching, which, explicitly or by implication, suggest the need for, and possible ways of improving the teaching of elementary college mathematics."

In the opinion of the reviewer, Dr. Seidlin has succeeded admirably in carrying out this professed purpose, and has produced a book which will be valuable to all college teachers.

The first part of the book is a report on the author's visits to 150 college classes, where he was accustomed to record stenographically the actual proceedings. He soon found that all classes could be classified into seven categories: the pure lecture, the lecture with questions and interruptions, the recitation at which time is divided between instructor and students with spontaneous questions by the latter, the combination of blackboard and oral recitation, the question and answer development, the ground-covering, textbook-repeating mechanical recitation, and the simple blackboard recitation. He reports in extenso one or more actual classes of each type, with pertinent criticisms. It comes as a shock to one who has been accustomed to think of the two last-named forms of recitation as rather low and comparatively useless types of activity, to learn that of the 150 classes visited, 106, or more than 70%, were of these types. Apparently the greater part of our college teaching, in the more elementary courses, is carried on in the recitation style which one usually associates with the high school. Dr. Seidlin later in the book evaluates the different types of recitation, and arrives at the conclusion that the two methods in question are the least valuable of the seven. He also shows an interesting relation between the

type of recitation which predominates at each institution, and the percentage of students who take major work in mathematics. The latter figure shows a clear inverse relationship to the prevalence of the purely mechanical recitation forms.

The second chapter is devoted to "Violations of Some Principles of the Learning Process," with full quotations from the author's record of classes he has attended. Every teacher of mathematics will read this chapter with much searching of the heart. With this chapter may be associated a later one, in which are quoted several instances of serious errors of instructors, due either to ignorance or to inexcusable carelessness.

There is a chapter based largely on interviews with heads of departments, dealing with the essential qualities of good teaching and the factors on which it depends. Another chapter shows by numerous quotations how frequently we instructors require certain things because either "It is in the textbook; the author says so;" etc., or because "it will be included in the examination."

There is an evaluation of attempts to improve teaching, and a final chapter of Summary and Conclusions. It would be well if these could be quoted verbatim, and it is hoped that many teachers will be able to read the book. It is not written in the technical jargon of pedagogy, but is a serious, well-considered study of many questions that are of great importance in the minds of all good college teachers.

ROGER A. JOHNSON

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3528. *Proposed by Albert A. Bennett, Brown University.*

If a, b, c, be complex numbers such that

$$|a| = |b| = |c| = r \neq 0,$$

then $|(ab+bc+ca)/(a+b+c)| = r$. Generalize.

3529. *Proposed by Vladimir F. Ivanoff, San Francisco, California.*

A ship is sailing with a speed and direction, v_1; the wind blows apparently (judging by the vane on the mast) in the direction of a vector, $\boldsymbol{a}$; on changing the

direction and the speed of the ship from v_1 to v_2, the apparent wind is in the direction of a vector, $\boldsymbol{b}$.

Find the vector velocity of the wind.

3530. *Proposed by Allen A. Shaw, University of Arizona.*

Prove that the product of any three consecutive integers is a multiple of 504, if the middle integer is a perfect cube.

3531. *Proposed by Eugene M. Berry, Lynchburg College.*

We have given a horizontal bar of length l, with a (vertical) leg at each end of the bar. The legs are short and of equal length. The front leg comes to a point at the bottom while the foot of the rear leg is a knife edge which is parallel to the bar. If this is placed with the legs on paper find the curve traced by the knife edge when the front leg traces a given circle.

3532. *Proposed by R. E. Gaines, University of Richmond.*

If the triangle PQR is inscribed in an ellipse so as to cut off three segments of equal area, and if tangents are drawn at P, Q, and R, the triangle thus formed will be inscribed in a second ellipse which will be divided in the same way.

3533. *Proposed by R. E. Gaines, University of Richmond.*

The generating circle of a cycloid in one position is tangent to one arch of the cycloid and intersects the next following arch in P_1 and P_2; another such circle intersects the same arch in P_2 and P_3; another, in P_3 and P_4; and so on indefinitely. If $\alpha_1, \alpha_2, \alpha_3, \cdots$ are the angles which the chords P_1P_2, P_2P_3, $P_3P_4, \cdots$ make with the base of the cycloid, then

$$6\sum_{n=1}^{\infty}\alpha_n = 4\pi - 3^{3/2}.$$

3534. *Proposed by J. V. Uspensky, Stanford University.*

Show that

$$\left[\frac{d^{n-1}}{dt^{n-1}}\left(\frac{t-1}{t^r-1}\right)\right]_{t=1} = [1-r][1-2r]\cdots[1-(n-1)r]r^{-n}.$$

SOLUTIONS

275. [1917, 467; 1931, 342]. *Proposed by V. M. Spunar, Chicago, Ill.*

A square, side $2a$, is represented by the equation $x^n+y^n=a^n$ $(n=\infty)$. Find a like formula for an equilateral triangle.

Solution by Ralph P. Agnew, Princeton University

This problem may be regarded from two quite different points of view which depend on different interpretations of the word "represents."

From one point of view, we consider the curves C_n having the equations

$x^n+y^n=a^n$, $n=1, 2, 3, \cdots$, $a>0$. When n is even, it is easy to determine the intersection of C_n and an arbitrary half-ray emanating from the origin, and to show that as n increases through even integral values this intersection approaches the point where the half-ray meets the square of side $2a$. Hence if we let C_∞ represent the square of side $2a$, we may write $\lim_{n\to\infty} C_{2n}=C_\infty$ and use this fact as a basis for the assertion that $x^{2n}+y^{2n}=a^{2n}$, $n=\infty$, "represents" C_∞. When n increases through odd values, C_n does not approach C_∞, but only that part of C_∞ which lies in the half-plane $x+y\geqq 0$. With this interpretation of the term "represents," it is easy to solve the problem at hand. The transformation

$$\begin{cases} x' = x \\ y' = (x+a)y/(3^{1/2}a), \end{cases}$$

which carries the line $y=a$ into the line $y'=(x'+a)/3^{\frac{1}{2}}$, carries the square C_∞ into the equilateral triangle C_∞' whose vertices are $(-a, 0)$, $(a, 2a/3^{\frac{1}{2}})$, and $(a, -2a/3^{\frac{1}{2}})$. This transformation carries the curves $C_n: x^n+y^n=a^n$ into the curves

$$C_n': x'^n + [3^{1/2}ay'/(x'+a)]^n = a^n$$

and it is easily shown that $\lim_{n\to\infty} C_{2n}' = C_\infty'$; hence the equation

$$x^{2n} + [3^{1/2}ay/(x+a)]^{2n} = a^{2n}, \; n=\infty,$$

may be said to represent an equilateral triangle.

From another point of view, we may regard ∞ as a member of the real number system which satisfies the conditions $z^\infty=0$ when $-1<z<1$ and $(z^2)^\infty=1$ when $z=\pm 1$. Then the equation $x^{2n}+y^{2n}=a^{2n}$, $n=\infty$, when written in the form

$$(x^2/a^2)^\infty + (y^2/a^2)^\infty = 1,$$

represents the square C_∞ with its vertices omitted; for if $|x|>a$, y is undefined; if $|x|<a$, $y=\pm a$; and if $x=\pm a$, then y may be any number satisfying the condition $-a<y<a$. The assertion that $x^n+y^n=a^n$, $n=\infty$ represents C_∞, as well as the assertion that $(x/a)^n+(y/a)^n=1$, $n=\infty$ represents C_∞, can be criticized; for when $z=-a$, $(z/a)^n$, $n=\infty$, becomes meaningless and the left and lower sides of the square do not appear in the graph. A similar analysis shows that the equation

$$[x^2/a^2]^\infty + [3y^2/(x+a)^2]^\infty = 1$$

represents an equilateral triangle with its vertices omitted.

The first point of view furnishes a general method for solving problems of this character, but scarcely justifies use of the term "represents." The second view does justify use of the term, but fails to give a representation of all the points of the square and triangle.

3478. [1931, 112]. *Proposed by Vladimir F. Ivanoff, San Francisco. California.*

A segment M_0M_n of unit length is divided into equal parts by points M_1, $M_2, \cdots, M_{n-1}$. Another segment M_0C of length $\sqrt{(p/n)}$ (p being a given

constant) is drawn perpendicular to M_0M_n, and C is joined with the points of division. M_1M_2' is drawn parallel and equal to CM_2, $M_2'M_3'$ parallel and equal to CM_3, and finally $M_{n-1}'M_n'$ parallel and equal to CM_n. Find the limit of the locus of the vertices of this broken line as $n\to\infty$, and consequently $M_0C\to 0$.

Solution by Ralph P. Agnew, Princeton University

Further interest is added to this interesting problem, and a more complete notation is obtained, by formulating it as follows: Corresponding to each positive integer n, let $D_1^{(n)}, D_2^{(n)}, D_3^{(n)}, \cdots$ represent the points $(1/n, 0)$, $(2/n, 0)$, $(3/n, 0), \cdots$ of a plane rectangular coordinate system, and let $C\equiv C^{(n)}$ be the point $[0, -(p/n)^{\frac{1}{2}}]$. Let $f_n(x)$ be the function whose graph is obtained by drawing $C^{(n)}D_1^{(n)}$ and then by drawing successively $D_1^{(n)}P_2^{(n)}$ parallel and equal to $C^{(n)}D_2^{(n)}$, $P_2^{(n)}P_3^{(n)}$ parallel and equal to $C^{(n)}D_3^{(n)}$, $P_3^{(n)}P_4^{(n)}$ parallel and equal to $C^{(n)}D_4^{(n)}$, etc. Thus, after we define $P_0^{(n)}\equiv C^{(n)}$ and $P_1^{(n)}\equiv D_1^{(n)}$, the graph of $f_n(x)$ is a broken line which extends indefinitely upward and to the right, its vertices being $P_0^{(n)}, P_1^{(n)}, P_2^{(n)}, \cdots$. We are required to determine the behavior of $f_n(x)$ and its vertices as $n\to\infty$.

On comparing the above formulation with that of the proposer, we see that the difference lies in the fact that we define $P_r^{(n)}$ for $r=0, 1, 2, \cdots$ while the proposer defined $P_r^{(n)}$ only for $r=1, 2, \cdots, n$. Let $f_n^*(x)$ represent the function, whose graph is the broken line $P_1^{(n)}P_2^{(n)}\cdots P_n^{(n)}$, which the proposer offered for consideration. Then $f_n^*(x)=f_n(x)$ for all x for which $f_n^*(x)$ is defined.

For a fixed n, the coordinates of $P_0^{(n)}, P_1^{(n)}, P_2^{(n)}, \cdots$ are easily evaluated; the abscissa and ordinate of $P_r^{(n)}$ are respectively $r(r+1)/(2n)$ and $(r-1)[p/n]^{\frac{1}{2}}$.
Hence

$$f_n\left(\frac{r(r+1)}{2n}\right) = (r-1)\left[\frac{p}{n}\right]^{1/2}.$$

If the integers r and n are increased in such a way that the ratio $r(r+1)/(2n)$ maintains a constant rational value, say ξ, then the expression $(r-1)[p/n]^{\frac{1}{2}}$ tends to the limit $(2p\xi)^{\frac{1}{2}}$; hence we are led to suspect that $f_n(x)\to(2px)^{\frac{1}{2}}$ as $n\to\infty$.

We will now show that $f_n(x)$ converges uniformly over the entire range $x\geqq 0$ to $(2px)^{\frac{1}{2}}$ as $n\to\infty$. Let ξ be any number $\geqq 0$; then, corresponding to each fixed positive integer n, there is an integer $r=r_n(\xi)\geqq 0$ such that

$$\frac{r(r+1)}{2n} \leqq \xi < \frac{(r+1)(r+2)}{2n}.$$

Since $(2px)^{\frac{1}{2}}$ and $f_n(x)$ are both increasing functions of x, we have

$$\left[2p\,\frac{r(r+1)}{2n}\right]^{1/2} \leqq (2p\xi)^{1/2} < \left[2p\,\frac{(r+1)(r+2)}{2n}\right]^{1/2}$$

and

$$f_n\left(\frac{(r+1)(r+2)}{2n}\right) > f_n(\xi) \geqq f_n\left(\frac{r(r+1)}{2n}\right).$$

After subtracting and simplifying, we obtain

$$(p/n)^{1/2}[(r^2+r)^{1/2}-r] < (2p\xi)^{1/2} - f_n(\xi) < (p/n)^{1/2}[(r^2+3r+2)^{1/2}-(r-1)].$$

The function $[(r^2+r)^{\frac{1}{2}}-r]$ is $\geqq 0$ when $r \geqq 0$; and the function $[(r^2+3r+2)^{\frac{1}{2}} -(r-1)]$ is, for $r\geqq 0$, an increasing function of r which $\to 5/2$ as $r\to\infty$. Hence

$$0 < (2p\xi)^{1/2} - f_n(\xi) < (5/2)(p/n)^{1/2}.$$

Since $(p/n)^{\frac{1}{2}}\to 0$ as $n\to\infty$, and since this approach is independent of the choice of ξ the uniform convergence of $f_n(x)$ to $(2px)^{\frac{1}{2}}$ over the interval $x\geqq 0$ is established.

Since the function $f_n(x)$ converges uniformly to $(2px)^{\frac{1}{2}}$, all of the limit points of vertices of the functions $f_n(x)$ must lie on the semi-parabola S whose equation is $y=(2px)^{\frac{1}{2}}$. In conclusion, we will show that each point of S is a limit point of vertices of functions $f_n^*(x)$ and *a fortiori* a limit point of vertices of the functions $f_n(x)$. It is sufficient to show that each point of S with a rational abscissa is such a limit point. Let an arbitrarily chosen positive rational number be written in the form a/b where a and b are positive integers. Then $P_{r(k)}^{(n(k))}$, where $r(k)=2ka$ and $n(k)=kb(2ka+1)$, is, for each positive integer k, a vertex of $f^*_{n(k)}(x)$ whose abscissa is a/b; hence as $k\to\infty$ and $n(k)\to\infty$, the vertex $P_{r(k)}^{(n(k))}$ approaches the point on S whose abscissa is a/b and the result is established.

A Note by Otto Dunkel. In the first part of the above proof an additional inequality may be written. The segment $P_r^{(n)}P_{r+1}^{(n)}$, which contains the point $[\xi, f_n(\xi)]$, lies within a vertical strip with the base ends $x_1=\xi-\epsilon$, $x_2=\xi+\epsilon$, where $\epsilon=[(2\xi)^{\frac{1}{2}}+1]n^{-\frac{1}{2}}$. This, with the above inequality, shows that, as $n\to\infty$, $r=r_n(\xi)$, each point of $P_r^{(n)}P_{r+1}^{(n)}$ approaches ξ as a limit. This provides a second proof of the last part of the solution.

Also solved by A. Pelletier, R. Rosenbaum, F. Underwood, F. L. Wilmer, and the Proposer.

3479. [1931, 170]. *Proposed by C. A. Rasmussen, University of Alabama.*

If B and C be two fixed points and A a variable point at which BC subtends a given angle and if E and F be the feet of the perpendiculars from B and C, respectively, to the opposite sides of the triangle ABC, determine the path of a point P, the intersection of FP with EP, if $\angle BEP$ and $\angle PFC$, respectively, remain equal to $\angle ECB$ and $\angle FBC$.

Solution by Andrew G. Clark, Colorado Agricultural College.

Let θ be the constant angle at A, and $2a$ and O be the length and midpoint, respectively, of the line BC.

From elementary geometry it is known that $EO = FO = a$. Now, a consideration of the angles of the configuration shows that angles PEO and OFP are right angles, that $\angle EOF$ and $\angle EPF$ are supplementary, and that the latter equals 2θ. It follows that angles PEF and EFP are each equal to $90° - \theta$, and hence $EP = FP$.

It now follows at once that EPO and FPO are congruent right triangles, invariant to the variation of the point A except as to position.

Therefore, the point P must describe a circle about O as a center, and with radius equal to $a \csc\theta$.

Editorial Note. In order to remove the uncertainty of the problem as to the construction of P, we may say that P is determined by drawing the angles CFC' and BEB' so that these angles are equal in sense and magnitude to the angles CBA and BCA, respectively. Then P is the intersection of the sides of the angles FC' and EB'. In this way P is defined for the two cases, $\angle BAC = \theta \leqq 90°$ and $\angle BAC = 180° - \theta$, where in the second case P is the intersection of $B'E$ and $C'F$ produced. With this definition P traces the entire circumference of the circle found above.

Also solved by L. Green, O. J. Ramler, and F. Underwood.

3483. [1931, 170]. *Proposed by Mabel M. Young, Wellesley College.*

A is a fixed point, B a variable point on a circle with center at O. Show that the locus of the orthocenter of triangle AOB is a strophoid.

Solution by Byron D. Roberts, Parsons College

Choose coordinate axes with origin at O. The equation of the circle is then $x^2 + y^2 = r^2$. Let the fixed point A be the point $(r, 0)$, and the variable point B have coordinates (x, y). Let (α, β) be the orthocenter.

The obvious relations are then

$$\alpha = x \text{ and } \frac{\beta}{\alpha} = \frac{r - x}{y} = \cot \angle OAB.$$

Eliminate x and y from these two equations and the equation of the circle and get

$$\beta^2 = \alpha^2 \frac{r - \alpha}{r + \alpha} \cdot$$

the equation of the strophoid.

A Note by the Editors. The strophoid is usually defined by means of two straight lines perpendicular at O. From a fixed point A on one a straight line is drawn cutting the other in K, and the point P is taken in two ways on AK so that $KP = KO$. The locus of the points P is the strophoid.

If from A and P perpendiculars are drawn to PO and AO, respectively, they meet in the orthocenter B of AOP. It is easily seen from a figure that PO bisects the angle APB, and hence APB and AOB are isosceles triangles. Thus B lies on the circle with center O and radius OA, and P is the orthocenter of AOB. The other point on AP is given in the same way by the other extremity B' of the diameter BOB'.

Also solved by Leo Aroian, A. G. Clark, Ralph Deutsch, A. E. Gault, R. A. Johnson, L. S. Johnston, Theodore Lindquist, A. Pelletier, R. C. Staley, Hazel Schoonmaker, F. Underwood, Kamcheung Woo, Roscoe Woods, one unsigned solution, and the Proposer.

The following solutions were received after selections were made in September: W. B. Campbell, 3471; Dorothy M. Wilson, 3470; G. A. Williams, 238; D. C. Duncan, 275, and 432; R. P. Agnew, 434.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

The American Institute of Physics, an agency of cooperation in the interest of physics, has recently been established by the American Physical Society, the Optical Society of America, the Acoustical Society of America, and the Society of Rheology. This Institute will perfect details of organization and arrange for cooperative publication of physics journals, and assist the newspapers to disseminate accurate news of important developments and applications of physics. Henry Barton has been made the director, and John T. Tate the advisor on publications.

The Henry Burchard Fine Hall of Mathematics of Princeton University was dedicated with appropriate ceremonies on October 22, 1931. On this occasion papers were read by Professors G. D. Birkhoff and G. A. Bliss, both formerly members of the department of mathematics of Princeton.

The Frederick Ives medal of the Optical Society of America has been awarded to Dr. Theodore Lyman, emeritus professor of mathematics and natural philosophy at Harvard.

The John Fritz gold medal, conferred by the four American societies of civil, mining and metallurgical, mechanical, and electrical engineers, has been awarded for 1932 to Professor M. I. Pupin, of Columbia University, for his achievements as "scientist, engineer, author, inventor of the tuning of oscillating circuits and the loading of telephone circuits by inductance coils."

Professor Alonzo Church, of Princeton University, lectured before the Mathematical Clubs of Rutgers University and the New Jersey College for Women, December 4, 1931 on the subject "A Finite Geometry."

Colonel C. P. Echols, professor of mathematics at the United States Military Academy at West Point, has retired from the Army.

Dr. Irwin Roman, mathematical physicist of the Geophysical Research Corporation, has been appointed assistant professor of mathematics at the Michigan College of Mining and Technology.

Professor Elijah Swift, head of the department of mathematics in the College of Arts and Sciences of the University of Vermont, has been made dean of the College.

Professor Charlotte A. Scott, formerly professor of mathematics at Bryn Mawr College, died November 8, 1931, at the age of seventy-three.

Professor G. O. James, of Washington University, St. Louis, died November 24, 1931.

Mr. E. A. Kholodovsky, of the department of mathematics of Columbia University, died in October, 1931.

Professor G. W. Myers, of the University of Chicago, died November 23, 1931.

The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The prize will be awarded hereafter every three years. The last award was in December, 1929, to Professor T. H. Hildebrandt. The next award will be in December, 1932, for the period 1929-1931.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of......................................Dollars, to be known as the................................Fund, and to be used

for[2] { Endowment—the income only of which may be expended.
Special Projects—for which both principal and income may be expended.

Witness:...................... Signature......................

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.
[2] Indicate which one of the two purposes is desired, and omit the other.

The Association needs funds for scientific publications and for the promotion of scientific activities.

CONTENTS

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Sixteenth Summer Meeting of the Association, Los Angeles, Calif., Aug. 29-30, 1932.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1932 and reported to the Secretary.

ILLINOIS, Urbana.
INDIANA.
IOWA.
KANSAS.
KENTUCKY.
LOUISIANA-MISSISSIPPI, March 11-12.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May.
MICHIGAN, Ann Arbor, March 19.
MINNESOTA.
MISSOURI.
NEBRASKA.
OHIO, Columbus, Ohio, April 7.
PHILADELPHIA, Philadelphia, Pa., Nov. 26.
ROCKY MOUNTAIN, Laramie, Wyo.
SOUTHEASTERN, Gainesville, Fla., Mar. 18-19.
SOUTHERN CALIFORNIA, San Diego, March 26.
TEXAS, Austin, Jan. 30.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXIX, 1932

NUMBER 3, MARCH

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND ITHACA, N.Y.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The prize will be awarded hereafter every three years. The last award was in December, 1929, to Professor T. H. Hildebrandt. The next award will be in December, 1932, for the period 1929-1931.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

THE SIXTEENTH ANNUAL MEETING OF THE ASSOCIATION

The sixteenth annual meeting of the Mathematical Association of America was held at New Orleans, Louisiana, on Wednesday and Thursday, December 30 and 31, 1931, in affiliation with the American Association for the Advancement of Science and the American Mathematical Society. Two hundred twenty-five were in attendance at the meetings, including the following one hundred thirty members of the Association:

V. W. Adkisson, University of Arkansas
Nola L. Anderson, Sophie Newcomb College
Beulah M. Armstrong, University of Illinois
C. S. Atchison, Washington and Jefferson College

R. W. Babcock, Kansas State Agricultural College
E. A. Bailey, LaGrange College
F. R. Bamforth, Ohio State University
E. T. Bell, California Institute of Technology
B. A. Bernstein, University of California
T. A. Bickerstaff, University of Mississippi
G. D. Birkhoff, Harvard University
Leora Blair, Louisiana State Normal College
G. A. Bliss, University of Chicago
L. M. Blumenthal, Rice Institute
H. E. Bray, Rice Institute
Myrtle C. Brown, State Teachers College, Denton, Texas
H. E. Buchanan, Tulane University

W. D. Cairns, Oberlin College
R. D. Carmichael, University of Illinois
J. W. Cell, Southern Methodist University
L. W. Cohen, University of Kentucky
J. B. Coleman, University of South Carolina
Lennie P. Copeland, Wellesley College
H. M. Cox, Georgia School of Technology
A. T. Craig, University of Iowa
D. R. Curtiss, Northwestern University

R. D. Daugherty, Kansas State Agricultural College
H. T Davis, Indiana University
B. F. Dostal, University of Florida
Otto Dunkel, Washington University
W. L. Duren, Jr., Tulane University
E. P. R. Duval, University of Oklahoma
Nat Edmonson, Jr., Texas Technological College

L. P. Eisenhart, Princeton University
G. C. Evans, Rice Institute
G. W. Evans, Lynn, Mass.

B. F. Finkel, Drury College
C. H. Fischer, University of Iowa
T. M. Focke, Case School of Applied Science
L. R. Ford, Rice Institute
Tomlinson Fort, Lehigh University
Bertha E. Frankenbush, Miller's High School, New Orleans, La.

F. C. Gentry, Tulane University
F. J. Gerst, Loyola University, Chicago, Ill.
J. S. Gold, Bucknell University
Michael Goldberg, Bureau of Ordnance, Navy Dept., Washington, D. C.

W. L. Hart, University of Minnesota
J. O. Hassler, University of Oklahoma
Sarah E. Haughton, University of Albama
E. R. Hedrick, University of California at Los Angeles
Deborah M. Hickey, Delta State Teachers College
Einar Hille, Princeton University
H. M. Hosford, University of Arkansas
Jewell C. Hughes, Hunter College
C. A. Hutchinson, University of Colorado

Louis Ingold, University of Missouri
M. H. Ingraham, University of Wisconsin

Dunham Jackson, University of Minnesota
Rosa L. Jackson, Alabama College
B. W. Jones, Cornell University

F. W. Kokomoor, University of Florida
H. W. Kuhn, Ohio State University

C. G. Latimer, University of Kentucky
F. A. Lewis, University of Alabama
R. G. Lubben, University of Texas

Dorothy McCoy, Belhaven College
E. J. McShane, University of Chicago
A. C. Maddox, Louisiana State Normal College
Israel Maizlish, Centenary College
T. E. Mason, Purdue University
E. D. Meacham, University of Oklahoma
R. L. Menuet, Tulane University
W. L. Miser, Vanderbilt University
Mrs. W. L. Miser, Ward-Belmont School
B. E. Mitchell, Millsaps College
C. N. Moore, University of Cincinnati
T. W. Moore, Indiana University
Eugenie M. Morenus, Sweet Briar College
Richard Morris, Rutgers University
A. B. Morton, Georgia School of Technology
Thirza A. Mossman, Kansas State Agricultural College
E. J. Moulton, Northwestern University

I. C. Nichols, Louisiana State University

C. O. Oakley, Brown University
Jesse Osborn, Harris Teachers College
W. P. Ott, University of Alabama
F. W. Owens, Pennsylvania State College

Z. M. Pirenian, University of Florida

Tibor Radó, Ohio State University
S. E. Rasor, Ohio State University
S. W. Reaves, University of Oklahoma
C. H. Richardson, Bucknell University
R. G. D. Richardson, Brown University
H. L. Rietz, University of Iowa

S. T. Sanders, Louisiana State University
Hazel E. Schoonmaker, Hartwick College
William Sell, University of Alabama
C. R. Sherer, Texas Christian University
W. A. Shewhart, Bell Telephone Laboratories, New York
H. A. Simmons, Northwestern University
T. M. Simpson, University of Florida
Edmond Siroky, Washington University
H. E. Slaught, University of Chicago
C. D. Smith, Mississippi A. and M. College
Clara E. Smith, Wellesley College
H. L. Smith, Louisiana State University
G. W. Snedecor, Iowa State College
Virgil Snyder, Cornell University
Mary C. Spencer, Sophie Newcomb College
Eugene Stephens, Washington University
R. C. Stephens, Knox College
W. T. Stratton, Kansas State Agricultural College
J. L. Synge, University of Toronto

J. D. Tamarkin, Brown University
J. F. Thomson, University of Michigan
Mrs. H. L. Titsworth, Sophie Newcomb College
B. A. Tucker, Southeastern Louisiana College
H. W. Tyler, Library of Congress, Washington, D. C.

H. S. Vandiver, University of Texas
R. W. Veatch, University of Tulsa
Oswald Veblen, Princeton University

J. H. Weaver, Ohio State University
W. P. Webber, Louisiana State University
J. H. M. Wedderburn, Princeton University
F. M. Weida, George Washington University
Marie J. Weiss, Sophie Newcomb College
C. W. Wester, State Teachers College, Cedar Falls, Iowa
K. P. Williams, Indiana University
W. H. Wilson, University of Florida
E. Kathryn Wyant, Northeastern Teachers College, Tahlequah, Okla.

As indicated above the attendance at the New Orleans meeting was not so large as is usual at the annual meetings. This is not surprising in view of the fact that the membership in the various states of the South is not so large as in other parts of the country. It was, however, a disappointment to the officers of the Mathematical Association that, so far as the registration indicates, there were only one member each from North Carolina, South Carolina and Virginia, two from Tennessee, three from Georgia, five from Alabama, five from Mississippi, and nine from Louisiana outside of New Orleans. This does not count a number of visitors who are not members of the Association; but one large purpose in holding the mathematical meetings at New Orleans was to bring a strong

national program close to the center of this region and thus afford our members in the South a good opportunity to attend a national meeting and to meet teachers personally from the other parts of the country. This lack of attendance is in part to be explained, doubtless, by the financial status of the country at the present time. Many of those present spoke of the great value of the meetings and all found pleasure in making new personal contacts.

The sessions of the American Association for the Advancement of Science began on Monday evening with an address in the Municipal Auditorium by Doctor R. P. Strong of Harvard University in the absence through sickness of the retiring president, Doctor Thomas Hunt Morgan, and of the president for the year, Doctor Franz Boas. The lecture was followed by a general reception given by officers of the American Association and the officers of the general local committee. Lectures of general interest were given by prominent scientists each evening during the week; these included an Edison memorial program on Wednesday evening and an address on Friday evening by Doctor Irving Fisher of Yale University on "First principles of booms and depressions."

The Council of the American Association met Monday afternoon and each morning thereafter for the transaction of general business. Professor H. H. Mitchell of the University of Pennsylvania was elected vice-president and chairman of Section A for the year 1932. Professor D. R. Curtiss of Northwestern University was re-elected a member of the Executive Committee.

The mathematicians were comfortably housed in one of the dormitories of Sophie Newcomb College, and had their meals together in the same building. It was a very unusual and delightful experience to have southern cooking, including southern coffee. Many of the visitors made their way to the famous restaurants in the French Quarter for some of their meals. At the lunch on Monday, President Dinwiddie of Tulane University was present and gave a hearty welcome to the mathematics group. A resolution was adopted by rising vote at the Thursday afternoon session of the Association expressing to the local committee of the Tulane and Sophie Newcomb faculty the appreciation of the members for the great hospitality shown, and the ample provision made to insure the comfort and convenience of the visitors.

The annual dinner was held at the Patio Royale in the French Quarter with one hundred sixty-five present. Under the toastmastership of President Eisenhart of the Society, Professor Buchanan welcomed the guests and expressed the hope, certainly realized in high degree, that the week was proving agreeable; he spoke in recognition of the value of the meetings of the two organizations to the mathematics teachers of the South. Professor Slaught described as a dream a picture of the celebration in January 1966 of the fiftieth anniversary of the founding of the Mathematical Association, with Professors Hedrick, Cairns, Slaught, Finkel and many others present and in the full bloom of their years, as at the time of the organization at the beginning of 1916. He prophesied the future growth and prosperity of the publications and other activities of the Society and Association in these fifty years, and spoke also of the fine spirit which

is continually existing in the relations of these organizations in their joint work over the country. After brief remarks by Professor Latimer, the chairman of the program committee, and by Professor Radó, who was introduced as a representative of those born abroad who have adopted this as their country, Professor Jackson, retiring as an editor of the Transactions, described the financial stress which the Society has undergone the past few years as a not unmixed evil. We should have, he said, a collective sense that the work we are doing is a part of the whole work about us, that the work which will be read with interest by a few persons may be more inspiring than work that might appeal to many more. We should learn to estimate not so much the work of others as the work of ourselves and its possible value to others. The value of research is to the man who does the research, in relation to his college or university. Everyone should do so much independent investigation of mathematical ideas that no one person should expect to publish everything that he finds out, but pruning and evaluating he should then present his results to the mathematical journals.

By vote of the members present at the dinner, the Secretaries were instructed to send a telegram of greeting to Professor J. W. Young with an expression of hope for an early recovery from the illness which occasioned his resignation as chairman of the Joint Committee on Funds of the two organizations.

The American Mathematical Society held sessions on Monday morning and afternoon and Wednesday morning for the reading of papers. On Tuesday morning the Society had a joint session with the American Physical Society at which Professor G. D. Birkhoff of Harvard University spoke on "Stability and instability of physical systems," and Doctor W. F. G. Swann of the Bartol Research Foundation gave an address on "The significance of the fundamental concepts of modern atomic theories"; abstracts of these two addresses appeared in *Science* for February 5. At the joint session of the Society with the Physical Society and Section A on Tuesday afternoon, Professor G. A. Bliss gave his retiring address as the Vice-President of Section A, his subject being "The calculus of variations of the quantum theory"; this address will appear in the *Bulletin* of the Society. The Ninth Josiah Willard Gibbs lecture on "Statistical mechanics and the second law of thermodynamics" was given by Professor P. W. Bridgman of Harvard University later Tuesday afternoon. A short abstract of this lecture will be found in *Science* and the paper is to appear in the *Bulletin* of the Society.

At the close of the Wednesday morning program of the Society, the Cole Prize in the Theory of Numbers was presented to Professor H. S. Vandiver of the University of Texas for his work on Fermat's Last Theorem. By invitation he gave a resumé of the advances which he himself has made toward the solution of this problem.

The program of the Mathematical Association consisted of a joint session with the Society on Thursday morning and a separate session on Thursday afternoon, with President E. T. Bell in the chair. A good program was provided by the committee consisting of Professors Julia Dale, H. L. Smith and C. G.

Latimer, chairman. The program follows, together with abstracts of some of the papers, numbered in accordance with their place on the program.

JOINT SESSION OF THE ASSOCIATION WITH THE SOCIETY

1. "Non-analytic functions of a complex variable" by Professor E. R. HEDRICK, University of California at Los Angeles, retiring president of the Society.

2. "Some recent researches in the theory of numbers" by Professor R. D. CARMICHAEL, University of Illinois, representing the Association.

1. The address by Professor Hedrick will appear in an early issue of the *Bulletin* of the Society.

2. The address by Professor Carmichael appears in this issue of the MONTHLY.

SEPARATE SESSION OF THE ASSOCIATION

1. "Maximum efficiency in freshman mathematics" by Professor C. R. SHERER, Texas Christian University.

2. "A method of handling certain elementary integrals" by Professor OTTO DUNKEL, Washington University.

3. "Fourier series" by Professor DUNHAM JACKSON, University of Minnesota.

1. The average high school graduate has been thoroughly steeped in the idea of avoiding anything that looks or sounds mathematical. Under the influence of the aim of practicality, reorganizations of curricula have changed mathematics, both in its position in the curricula and with regard to course content. The need for serious consideration of the situation with a view to retaining mathematics as one of the essential tool courses, and even as a cultural subject, is obvious.

The following plan was initiated in Texas Christian University in the fall semester of 1930–31. Its purpose was two-fold:

> 1. To aid those students who were poorly prepared in their high school mathematics or who had forgotten most of the fundamental principles necessary for successful study in college mathematics.
>
> 2. To stimulate and motivate those students of exceptional ability and capacity.

Three sections of freshman mathematics are scheduled for the same hour; for example, eight o'clock Monday, Wednesday, and Friday. In order that schedule problems shall be minimized, all students registering for that class cannot register for another course at the same hour on Tuesday, Thursday, and Saturday. The three sections meet together for the first class hour, where the proper psychological atmosphere is very carefully prepared by a staff member who understands and approves the plan. A comprehensive review of elementary algebra is given for a period of three weeks. This is followed by a suitable uniform placement examination.

Guided by the results of this examination, the high school record, the general intelligence tests, and, in doubtful cases, by personal interviews, A, B, and C sections are formed. The A and B sections are composed of the exceptional and average students respectively, while the C section is comprised of all those whose ability or preparation is below normal. The A and B sections continue with the normal schedule of three days per week, but the C section meets five times for three hours credit. The content of the course for the C section is so constructed that an ambitious student will get a foundation sufficient to continue in the next course.

In order that the superior or exceptional people may advance in proportion to their capacity, a special class is formed for the second semester. They are chosen on the following bases:

1. Record in the first semester's work.
2. Special examination to be taken by all those who wish to become candidates.
3. Recommendation by the faculty.
4. Personal conferences through which aims and attitudes are discovered.

The general class procedure is built upon the seminar plan. Since the membership of the class is small, it is possible to direct a part of their study through individual conferences.

2. This paper presents a method of evaluating the integrals (1) $\int(x^2+2Ax+B)^{-1/2}\,dx$, (2) $\int(x^2+2Ax+B)^{1/2}\,dx$ by use of a simple algebraic substitution and the principle of "composition and division" in elementary algebra. The method appears to involve less computation than the usual methods. This paper appears in this issue of the MONTHLY.

3. Professor Jackson gave an outline of a course in Fourier Series which he has been giving the first semester of the present academic year, adapted to undergraduates who have had but one year of the calculus. This paper made it quite manifest that it is readily possible to carry on such a course for juniors as well as seniors and graduate students. It is to be hoped that Professor Jackson will make this material available for the teachers of undergraduate courses.

MEETINGS OF THE BOARD OF TRUSTEES

Seven members of the outgoing Board of Trustees and eight members of the incoming Board were present at the New Orleans meetings.

The following seventeen persons and three institutions were elected to membership on applications duly certified:

To Individual Membership

ANN S. BUCHANAN, A.M. (La. State Univ.) Instr., Southwestern Louisiana Institute, Lafayette, La.

R. V. CHURCHILL, Ph.D. (Michigan) Asst. Prof., Univ. of Michigan, Ann Arbor, Mich.

H. M. Cox, A.M. (Duke) Instr., Georgia School of Tech., Atlanta, Ga.
A. T. Craig, Ph.D. (Iowa) Asso., Univ. of Iowa, Iowa City, Iowa
J. H. Cross, A.M. (Texas Tech.) Asst., Dept. of Physics, Texas Tech. Coll., Lubbock, Tex.
Sister Francis Xavier, Ph.D. (Fordham) Prof., St. Joseph's Coll. for Women, Brooklyn, N. Y.
F. C. German, A.M. (Illinois) Asso. Prof., State Teachers Coll., Pittsburg, Kans.
Sarah E. Haughton, A.M. (Alabama) Instr., Univ. of Alabama, University, Ala.
R. L. Korgen, A.M. (Harvard) Instr., Bowdoin Coll., Brunswick, Maine.
O. C. Kreider, M.S. (Iowa State Coll.) Teacher, Ellsworth Jr. Coll., Iowa Falls, Iowa
J. E. LaFon, A.M. (Oklahoma) Asst., Univ. of Oklahoma, Norman, Okla.
M. C. Ledesma, Senior, School of Engineering, Milwaukee, Wis.
Sister Charles Mary Morrison, Ph.D. (Catholic Univ.) Prof., Nazareth Coll., Louisville, Ky.
V. F. Murray, M.A. (St. Andrew's Univ., Scotland) Hoboken, N. J.
Viola Perry, A.M. (Peabody) Head of Dept., S. Georgia Teachers Coll., Collegeboro, Ga.
H. C. Searcy, Jr., A.M. (So. Methodist Univ.) Instr., Jr. Coll., Edinburg, Tex.
Catharine L. Walker, A.M. (Columbia) Instr., Jr. Coll., Fort Smith, Ark.
Henry Walther, B.S. (Cooper Union) Research Engr., Bell Telephone Labs., New York, N. Y.

To Institutional Membership

University of New Mexico, Albuquerque, N. M.
Bennett College for Women, Greensboro, N. C.
North Carolina State College of Agriculture and Engineering, Raleigh, N. C.

The financial report of the Secretary-Treasurer for the year 1931 was accepted. Professor Slaught for the finance committee, with Professors C. A. Hutchinson and F. W. Owens, had examined the report and the evidences of assets and declared the report satisfactory.

The Trustees accepted with regret the resignation of Professor J. W. Young as chairman and member of the Joint Committee on Funds, on account of illness. Because the general plans of this committee are necessarily at a standstill at present, it was voted that the chairmanship of the committee remain vacant for the present.

It was voted to approve the following associate editors of the Monthly for the year 1932, as nominated by Professor Carver:

W. F. Cheney	R. E. Gilman	J. R. Musselman
N. A. Court	R. A. Johnson	H. L. Olson
Otto Dunkel	B. W. Jones	D. E. Smith
H. S. Everett	H. W. Kuhn	F. M. Weida
B. F. Finkel		

It was voted to nominate Professor H. L. Rietz as representative of the Association on the National Research Council for a three-year term from June, 11, 1932, in succession to Professor R. D. Carmichael.

It was voted that as soon as practicable the dues of our foreign members should be put on a basis of U. S. exchange.

It was voted to accept with thanks the invitation of Harvard University to

hold the meeting of the Mathematical Association there in connection with the meeting of the American Association in December 1933. It was also voted to hold the meeting of the Mathematical Association at Atlantic City in December 1932 in connection with the meeting there of the American Association.

Some other items of business were transacted which had to do with general and sectional activities.

Annual Business Meeting of the Association

The Secretary announced the names of those who had been elected to membership at the meeting of the Trustees. He reported also the deaths of the following members:

M. Sue Burney, Greenwich, Conn. Formerly instructor of mathematics, Junior College, St. Joseph, Mo. (August 20, 1931).

Annie C. Clark (Mrs. Theron Clark) Head of mathematics dept., Holmby College, Los Angeles, Calif. (January 21, 1931)

G H. Cresse, Professor of mathematics, University of Arizona (May 3, 1931).

S. M. Hadley, Dean and professor of mathematics, Penn College (Retired June, 1930) (November 10, 1931)

Charles Haseman, Professor of mathematics and mechanics, University of Nevada (July 9, 1931)

A. A. Himwich, M.D. New York, N. Y. (April 18, 1931)

H. L. Hodgkins, Dean and professor of mathematics, George Washington University (February 13, 1931)

F. J. Holder, Professor of mathematics and dean of School of Commerce, Mercer University, Macon, Ga. (December 30, 1931)

Alexander Knisely, Columbia City, Ind. (September 29, 1931)

Bessie I. Miller, Formerly professor of mathematics, Rockford College, Ill. (February 4, 1931)

C. L. E. Moore, Associate professor of mathematics, Massachusetts Institute of Technology (December 5, 1931)

G. W. Myers, Professor of mathematics, University of Chicago (November 23, 1931)

G. D. Olds, President emeritus, Amherst College (May 11, 1931)

C. I. Palmer, Professor of mathematics and dean of students, Armour Institute of Technology (April 9, 1931)

C. W. Watts, Professor of mathematics, Virginia Military Institute (July 10, 1931)

W. A. Zehring, Professor of mathematics, Purdue University (May 1, 1931)

The election of officers for the year 1932 resulted in the following, as reported by the tellers, Professors H. M. Hosford and F. M. Weida:

For Vice-Presidents: W. H. Bussey, 307 votes; Abraham Cohen, 259 votes; G. C. Evans, 286 votes; Mayme I. Logsdon, 184 votes.

For additional members of the Board of Trustees, to serve until January 1935: C. R. Adams, 223 votes; W. C. Brenke, 212 votes; H. E. Buchanan, 280 votes; W. B. Ford, 336 votes; E. R. Hedrick, 406 votes; W. R. Longley, 250 votes; W. E. Milne, 178 votes; J. L. Walsh, 186 votes.

The following were accordingly declared elected:

Vice-Presidents: W. H. Bussey, University of Minnesota; G. C. Evans, Rice Institute.

Additional members of the Board of Trustees: H. E. Buchanan, Tulane

University; W. B. FORD, University of Michigan; E. R. HEDRICK, University of California at Los Angeles; W. R. LONGLEY, Yale University.

By action of the Association the Secretary was instructed to write the following letter to the retiring editor-in-chief, Professor W. H. Bussey:

MY DEAR PROFESSOR BUSSEY:

I have the pleasure of informing you that at the session of the Mathematical Association of America on Thursday afternoon, December 31, 1931, the Association members adopted heartily a resolution expressing our appreciation of the notable work which you have done during the past five years as editor-in-chief of the American Mathematical Monthly. They asked me to express to you our recognition of the high ideals that have characterized the Monthly during your term of office and to thank you for the care and the devotion which you have shown in performing so ably the duties of the office.

Yours cordially,
(Signed) W. D. CAIRNS, *Secretary*

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DECEMBER 14, 1931

RECEIPTS

Balance Dec. 15, 1930		$9,236.65
1929 indiv. dues	48.00	
1930 indiv. dues	334.40	
1930 instit. dues	28.50	
1931 indiv. dues	6,800.72	
1931 instit. dues	787.60	
1931 subscriptions	879.09	
Initiation fees	*252.00*	
Life memberships	125.47	
Advertising	639.00	
Sale *Register*	1.50	
Sale copies of MONTHLY	85.50	
Sale First Carus Mon.	30.00	
Sale Second Carus Mon.	36.25	
Sale Third Carus Mon.	40.00	
Sale Fourth Carus Mon.	45.00	
Advance sale Fifth Mon.	1.25	
Sale Catalog	.50	
Sale Rhind Papyrus	990.00	
Transfer from Carus Mon. certif.	83.75	
Carrying charges Papyrus	27.80	
Received *Annals* subscriptions	5.00	
Int. Oberlin Savgs. Bk.	99.26	
Int. Peoples Bkg. Co.	83.57	
Int. Liberty Bonds	42.51	
Int. Hardy Fund	120.00	
Int. certifs. of deposit	19.34	
Int. from Genl. Endowment Fund Bonds	392.50	
Int. Carus Fund	50.00	
Int. Chace Fund	132.50	

EXPENDITURES

Publisher's bills (Nov. '30–Oct. '31)		$5,278.71
President's office		44.25
Joint Committee on Funds		342.63
Manager's office		72.22
Editor-in-chief's office 1931		500.73
Editor-in-chief's office 1932		172.38
Committee on Geometry		4.65
Committee on Membership		*86.50*
Register		167.00
Printing library catalog		161.49
Secretary-Treasurer's office:		
Postage	374.01	
Bond	5.00	
Safety deposit	4.00	
Office supplies	112.01	
Express, tel., etc.	79.25	
Clerical work	1,685.25	
Printing	347.06	
Library expense	146.20	
Ins. back copies of MONTHLY	11.00	
Paid copies MONTHLY	33.05	
Cleveland meeting	50.00	
Minneapolis meeting	75.00	
Refund instit. dues	4.50	
Refund 1931 subscriptions	7.20	
Refund Mon. purchase	1.25	2,934.78
Annals subvention		375.00
Paid to sections from initiation fees		85.70
Forwarded *Annals* subscriptions		7.50
Paid *Annals* subscriptions		5.00

Int. Chauvenet Fund...	25.00	
Int. from investment of current funds........	180.81	12,386.82
Total 1931 receipts..............		$21,623.47
Total expenditures..............		$11,812.84
Balance to the end of 1931 business.		9,810.63
Received on 1932 business........		740.47
Book balance Dec. 14, 1931.......		$10,551.10

Paid B. F. Finkel int. Hardy Fund.	120.00
Sustaining memb. in American Math. Society................	100.00
Carrying charges Papyrus........	27.77
Expense acct. Carus Mons........	225.00
Transfer to Carus Mon. Fund.....	124.27
Transfer to Chace Fund..........	977.26
Total expenditures..............	$11,812.84
Cash on hand....................	10.03
Checking account................	1,299.44
Oberlin Savgs. Bk. acct...........	2,556.26
Peoples Bkg. Co. acct............	2,105.38
Liberty Bonds..................	1,000.00
Iowa Ry. & Light Co. 5% Bonds...	3,000.00
Certif. of deposit................	496.24
Part certif. of deposit............	83.75
Bank balance Dec. 14, 1931......	$10,551.10

Exhibit of the Funds of the Association

Carus Monograph Fund

Balance Dec. 15, 1930		$5,756.19
Receipts: Sales	$152.50	
Interest	241.77	394.27
		$6,150.46
Expenditures		225.00
		5,925.46
Certificates of deposit	4,925.46	
Cleveland Trust Securities Co. Gold Bond	1,000.00	
Balance Dec. 14, 1931		$5,925.46

Arnold Buffum Chace Fund

Balance Dec. 15, 1930		$2,930.01
Receipts: Sales	990.00	
For carrying charges	27.80	
Interest	174.94	1,192.74
		$4,122.75
Expenditures: For carrying charges		27.77
		$4,094.98
Iowa Rwy. and Light Co. 5% Bond	$1,000.00	
Western United Gas and Elec. Co. Bonds	1,500.00	
Certificates of deposit	1,449.71	
Cash in bank	145.27	
Balance Dec. 14, 1931		$4,094.98

CHAUVENET PRIZE FUND

Balance Dec. 15, 1930		$550.00
Interest	25.00	
Amount set aside for 1931	20.00	45.00
		$595.00
Iowa Rwy. & Light Co. 5% Bond	500.00	
Cash in bank	95.00	
Balance Dec. 14, 1931		$595.00

LIFE MEMBERSHIP FUND

Liability on life memberships Dec. 15, 1930	$446.31
Received on life membership payments	125.47
	$571.78
To be transferred to current funds, surplus	14.79
Liability on life memberships as of Jan. 1, 1932	$556.99

GENERAL ENDOWMENT FUND

Balance Dec. 15, 1930		$8,000.00
Liberty Bond	$1,000.00	
Land Trust Certificate	1,000.00	
Cleveland Trust Investment Co. Gold Bond	1,000.00	
Idaho Power Co. 5% Bonds	2,000.00	
Northwestern Elec. Co. Bonds	3,000.00	
Balance Dec. 14, 1931		$8,000.00

Of the funds on hand, indicated in the first division of this financial report, $95.00 belongs to the Chauvenet Prize Fund; $145.27 belongs to the Arnold Buffum Chace Fund; and $556.99 is held as a Life Membership Fund, representing the liability on life memberships already paid for, as of date Jan. 1, 1932. Aside from these amounts, the Carus Monograph Fund is carried as a separate fund in the form of two certificates of deposit which bear 4%, compounded quarterly, to the amount of $4,925.46 and a Gold Bond of $1,000.00; $1,449.71 of the Arnold Buffum Chace Fund is similarly carried in the form of two certificates of deposit and $2,500.00 in securities; the contribution of Professor Walter B. Ford to the Chauvenet Fund is carried as a 5% Gold Bond; and the sum of $8,000.00 is held in reserve as the General Endowment Fund, in securities as listed above.

When the accounts were closed Dec. 14, 1931, there remained on the total business for 1931 the following items:

BILLS RECEIVABLE (partly estimated)		BILLS PAYABLE (partly estimated)	
1931 individual dues	$200.00	Publisher's bills (Nov. Dec. '31)	$1,220.00
Advertising	100.00	President's office	30.00
		Manager's office	25.00
	$300.00	Editor-in-chief's office	140.00

Other editors' postage	30.00
Printing *Register*	550.78
Secretary-Treasurer's office	300.00
Annals subvention	75.00
Initiation fees due to sections	900.00
Chauvenet Prize Fund	95.00
Chace Fund	145.27
Life Membership Fund	556.99
	$4,068.04

If to the balance on 1931 business shown in the report, $9,810.63, there be added the bills receivable, $300.00, and there be subtracted the estimated bills payable, $4,068.04, there results an estimated final balance on 1931 business of approximately $6,040, which represents the accumulated surplus in current funds. This is a satisfactory showing for the year when compared with the corresponding figure of $5,280 for a year ago. It is however a disquieting feature of our status that more members are in arrears, too often without any word to the secretary-treasurer. It is very much to be hoped that those who find it necessary to postpone the payment of their dues will inform the office as to when payment may be expected, for arrangements may ordinarily be made in such cases so that the members may retain full standing. In the absence of such information the secretary-treasurer needs, even reluctantly, to conform to the rule of the Trustees, according to which members who are more than one year in arrears do not receive the MONTHLY and members who are more than two years in arrears and have not responded to letters from the office of the secretary-treasurer are omitted from the list of members. It is quite preferable to keep in touch with the secretary-treasurer.

W. D. CAIRNS, *Secretary-Treasurer*

THE TWENTIETH MEETING OF THE IOWA SECTION

The twentieth meeting of the Iowa Section of the Mathematical Association of America was held with the Iowa Academy of Science at the Davenport Public Museum and St. Ambrose College, Davenport, Iowa, on May 1 and 2, 1931. The meetings were held in the Assembly Room of Lewis Hall, St. Ambrose College.

The attendance was about forty, including the following seventeen members of the Association: E. W. Chittenden, L. M. Coffin, Julia T. Colpitts, N. B. Conkwright, J. M. Earl, F. M. McGaw, E. E. Moots, J. F. Reilly, H. L. Rietz, B. D. Roberts, E. R. Smith, G. W. Snedecor, C. W. Strom, John Theobald, L. E. Ward, C. W. Wester, Roscoe Woods.

The Section chairman, Professor G. W. Snedecor, presided at both the Friday afternoon and Saturday morning sessions. The Section joined with the Iowa Academy of Science at dinner on Friday evening.

At the business session officers for 1931–1932 were elected as follows: Chair-

man, C. W. Strom, Luther College; Vice-chairman, B. D. Roberts, Parsons College; Secretary-Treasurer, J. F. Reilly, University of Iowa.

The program consisted of nineteen papers, as follows:

1. "Complete systems under certain finite groups" by Professor C. W. Strom, Luther College.
2. "Some properties of a class of polynomials generated in the successive differences of the Poisson function" by Professor E. R. Smith, Iowa State College.
3. "The convergence of a sequence of polynomials" by Professor J. M. Earl, University of Omaha.
4. "Concerning some properties of the distribution of geometric means of large samples" by Dr. A. T. Craig, University of Iowa, by invitation.
5. "Review of solar observations at Alta, Iowa, for the five years, 1926 to 1930" by Mr. D. E. Hadden, Astronomical Observatory, Alta, Iowa, (by title).
6. "A method of solving numerical equations" by Mr. S. A. Corey, Des Moines, presented by Professor D. L. Holl, Iowa State College.
7. "The standard error of the mean of 'matched' samples from a normal universe" by Dr. S. S. Wilks, University of Iowa, by invitation.
8. "A normalized orthogonal system of functions" by Mr. Fred Robertson, Iowa State College, presented by Professor E. S. Allen.
9. "Simpson's rules for the approximate value of an integral" by Professor J. F. Reilly, University of Iowa.
10. "Determination of the Gram-Charlier series as a solution of Pearson's differential equation" by Professor E. R. Smith, Iowa State College.
11. "The standard deviation of a standard regression coefficient" by Professor G. W. Snedecor, Iowa State College.
12. "On the reduction of a principal part of an unsymmetric kernel to a canonical form" by Professor E. W. Chittenden, University of Iowa.
13. "Grades in a certain course in mathematics as a criterion of teaching ability in the preceding course" by Miss Gertrude Cox, Iowa State College, by invitation.
14. "Some noteworthy solar disturbances observed at Alta, Iowa, during the years 1890 to 1930" by Mr. D. E. Hadden, (by title).
15. "On the problem of n-bodies with a special law of force" by Mr. J. J. Hinrichsen, Iowa State College, by invitation.
16. "Transverse oscillations of a clamped beam" by Mr. A. W. Davis, Iowa State College, by invitation.
17. "The scale in certain conformal representations of the earth upon a plane" by Mr. H. T. Hurley, Storm Lake, by invitation.
18. "Some examples illustrating exceptional cases in elementary differential equations" by Dr. N. B. Conkwright, University of Iowa.
19. "A statistical investigation of the relation between planetary and lunar positions and temperatures in Des Moines" by Mr. Larry Page, Des Moines, by invitation.

Abstracts of some of these papers follow:

1. In this paper Professor Strom determined the complete systems of invariant polynomials under the following groups: (1) the cyclic groups, (2) the solvable groups, (3) the simple group of degree 7 and order 168.

2. The function $\psi(x)=e^{-\lambda}\lambda^x/x!$, defined for positive integral values of x, has its nth difference of the form $P_n(x)\psi(x)$, where

$$P_n(x) = 1 - \frac{nx}{\lambda} + \frac{n(n-1)}{1.2}\,\frac{x(x-1)}{\lambda^2} + \cdots + (-1)^n \frac{x(x-1)\cdots(x-n+1)}{\lambda^n}.$$

Professor Smith shows that $\sum\psi_n(x)P_m(x)=0$ when $m\neq n$, and $=m!/\lambda^m$ when $m=n$, and because of this biorthogonal property the expansion of certain functions, such as the frequency function, in terms of a series of these polynomials may be readily obtained.

3. Professor Earl in his paper was concerned with a generalization of expansions in terms of linear combinations of the classical trigonometric polynomials.

5. In his first paper Mr. Hadden reviewed the records he had made at Alta of sunspot activity for the five year period ending 1930, fixing the date of maximum activity as September, 1928.

6. In this paper Mr. Corey considers numerical solutions of $f(x)=0$, in which the function is algebraic or transcendental, by iteration methods. By writing the equation in the form $f(x)+g(x)-yg(x)=0$ the roots are expressed as a power series in y by means of a MacLaurin's development. When an approximate value of a root is known a more accurate value may be readily obtained by making a proper choice of $g(x)$. The method may be used to locate both real and imaginary roots.

8. The purpose of Mr. Robertson's paper was to form a normalized orthogonal system of functions over the range zero to infinity for the system of functions

$$Q_n(x,z) = \left[1 - \left\{1 + xz + \frac{x^2z^2}{2!} + \cdots + \frac{x^{n-1}z^{n-1}}{(n-1)!}\right\}e^{-xz}\right]z^{-n}.$$

9. In the derivation of Simpson's rules for the approximate value of an integral Professor Reilly suggested the use of finite rather than infinite expansions, thereby obtaining a reliable, if somewhat indefinite limit for the error.

10. In this paper Professor Smith shows that the solution of the equation $dy/dx=(\alpha+x)y/(\alpha+\beta x+\gamma x^2)$ may be expressed by means of a series of the Gram-Charlier type. A convenient recurrence relation between the successive coefficients makes it possible to express them in terms of the parameters of the differential equation. The method admits of generalization.

15. In his paper Mr. Hinrichsen considers the problem of n bodies with the potential function defined by $\mu=\sum m_im_j/r_{ij}^d$, $d>2$. He establishes two theorems concerning real motions of the n bodies, showing that under certain conditions the motion will terminate in a collision, while under other conditions all n bodies separate indefinitely from one another.

16. In Mr. Davis's paper the frequency numbers corresponding to the different modes of the transverse vibrations of a beam are determined by a minimizing process given by W. Ritz. The assumed function in the integral to be minimized satisfies the boundary conditions and contains n parameters, the determinant of which leads to the frequency equations. In the simpler cases these results are compared with the well known accurate results.

17. A combination of expanding scales for vertical and horizontal distances on Mercator's map gives a scale applicable to distances in any direction, the scale of middle scalar distance between the two points being used as a measure of the distance. Mr. Hurley constructed such a scale for the stereographic map, measuring slant distances on the scale of the bounding latitude. These two scales together give the distance between any two points on the earth within a limit of, say, 3000 miles.

18. In his paper Dr. Conkwright discussed singular solutions of certain equations which do not represent envelopes, singular solutions of simultaneous equations, and of total differential equations.

JOHN F. REILLY, *Secretary*

FIFTEENTH ANNUAL MEETING OF THE KENTUCKY SECTION

The fifteenth annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky, Lexington, Kentucky, on Saturday, May 9, 1931. The chairman, Professor J. M. Davis, presided at both morning and afternoon sessions.

There were thirty-eight persons present, including the following seventeen members of the Association: P. P. Boyd, J. M. Davis, H. H. Downing, A. R. Fehn, Lydia K. Fremd, Charles Hatfield, C. G. Latimer, Elizabeth LeStourgeon, Mayme I. Logsdon, C. A. Maney, W. L. Moore, R. S. Park, Sallie Pence, D. W. Pugsley, E. L. Rees, Guy Stevenson, H. M. Yarbrough.

The Section was fortunate in having as its guest Professor Mayme I. Logsdon of the University of Chicago. During the intermission the members and guests had an excellent opportunity to become acquainted in the faculty club room placed at their disposal. A delightful luncheon was served by the University to all present. The following officers were elected for the coming year: Chairman, R. S. Park, Eastern State Teachers College; Secretary, A. R. Fehn, Centre College.

The program of the meeting follows, with abstracts of some of the papers:

1. "Curvature in the Einstein space-time world" by Professor E. L. Rees, University of Kentucky.

2. "The arithmetics of certain generalized quaternion algebras" by Professor J. M. Boswell, Georgetown College, by invitation.

3. "Brief outline of thesis on non-Euclidean geometry" by Miss Alleen Lemons, University of Kentucky, by invitation.

4. "Finite geometries" by Professor W. L. Moore, University of Louisville.

5. "On the class numbers of cubic fields" by Professor C. G. Latimer, University of Kentucky.

6. "Some concepts of mathematical physics" by Professor J. G. Black, Morehead State Teachers College, by invitation.

7. "The slide rule as an aid in teaching mathematics" by Professor D. W. Pugsley, Berea College.

8. "On the place of mathematics in a liberal arts education" by Professor C. A. Maney, Transylvania College.

9. "Reorganization of material for freshman mathematics" by Professor Mayme I. Logsdon, University of Chicago.

1. In this paper the theory of relativity was discussed from the point of view of the curvature of the space-time world and the fundamental importance of this curvature in the theory was emphasized.

4. Dr. Moore's paper is a preliminary report on a method of constructing certain finite plane geometries of $n+1$ points on a line. Let P_{ij} $(i=0, 1, 2, \cdots, n;$ $j=1, 2, \cdots, n)$ and P_{00} be the n^2+n+1 points in the geometry, then the n^2+n+1 lines are selected from the square array as follows:

$$\begin{array}{lll} P_{00} & P_{00} \cdots & P_{00} \\ P_{01} & P_{11} \cdots & P_{n1} \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot \\ P_{0n} & P_{1n} \cdots & P_{nn}: \end{array}$$

I. The points in each column, giving $n+1$ lines.

II. P_{01} and the points in each row of the square array from which the first row and column have been deleted, giving n lines.

III. P_{0l} $(l=2, 3, \cdots, n)$ and points from each row and column chosen such that no pair occurs in any previous selection, giving $n(n-1)$ lines.

All such geometries can be easily constructed for which there exists a group G_n of order n which is left invariant by a group G_{n-1} of order $n-1$. Those geometries for which the group G_{n-1} is Abelian were found to be Desarguean while those in which the group G_{n-1} is non-Abelian were found to be non-Desarguean. The author has not yet been able to satisfy himself that all finite plane geometries are given by this construction nor that the relation between the character of the group G_{n-1} and the geometry is a general one.

6. Starting with the inverse square formulas for "action at a distance" in the case of electricity, magnetism and gravitation, the merging of electricity and magnetism into the electromagnetic equations of Maxwell was discussed. Mentioning the work of Newton in gravitation and light, the different portions of the light spectrum as known at present were described. The postulate of an ether as a light bearing medium and the very accurate yet unsuccessful attempt to detect absolute motion through it led to the relativity theory which indicated that light had mass and also that mass was convertible into light or radiation.

Millikan's cosmic ray work was reviewed and the analysis of his absorption curves into four components corresponding to the building up of the elements helium, oxygen, silicon and iron according to Einstein's equation for the change of mass into radiation was explained.

7. Professor Pugsley's paper took up briefly the development of the logarithmic slide rule, the question of accuracy, and the usefulness of the slide rule. It was pointed out that the usefulness of the slide rule is three-fold, namely, it can be used to create student interest, it saves time and drudgery in certain computations, and it is very helpful in checking computations by both student and instructor. The discussion was confined to the topic indicated in the title, only brief mention being made of the practical usefulness in other fields.

8. Exhibits were presented by Professor Maney showing that elementary mathematics, especially statistics, is being used increasingly in research in the fields of education, sociology, business administration, and economics. More and more mathematics is being reorganized as the necessary language of scientific thinking. The study of mathematics is therefore a basic part of a liberal education.

A. R. FEHN, *Secretary*

SOME RECENT RESEARCHES IN THE THEORY OF NUMBERS[1]

By R. D. CARMICHAEL, University of Illinois

1. *Introduction.* Recent advances in the theory of numbers have been so numerous and varied that no discussion of all of them in the space of an hour could have that unity which is necessary to the comfort of either the speaker or the hearer. Hence there must be a selection of material. The choice will be determined by the need of procuring unity and by the present interests of the speaker.

Although there have been some recent advances in the theory of integers defined by recurrence relations, and the speaker has himself been interested in this matter, the subject will be omitted for the sake of greater unity. The contributions which have been made in recent years, and particularly in America, to a treatment of Fermat's last theorem will be left entirely out of consideration. No analysis will be given of progress in the theory of algebraic numbers. The most important of all omitted contributions are those of Dickson and his followers. In these researches we have a remarkable characteristic development of the theory of numbers having the closest association with classical points of view and dealing with some of the most interesting questions which have ever arisen in the theory of numbers.

Passing over still other subjects, we shall confine our attention almost en-

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[1] An address delivered by invitation at a meeting of the Mathematical Association of America, at New Orleans, on December 31, 1931.

tirely to positive integers and largely to the asymptotic properties of number-theoretic functions and the expansion of the latter in infinite series. Such problems frequently begin from what seem to be simple ideas but their treatment often requires the most abstruse methods. In the main the problems which we shall examine have this quality. But there is one of them (that treated in §VI) which deals with a question that has not yet passed beyond the most elementary stage. It therefore invites further treatment. In fact, it is of such character that the amateur may well be able to do some work of value, particularly if his results are put into the hands of one trained in the deeper methods of the theory of numbers.

I. THE WORK OF SRINIVASA RAMANUJAN

2. *Srinivasa Ramanujan.* During our generation no more romantic personality than that of Srinivasa Ramanujan has moved across the field of mathematical interest. Indeed it is true that there have been few individuals in human history and in all fields of intellectual endeavor who draw our interest more surely than Ramanujan or who have excited more fully a certain peculiar admiration for their genius and their achievements under adverse conditions. There is nothing particularly noteworthy about Ramanujan's ancestry to account for his great gifts. His father and paternal grandfather were petty accountants in Kumbakonam, an important town in the Tanjore district in India. His mother was a woman of strong common sense. For some time after marriage she was without children. Her father prayed to a famous local goddess to bless his daughter with children; and shortly afterwards Ramanujan, her eldest child, was born.

I can not pause to speak of his meager and somewhat irregular school and college education. On January 16, 1913, at 25 years of age, he wrote to G. H. Hardy a letter containing the following words:

> I have had no University education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at Mathematics. I have not trodden through the conventional regular course which is followed in a University course, but I am striking out a new path for myself.

On February 27 of the same year he wrote to Hardy as follows:

> I have found a friend in you who views my labours sympathetically. This is already some encouragement to me to proceed. . . . I find in many a place in your letter rigorous proofs are required and you ask me to communicate the methods of proof. . . . I told him that the sum of an infinite number of terms of the series $1+2+3+4+\cdots=-1/12$ under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal. . . . What I tell you is this. Verify the results I give and if they agree with your results . . . you should at least grant that there may be some truths in my fundamental basis. . . .
>
> To preserve my brains I want food and this is now my first consideration. Any sympathetic letter from you will be helpful to me here to get a scholarship either from the University or from Government. . . .

These passages from Ramanujan's letters will give a faint indication of the human side of a correspondence which stands as a remarkable one in the history

of our science. Along with the letters numerous astonishing mathematical results were communicated by Ramanujan to Hardy. Many of these were right; not a few were wrong; even the errors themselves were sometimes brilliant. The association thus opened up with Hardy became one of the prime factors in the life of Ramanujan. He went to England in 1914.

Hardy has put on record the fact that his appearance set a great puzzle for solution. What was to be done in teaching such an intellect the spirit and methods of modern mathematics? In some directions his knowledge was profound. In others his limitations were quite as startling. He could work out modular equations and theorems of complex multiplication, to orders unheard of. His mastery of continued fractions was remarkable. He had found for himself the functional equation of the Zeta-function and dominant terms in the case of many famous problems in the analytic theory of numbers. Yet he had never heard of a doubly periodic function or of Cauchy's theorem. Indeed, he had but the vaguest idea of a function of a complex variable. His conception of mathematical proof was quite inadequate. His results, new or old, right or wrong, had been obtained by argument, intuition, induction, mingled in a way which he himself could not coherently describe. Many of his proofs were invalid. Not a few of his results were false. His approximations were not as close as he supposed. But, notwithstanding the fact that he had never seen a French or German book and that his command of English was meager, he had conceived for himself and had treated in an astonishing way problems to which for a hundred years some of the finest intellects in Europe had given their attention without having reached a complete solution. That such an untrained mind made mistakes in dealing with such questions is not remarkable. What is astonishing is that it ever occurred to him to treat these problems at all.

In a few years after reaching England Ramanujan had a fair knowledge of the theory of functions and the analytic theory of numbers. He had learned to know when he had proved a theorem and when he had not. And his flow of original ideas kept up without abatement during this period of acquisition.

In the spring of 1917 he began to show evidence of being unwell. He went to a Nursing Home at Cambridge early in the summer of 1917 and was never afterwards out of bed for long at a time. He died April 26, 1920, a little more than seven years after writing his initial letter to Hardy, the event which brought him for the first time into touch with modern mathematical ideas. During this brief interval, about one-third of which was spent in bed on account of illness, he added to the achievements made unaided during the first years of his life a body of contributions which is sure to exert a marked influence on the development of certain chapters of mathematics, indeed has already exerted such an influence.

Under the title *Collected Papers of Srinivasa Ramanujan* there has been brought together in a single volume everything published by Ramanujan with the exception of a few solutions of questions proposed by other mathematicians and answered by him in the *Journal of the Indian Mathematical Society*. He left

behind him a large mass of unpublished material in note-books now famous. This is in process of being edited for publication.

Such is a brief statement concerning the human aspects of a remarkable mathematician some of whose contributions to the theory of numbers we shall now describe very briefly.

3. *On certain arithmetical functions.* We shall entirely pass over several of Ramanujan's contributions to the theory of numbers, including his investigation of the expression of integers in the form $ax^2+by^2+cz^2+dt^2$ and his theory of highly composite numbers (the latter work, in Hardy's judgment, belonging perhaps to a backwater of mathematics) and shall take up first a paper of 1916 relating to several arithmetical functions (*Collected Papers*, pp. 136–162).

Let $\sigma_s(n)$ denote the sum of the sth powers of the divisors of the positive integer n and let $\sigma_s(0)=\frac{1}{2}\zeta(-s)$, where $\zeta(s)$ is the Riemann Zeta-function $1^{-s}+2^{-s}+3^{-s}+\cdots$. Write

$$\sum\nolimits_{r,s}(n) = \sigma_r(0)\sigma_s(n) + \sigma_r(1)\sigma_s(n-1) + \cdots + \sigma_r(n)\sigma_s(0).$$

Then Ramanujan proves that

$$\begin{aligned}\sum\nolimits_{r,s}(n) &= \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\,\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}\,\sigma_{r+s+1}(n) \\ &\quad + \frac{\zeta(1-r)+\zeta(1-s)}{r+s}\,n\sigma_{r+s-1}(n) + O\left\{n^{2(r+s+1)/3}\right\},\end{aligned}$$

whenever r and s are positive odd integers; and he shows that the error term denoted by O is zero in the following nine cases: $r=1$, $s=1, 3, 5, 7, 11$; $r=3$, $s=3, 5, 9$; $r=5$, $s=7$.

In the proof of this deep-lying result skillful use is made of transformations involving identities from the theory of trigonometric and elliptic functions. Many striking relations are obtained. In the course of the argument the author is led to introduce (among others) the number-theoretic function $\tau(n)$ defined by the identity

$$\sum_{n=1}^{\infty}\tau(n)x^n = x\left\{(1-x)(1-x^2)(1-x^3)\cdots\right\}^{24}.$$

He establishes several remarkable properties of this function and conjectures others. Later writers have pushed the investigation further.

It is shown that a similar analysis may be applied to interesting arithmetical functions of a different kind and in particular to the function $r_{2s}(n)$ which denotes the number of ways of representing the positive integer n as a sum of $2s$ squares. But it is convenient to treat this in connection with the analysis of the next paper.

4. *Certain trigonometric sums and expansions in infinite series.* The memoir of 1918 reprinted on pages 179–199 of Ramanujan's *Collected Papers* should be

read in connection with the one treated in §3. The two taken together contain a contribution of great interest. The later one has the merit that much of it is easy to read.

Let $c_s(n)$ denote the sum of the nth powers of the primitive sth roots of unity, so that

$$c_s(n) = \sum_{\lambda} \cos \frac{2\pi\lambda n}{s}$$

where λ is prime to s and not greater than s. Ramanujan's principal object in this paper is to obtain expressions for a variety of well-known arithmetical functions of n in the form of the series

$$\sum_{s} a_s c_s(n)$$

where a_s is independent of n. A typical formula is

$$\sigma(n) = \frac{\pi^2 n}{6}\left\{\frac{c_1(n)}{1^2} + \frac{c_2(n)}{2^2} + \frac{c_3(n)}{3^2} + \cdots\right\}$$

where $\sigma(n)$ is the sum of the divisors of n. Many of the formulas obtained are elementary in the technical sense that they can be proved by a combination of processes involving only finite algebra and simple general theorems concerning infinite series. But some are of a deeper character and can be proved only by means of theorems which seem to depend essentially on the theory of analytic functions. The remarkable formula

$$c_1(n) + \tfrac{1}{2}c_2(n) + \tfrac{1}{3}c_3(n) + \cdots = 0$$

is typical of the latter class.

It is convenient to employ the symbol $\eta_s(n)$ to denote the sum of the nth powers of all the sth roots of unity, whence $\eta_s(n) = s$ or 0 according as n is or is not a multiple of s. Then we have

$$\eta_s(n) = \sum_{\delta} c_\delta(n)$$

where δ runs over the divisors of s. Thence it follows that

$$c_s(n) = \sum_{\delta} \mu(\delta')\eta_\delta(n)$$

where δ is a divisor of s, $\delta\delta' = s$ and $\mu(\nu)$ is defined by the identity

$$\sum \mu(\nu)\nu^{-s} = \{\zeta(s)\}^{-1}.$$

Ramanujan also employs the function $s_q(n)$,

$$s_q(n) = \sum_{\lambda}(-1)^{(\lambda-1)/2}\sin\frac{2\pi\lambda n}{q}, \quad q = 1, 2, 3, \cdots,$$

where the sum is taken for the $\phi(q)$ integers which are prime to q and do not exceed q. He also uses the function

$$\gamma_q(n) = c_q(n) \cos \tfrac{1}{2}\pi s(q-1) - s_q(n) \sin \tfrac{1}{2}\pi s(q-1),\ q = 1, 2, 3, \cdots,$$

where s is a given positive integer.

Using $\sigma_s(n)$ to denote the sum of the sth powers of the divisors of the positive integer n, he shows that

$$\sigma_s(n) = n^s\zeta(s+1)\left\{\frac{c_1(n)}{1^{s+1}} + \frac{c_2(n)}{2^{s+1}} + \frac{c_3(n)}{3^{s+1}} + \cdots\right\},\quad s > 0.$$

Let $p_1, p_2, p_3, \cdots$, be the prime divisors of n and write

$$\phi_s(n) = n^s(1 - p_1^{-s})(1 - p_2^{-s})(1 - p_3^{-s}) \cdots .$$

Then $\phi_1(n) = \phi(n)$, the number of integers not greater than n and prime to n. Then Ramanujan proves that

$$\frac{\phi_s(n)\zeta(s+1)}{n^s} = \frac{\mu(1)c_1(n)}{\phi_{s+1}(1)} + \frac{\mu(2)c_2(n)}{\phi_{s+1}(2)} + \frac{\mu(3)c_3(n)}{\phi_{s+1}(3)} + \cdots .$$

If $r_{2s}(n)$ denotes the number of ways of representing the positive integer n as a sum of $2s$ squares and $\delta_{2s}(n)$ denotes the function

$$\delta_{2s}(n) = \frac{\pi^s n^{s-1}}{(s-1)!} g_s(n)$$

where

$$g_s(n) = \frac{\gamma_1(n)}{1^s} + \frac{\gamma_4(n)}{2^s} + \frac{\gamma_3(n)}{3^s} + \frac{\gamma_8(n)}{4^s} + \frac{\gamma_5(n)}{5^s} + \frac{\gamma_{12}(n)}{6^s} + \frac{\gamma_7(n)}{7^s} + \cdots,\quad s > 1,$$

and $g_1(n)$ denotes half of what the second member of the last equation becomes for $s=1$, then, by aid of results in the paper described in our §3, Ramanujan shows that $r_{2s}(n) = \delta_{2s}(n)$ for $s = 1, 2, 3, 4$ and in general that

$$r_{2s}(n) = \delta_{2s}(n) + O(n^{s/2}).$$

This paper contains many other remarkable expansions of number-theoretic functions obtained by aid of the trigonometric sums $c_q(n)$, $s_q(n)$, $\gamma_q(n)$.

5. *Partitions of integers.* Let $p(n)$ denote the number of unrestricted partitions of n, that is, the number of ways of representing n as a sum of positive integers. Then, in accordance with a classic result, $p(n)$ is the coefficient of x^n in the expansion

$$f(x) = 1 + \sum_{n=1}^{\infty} p(n)x^n = \left\{(1-x)(1-x^2)(1-x^3)\cdots\right\}^{-1}.$$

Hardy has expressed the opinion that Ramanujan shows at his very best in the theory of partitions and the allied parts of the theories of elliptic functions and continued fractions. Besides his contributions in collaboration with Hardy (to be treated in §IV) and those in the papers already mentioned, the most important published work of Ramanujan in the theory of partitions is contained in the three articles appearing on pages 210–213, 214–215, 232–238 of his *Collected Papers*.

In the first of these papers Ramanujan conjectures the following theorem: If $\delta = 5^a 7^b 11^c$ and $24\lambda \equiv 1$ *mod* δ, then

$$p(\lambda),\ p(\lambda + \delta),\ p(\lambda + 2\delta),\ \cdots \equiv 0 \bmod \delta.$$

He found the theorem supported by all the available evidence but he was unable to obtain a general proof. He proved simply, however, that $p(5m+4) \equiv 0$ *mod* 5, $p(7m+5) \equiv 0$ *mod* 7. With greater difficulty he was able to prove some other independent cases of the conjectured theorem. In the course of this paper several identities were established, the most elegant of which is the following

$$p(4) + p(9)x + p(14)x^2 + \cdots = 5\frac{\{(1 - x^5)(1 - x^{10})(1 - x^{15}) \cdots\}^5}{\{(1 - x)(1 - x^2)(1 - x^3) \cdots\}^6},$$

from which it appears directly that $p(5m+4)$ is divisible by 5. In speaking of the beautiful formulas in Ramanujan's work, Hardy says that if he had to select one from all of them he would agree with Major MacMahon in choosing this one.

The third paper just mentioned has intimate contacts with several other papers of Ramanujan. It contains new proofs of the two congruences just mentioned and the first published proof of the congruence $p(11m+6) \equiv 0$ *mod* 11.

6. *Remarks.* This very inadequate sketch of Ramanujan's contributions to the theory of numbers must fail to call forth any proper appreciation of the character of his labors. This can be gotten only from a first-hand contact with the work itself and from a realization of the stimulus which it imparts. We can not here take the space to follow up the extensions of Ramanujan's work except as they may appear incidentally in later sections.

II. REPRESENTATIONS AS SUMS OF SQUARES

7. *Orthogonal arithmetical functions.* A study of Ramanujan's trigonometric sums and of his expansions in terms of them has led me (in an unpublished paper) to introduce the general notion of orthogonal arithmetical functions. If two functions $f(n)$ and $g(n)$ are defined for $n = 1, 2, 3, \cdots$, they will be called orthogonal when (1) they have positive integral periods α and β and when (2) the sum $\sum f(n)g(n)$ (for $n = 1, 2, \cdots, l$) has the value 0, l being the least common multiple of α and β. If two (finite or infinite) sets of functions $f_1(n), f_2(n), \cdots$, and $g_1(n), g_2(n), \cdots$, are such that $f_i(n)$ and $g_j(n)$ are orthogonal when $i \neq j$, then the two sets are said to be biorthogonal. If the single set $f_1(n), f_2(n), \cdots$,

is such that every function in it is orthogonal to every other function in it, then it is said to be a set of orthogonal arithmetical functions.

For the named biorthogonal sets we shall be interested mainly in the case when the supplementary condition

$$\sum_{n=1}^{l_i} f_i(n)g_i(n) \neq 0$$

is satisfied for each i, where l_i is the least common multiple of the periods of $f_i(n)$ and $g_i(n)$. For the case of a single set the supplementary condition takes the form

$$\sum_{n=1}^{\alpha_i} f_i^2(n) \neq 0$$

where α_i is the period of $f_i(n)$. It is surely satisfied when $f_i(n)$, for every fixed i, is real-valued for all n and is not identically zero.

In the cases of both biorthogonal and orthogonal sets satisfying the supplementary condition there is an expansion theory, analogous to that of Fourier series, for the expansion of suitable arithmetical functions $h(n)$ in the form

$$h(n) = a_1f_1(n) + a_2f_2(n) + a_3f_3(n) + \cdots,$$

where the coefficients a_i are independent of n.

Ramanujan's trigonometric sums have the orthogonality properties expressed by the following relations:

$$\sum_{n=1}^{qp} c_q(n)c_p(n) = 0 \text{ if } p \neq q, \quad \sum_{n=1}^{q} c_q^2(n) = q\phi(q);$$

$$\sum_{n=1}^{qp} c_q(n)s_p(n) = 0 \text{ if } p \neq q;$$

$$\sum_{n=1}^{qp} s_q(n)s_p(n) = 0 \text{ if } p \neq q, \quad \sum_{n=1}^{q} s_q^2(n) = \tfrac{1}{4}q \sum_{\lambda}\left\{(-1)^{(\lambda-1)/2} - (-1)^{(q-\lambda-1)/2}\right\}^2$$

where λ runs over the integers prime to q and not exceeding q. In particular,

$$\sum_{n=1}^{4q} s_{4q}^2(n) = 4q\phi(4q).$$

The corresponding orthogonality properties of $\gamma_q(n)$ are implied by the foregoing.

8. *Sums of squares*. In a paper in the *Annals of Mathematics*, 1931, I have employed these orthogonality properties and certain deep-lying results due to Ramanujan in establishing several asymptotic relations which will be reproduced in §§8–10.

With the notation of the latter part of §4, we have

$$\frac{1}{m}\sum_{n=1}^{m}\gamma_\rho(n)n^{1-s}\delta_{2s}(n) = \frac{\pi^s\epsilon_\rho(s)\phi(\rho)}{(s-1)!\rho^s} + O\left(\frac{1}{m}\right), \quad s > 3,$$

where $\epsilon_\rho(s)$ is 1, 0 or 2^s according as ρ is odd, $\rho\equiv 2 \bmod 4$ or $\rho\equiv 0 \bmod 4$; also,

$$\frac{1}{m}\sum_{n=1}^{m}\gamma_\rho(n)n^{1-s}r_{2s}(n) = \frac{\pi^s\epsilon_\rho(s)\phi(\rho)}{(s-1)!\rho^s} + C\left(\frac{1}{m}\right), \quad s > 3.$$

The weight factors $\gamma_\rho(n)$ are suitable for smoothing out (in the limit) the irregularities of each of the functions $n^{1-s}r_{2s}(n)$ and $n^{1-s}\delta_{2s}(n)$. In particular, if we take $\rho=1$ we have the following general theorem: The functions $n^{1-s}r_{2s}(n)$ and $n^{1-s}\delta_{2s}(n)$ are each in the mean (on the average) equal to $\pi^s/(s-1)!$ if $s>3$, the error term for the average of the values $n=1, 2, \cdots, m$ being $O(1/m)$.

9. *Sums of triangular numbers.* If $R_{2s}(n)$ denotes the number of ways of representing n as a sum of $2s$ triangular numbers, where s is a given positive integer, and if we write $R_{2s}(n)=D_{2s}(n)+E_{2s}(n)$, where

$$D_{2s}(n) = \frac{(\frac{1}{2}\pi)^s}{(s-1)!}(n+\tfrac{1}{4}s)^{s-1}\sum_{\mu=1}^{\infty}(2\mu-1)^{-s}(-1)^{(\mu-1)s}c_{2\mu-1}\left(\frac{4n+s}{\eta(s)}\right), \quad s > 1,$$

$\eta(s)$ being 1, 2 or 4 according as s is odd, is twice an odd number or is a multiple of 4, while $D_2(n)$ is half of what the last second member becomes for $s=1$, then Ramanujan has shown that $E_{2s}(n)=0$ for $s=1, 2, 3, 4$ while in general $R_{2s}(n)\sim D_{2s}(n)$. We have the following mean value relations:

$$\frac{1}{m}\sum_{n=1}^{m}c_{2\lambda-1}\left(\frac{4n+s}{\eta(s)}\right)(n+\tfrac{1}{4}s)^{1-s}D_{2s}(n) = \frac{(-1)^{(\lambda-1)s}(\frac{1}{2}\pi)^s\phi(2\lambda-1)}{(s-1)!(2\lambda-1)^s} + O\left(\frac{1}{m}\right), \quad s > 3,$$

$$\frac{1}{m}\sum_{n=1}^{m}c_{2\lambda}\left(\frac{4n+s}{\eta(s)}\right)(n+\tfrac{1}{4}s)^{1-s}D_{2s}(n) = O\left(\frac{1}{m}\right), \quad s > 3,\ s\equiv 0 \bmod 4,$$

where in each case λ ranges over the set $1, 2, 3 \cdots$.

10. *Ramanujan's function* $\sum_{r,s}(n)$. This function (defined in §3) has the following property:

$$\frac{1}{m}\sum_{n=1}^{m}c_\rho(n)n^{-r-s-1}\sum\nolimits_{r,s}(n) = \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\,\frac{\phi(\rho)\zeta(r+1)\zeta(s+1)}{\rho^{r+s+2}} + O\left(\frac{\log m}{m}\right)$$

whenever r and s are positive odd integers.

III. EXPANSIONS OF ARITHMETICAL FUNCTIONS IN INFINITE SERIES

11. *Generalizations of Ramanujan's trigonometric sums.* By a character $\chi(a)$ mod k we mean (as usual) a function $\chi(a)$ such that (1) $\chi(a)=0$ if a is not prime to k, (2) $\chi(1)=1$, (3) $\chi(a_1a_2)=\chi(a_1)\chi(a_2)$, (4) $\chi(a_1)=\chi(a_2)$ if $a_1\equiv a_2$ mod k. By $\chi(a)$, $\chi_1(a)$, $\chi_2(a)$, $\cdots$, we denote any characters moduli k, k_1, k_2, $\cdots$, respectively, and by $\chi_0(a)$, $\chi_{10}(a)$, $\chi_{20}(a)$, $\cdots$, we denote the principal characters

for these moduli, the principal character $\chi(a)$ modulo k being that character for which $\chi(a)=1$ or 0 according as a is or is not prime to k. If a symbol for a character is written with an argument which is not an integer it may have for this argument any conveniently assigned value, since it will appear in our formulas only when multiplied by a quantity having the value zero.The character which is equal to unity for all arguments will sometimes be replaced by 1. We use $\mu(a)$ and $\eta_q(n)$ in the senses defined in §4.

We introduce the functions

$$c_q^{(\lambda)}(n, \chi, \chi_1, \chi_2) = \sum_{d|q} \mu\left(\frac{q}{d}\right)\chi\left(\frac{q}{d}\right)\chi_1(d)\chi_2\left(\frac{n}{d}\right)\eta_d^{\lambda}(n)$$

where the sum is taken for the divisors d of q and where λ is positive. These functions reduce to Ramanujan's function $c_q(n)$ when $\lambda=1$ and $\chi(a)\equiv\chi_1(a)\equiv\chi_2(a)\equiv 1$. These new functions are rich in properties generalizing many of the properties of Ramanujan's functions $c_q(n)$. In particular, important expansions of number-theoretic functions may be derived by their aid, as I have shown in *Proc. Nat. Acad. Sc.* for September 1930 and in an unpublished paper. A few of the results obtained by aid of these expansions are given in the next two sections.

Incidental to these investigations the following remarkable relation was discovered:

$$\sum_{q=1}^{\infty} \frac{1}{q} c_q^{(\lambda)}(n, \chi_0, \chi_1, \chi_2) = 0.$$

When $\lambda=1$ and $\chi_0(a)\equiv\chi_1(a)\equiv\chi_2(a)\equiv 1$ the foregoing equation expresses a deep-lying result due to Ramanujan (see our §4).

12. *Functions of the divisors of an integer.* We introduce the function

$$\sigma_s(n, \chi_1, \chi_2) = \sum_{d|n} \chi_1(d)\chi_2\left(\frac{n}{d}\right)d^s$$

where the sum is taken for the divisors d of n. Here s denotes the complex variable $\sigma+it$ where σ and t are real. By $L(s, \chi)$ we denote the L-series

$$L(s, \chi) = \sum_{a=1}^{\infty} \frac{\chi(a)}{a^s}.$$

Then if $\sigma+\lambda>1$ we have the infinite expansions

$$\sigma_s(n, \chi_1, \chi_2) = n^s L(s+\lambda, \chi) \sum_{\nu=1}^{\infty} \frac{c_\nu^{(\lambda)}(n, \chi, \chi_2, \chi_1)}{\nu^{s+\lambda}},$$

$$\sigma_{-s}(n, \chi_1, \chi_2) = L(s+\lambda, \chi) \sum_{\nu=1}^{\infty} \frac{c_\nu^{(\lambda)}(n, \chi, \chi_1, \chi_2)}{\nu^{s+\lambda}}.$$

Numerous consequences may readily be drawn from these formulas. In particular, we have

$$\frac{1}{m}\sum_{n=1}^{m} c_\rho(n)\sigma_{-s}(n, 1, 1) = \frac{\phi(\rho)}{\rho^{s+1}}\left(1+\frac{1}{2^{s+1}}+\frac{1}{3^{s+1}}+\cdots\right)+O\left(\frac{1}{m}\right), \sigma > 2,$$

$$= \frac{\phi(\rho)}{\rho^{2\lambda}}\,\pi^{2\lambda}\,\frac{2^{2\lambda-1}B_\lambda}{(2\lambda)!}+O\left(\frac{1}{m}\right) \text{ if } s = 2\lambda - 1,$$

where in the last member λ is an integer greater than unity and the B's are the Bernoulli numbers $B_1=1/6$, $B_2=1/30, \cdots$. More generally, we have

$$\frac{1}{m}\sum_{n=1}^{m} c_\rho(n)\sigma_{-s}(n, \chi, 1) = L(s+1, \chi)\rho^{-s-1}\chi(\rho)\phi(\rho) + O(1/m), \sigma > 2,$$

an asymptotic relation special cases of which have appeared in several papers (see Dickson's *History of the Theory of Numbers*, Vol. I, pp. 291–4, 301, 322).

We have $\sigma_{-s}(n, \chi_0, 1) = \gamma_{-s}(n, k) =$ the sum of the sth powers of the reciprocals of those divisors of n which are prime to the modulus k. It may be shown that

$$\frac{1}{m}\sum_{n=1}^{m} c_\rho(n)\gamma_{-s}(n, k) = \frac{\chi_0(\rho)\phi(\rho)}{\rho^{s+1}}\sum_q{}' \frac{1}{q^{s+1}}+O\left(\frac{1}{m}\right), \sigma > 2,$$

where the prime in $\sum'$ indicates that q ranges over the integers which are prime to k.

To illustrate the meaning of these formulas we take $\rho=1$ in the third preceding equation and obtain the following theorem:

When λ is an integer greater than unity, the sum of the $(2\lambda-1)$th powers of the reciprocals of the divisors of n is in the mean (on the average) equal to $\pi^{2\lambda}2^{2\lambda-1}B_\lambda/(2\lambda)!$, the error term of the average for $n=1,2, \cdots, m$ being $O(1/m)$.

13. *Representations as sums of squares*. The foregoing properties of σ-functions are intimately connected with expansions of other number-theoretic functions. We merely state three particular results. Using $N[n=x^2+y^2]$ to denote the number of ways of representing n as a sum of two squares and employing similar notations for other cases, we have

$$N[n = x^2 + y^2] = 2\pi\left\{c_1(n) + \frac{1}{5}c_5(n) + \frac{1}{9}c_9(n) + \cdots\right\},$$

$$N[n = x^2 + 2y^2] = \pi(2)^{1/2}\left\{c_1(n) + \frac{1}{3}c_3(n) + \frac{1}{9}c_9(n) + \frac{1}{11}c_{11}(n) + \cdots\right\},$$

$$N[2^\alpha m = x^2 + y^2 + z^2 + t^2, m \text{ odd}] = g(\alpha)\pi^2 m\left\{c_1(m) + \frac{c_3(m)}{3^2} + \frac{c_5(m)}{5^2} + \cdots\right\},$$

where $g(\alpha)=1$ or 3 according as $\alpha=0$ or $\alpha>0$ and where in the second formula the subscripts appearing are those of the forms $8x+1$ and $8x+3$.

IV. THEORY OF PARTITIONS

14. *Older and newer problems.* From the time of Euler onwards to 1917 and 1918 when Hardy and Ramanujan (Ramanujan's *Collected Papers*, pp. 239–241, 244, 276–309) took up the problem, the theory of the partition of integers into positive integral summands had been developed almost entirely from an algebraic point of view. The results consisted mainly of formal identities, many of which are quite ingenious and of beautiful character. Of genuine asymptotic formulas it is fair to say that there were none, the only exceptions to this statement consisting of special results mainly of algebraic interest and essentially trivial from the point of view of asymptotic formulas. So true indeed is this statement that Hardy and Ramanujan found in the literature of the subject no allusion whatever to the question of the order of magnitude of the fundamental partition function $p(n)$, where $p(n)$ denotes the number of unrestricted partitions of n and is the coefficient of x^n in the expansion of the function

$$f(x) = 1 + \sum_{n=1}^{\infty} p(n)x^n = \{(1-x)(1-x^2)(1-x^3)\cdots\}^{-1}.$$

From Cauchy's theorem we have

$$p(n) = \frac{1}{2\pi i}\int_\Gamma \frac{f(x)}{x^{n+1}}\,dx$$

where Γ is a path which lies within the unit circle and encloses the point 0. The dominant idea in the method of Hardy and Ramanujan lies in the use of Cauchy's integral formula to obtain asymptotic properties of $p(n)$. This idea, extremely obvious as it is and dominating as it does a very large part of the modern analytic theory of numbers, appears never before to have been employed in the problem of partitions. Perhaps one reason for this is in the fact that since the time of Euler those who have investigated the theory of partitions have been interested primarily in algebra. But a more important reason lies in the extreme complexity of the generating function $f(x)$ near a point on the unit circle. The function does not exist outside of this circle, and every point of the circle is an essential singularity of the function. No part of the contour of integration can be so deformed as to become obviously negligible; every element of it requires special study; there is no obvious method of obtaining a dominant term in the value of $p(n)$.

This makes the difficulties appear very serious. But there is another fact which could lead one to hope for success. The formulas of the theory of the linear transformation of the elliptic functions afford a very powerful tool for studying the behavior of $f(x)$ near an assigned point of the unit circle. It is to an appropriate use of these formulas that the authors owe the startling success of their investigation.

15. *Elementary results.* Hardy and Ramanujan treat the problem of partitions not only by means of the deep-lying methods just referred to but also by

others of a more elementary character. In such problems it is desirable to distinguish clearly the stages to which progress can be made by progressively more deep-lying methods. The earlier results so obtained are likely to be completely superseded by the later; but it may be expected that the more elementary methods will often admit applications to other problems in which the more subtle analysis is impracticable.

By means of quite elementary methods depending on the Euler identity

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} = 1 + \frac{x}{(1-x)^2} + \frac{x^4}{(1-x)^2(1-x^2)^2} + \cdots .$$

Hardy and Ramanujan first prove the existence of positive constants A and B such that for all sufficiently large values of n we have

$$An^{1/2} < \log p(n) < Bn^{1/2}.$$

This raises the question whether there exists a constant C such that $\log p(n) \sim Cn^{1/2}$. By methods more difficult but not appealing to the theory of analytic functions they show that the answer is affirmative and that the value of C is $C=\pi(2/3)^{1/2}$. This means that

$$p(n) = \exp\left\{\pi(2n/3)^{1/2}(1+\epsilon)\right\}$$

where ϵ is small when n is large.

The next step of Hardy and Ramanujan was to replace the last result by the more precise formula

$$p(n) \sim (4n\cdot 3^{1/2})^{-1} \exp\left\{\pi(2n/3)^{1/2}\right\}.$$

For the proof of this result they required more powerful machinery, including the Cauchy integral formula for $p(n)$ already given and an important functional relation arising in the theory of elliptic functions. (This formula was independently discovered by Uspensky in 1920.)

16. *Asymptotic character of $p(n)$.* To obtain still more satisfactory results they found it necessary to construct an auxiliary function $F(x)$ regular on the unit circle except at $x=1$ and having there a singularity as nearly as possible like that of $f(x)$ at $x=1$. This procedure was suggested by the fact that though every point of the unit circle is a singularity of $f(x)$ yet in a certain sense the point $x=1$ is much the heaviest singularity. They hoped then to attain the goal by applying Cauchy's theorem to $f(x)-F(x)$ instead of to $f(x)$. Their hope was justified and they obtained the result

$$p(n) \sim (2\pi\cdot 2^{1/2})^{-1}\frac{d}{dn}(e^{C\lambda_n}/\lambda_n) + O(e^{Dn^{1/2}})$$

where $C=\pi(2/3)^{1/2}$, $\lambda_n=(n-1/24)^{1/2}$ and D is any number greater than $\frac{1}{2}C$.

Since the error term O in this formula is of exponential type and might therefore be expected ultimately to increase with very great rapidity, the authors were surprised to find that the formula gave exceedingly good results for fairly large values of n. They were therefore encouraged to push the approximation further. As they did so and as the successive formulas met in a striking way the test of numerical data, the authors were led to look for much more than they had first expected and indeed to seek a formula for $p(n)$ which not only exhibits its order of magnitude but may be used to compute its exact value. They succeeded in obtaining precisely this startling result.

During the process of their investigation the authors had the use of numerical data supplied by Major MacMahon and Mr. H. B. C. Darling; they acknowledge particular indebtedness to the former. They add: "It is certain that, without the encouragement given by the results of these calculations, we should never have attempted to prove theoretical results at all comparable in precision with those which we have enunciated."

The main theorem obtained is the following:

Suppose that

$$\phi_q(n) = \frac{1}{2\pi}(\tfrac{1}{2}q)^{1/2}\frac{d}{dn}(e^{C\lambda_n/q}/\lambda_n),\ C = \pi(\tfrac{2}{3})^{1/2},\ \lambda_n = (n - 1/24)^{1/2}$$

for all positive integers q; that p is a positive integer less than q and prime to q; that $\omega_{p,q}$ is a $(24q)$th root of unity, defined when p is odd by the formula

$$\omega_{p,q} = \left(\frac{-q}{p}\right)\exp\left[-\left\{\frac{1}{4}(2 - pq - p) + \frac{1}{12}\left(q - \frac{1}{q}\right)(2p - p' + p^2p')\right\}\pi i\right],$$

and when q is odd by the formula

$$\omega_{p,q} = \left(\frac{-p}{q}\right)\exp\left[-\left\{\frac{1}{4}(q - 1) + \frac{1}{12}\left(q - \frac{1}{q}\right)(2p - p' + p^2p')\right\}\pi i\right],$$

where (a/b) is the symbol of Legendre and Jacobi, and p' is any positive integer such that $1+pp'$ is divisible by q; that

$$A_q(n) = \sum_p \omega_{p,q}e^{-2np\pi i/q};$$

and that α is any positive constant and ν is the integral part of $\alpha n^{1/2}$.

Then

$$p(n) = \sum_{q=1}^{\nu} A_q(n)\phi_q(n) + O(n^{-1/4}),$$

so that $p(n)$ is, for all sufficiently large values of n, the integer nearest to

$$\sum_{q=1}^{\nu} A_q(n)\phi_q(n).$$

V. WARING'S PROBLEM

17. *Formulation and earlier history of the problem.* Let us consider the problem of the representation of positive integers n as sums of positive integral kth powers. There is always a solution since n is a sum of n ones. For the moment let us consider k fixed. When n is given there is then a *minimum* number of kth powers of which n is the sum. Thus 23 is the sum of nine cubes $(23=2.2^3+7.1^3)$ but of no smaller number. When k is fixed and n varies over all positive integers in order of increasing magnitude, will the minimum number of kth powers with sum n remain bounded? It does so when k is 2; for then it never exceeds 4. Waring's problem, in the restricted sense, is the problem of proving that this minimum number remains bounded in the general case.

To state the problem formally, let k be given. If a number m exists such that every positive integer n is a sum of not more than m kth powers, then there is a *least* value of m for which this is true. This least number, if it exists, depends on k and is uniquely determined when k is given. We denote it by $g(k)$. Thus, by definition, $g(k)$ is the least number, if such a number exists, for which it is true that every positive integer is the sum of not more than $g(k)$ kth powers. We have seen that $g(2)=4$.

In his *Meditationes Algebraicae* (Cambridge, 1770, pp. 204–5; 3rd edition, 1782, pp. 349–350) Waring stated that every positive integer is a sum of at most four integral squares, of at most 9 positive integral cubes, of at most 19 integral fourth powers, and so forth. There is a lack of precision concerning what is here meant by the words of continuation, but it appears that Waring intended to assert precisely the existence of $g(k)$. He implies that $g(2)=4$, $g(3)=9$, $g(4)=19$. The first had already been proved by Lagrange. The others we shall discuss later.

Let $[\alpha]$ denote the integral part of α. It was observed early and is easily verified that the number $([(3/2)^k]-1)2^k+2^k-1$, which is less than 3^k, requires at least $[(3/2)^k]+2^k-2$ kth powers for its representation, 2^k-1 of them being ones and the remaining being each 2^k. For $k=2,3,\cdots$, we therefore have 4, 9, 19, 37, 73, 143, 279, $\cdots$, as lower bounds for $g(2)$, $g(3)$, $\cdots$.

Waring gave no argument to support his statement. It seems that he intended to offer it only as a highly plausible speculation. But what is needed is proof; and that it has been terribly difficult to supply. In the time of Waring it was known that $g(2)=4$. In 1859 Liouville proved the existence of $g(4)$ and that its value is not greater than 53; his method is quite simple. The existence of $g(3)$ was first established in 1895 by Maillet who proved that $g(3)\leqq 17$. Other particular results followed by special methods. Waring's restricted problem of proving the existence of $g(k)$ for every positive integer k was first completely solved by Hilbert in 1909, 139 years after its formulation by Waring.

The numbers 23 and 239 require nine cubes each for their representation. There is no other number known requiring so many, and the matter has been tested numerically up to at least 40,000. In 1909 Landau proved the very beauti-

ful theorem that the number of positive integers requiring as many as 9 cubes is finite. This suggests the introduction of the number $G(k)$ defined as being the least number for which it is true that every number from a certain point onwards is a sum of not more than $G(k)$ kth powers. Since $g(k)$ exists it follows that $G(k)$ exists and that $G(k) \leqq g(k)$. The number $G(k)$ appears to be more fundamental than $g(k)$.

It may be further asked: What is the least integer $G_1(k)$ such that nearly all positive integers n are each expressible as a sum of not more than $G_1(k)$ positive integral kth powers? [To say that "nearly all" positive integers have a given property is to say that $\psi(n)/n \to 1$ as n becomes infinite where $\psi(n)$ is the number of positive integers not exceeding n and having the property in consideration.] From what precedes it is clear that $G_1(k)$ exists and that $g(k) \geqq G(k) \geqq G_1(k)$. It may be shown that $G_1(2) = 4$.

Hardy and Littlewood (see *Mathematische Zeitschrift* 23 (1925): 1–37 and the papers there cited) in recent years have made very remarkable contributions to the study of Waring's problem in the larger sense, namely, the investigation of the magnitude of the numbers $g(k)$, $G(k)$, $G_1(k)$. In §§18 and 19 we shall very briefly describe their work.

18. *The methods of Hardy and Littlewood.* In order to come to close grips with Waring's problem in the larger sense, to get into real contact with the actual values of $g(k)$, $G(k)$, $G_1(k)$, still more in order to treat the problem of the number of representations of a given integer as a sum of s kth powers, it is necessary to have powerful tools and to have in reach the means of handling them effectively. Only the modern theory of functions can be expected to yield the necessary assistance. It was from such considerations as these that Hardy and Littlewood were led to seek means of applying Cauchy's integral theorem to the problem; and it is through such application that they have obtained their results.

Let $r(n, k, s)$ denote the number of representations of n as a sum of s kth powers. Then, with suitable conventions concerning the way in which the number of representations is to be counted, we have

$$1 + \sum_{n=1}^{\infty} r(n, k, s)x^n = \{f(x)\}^s \equiv (1 + 2x^{1^k} + 2x^{2^k} + 2x^{3^k} + \cdots)^s.$$

The two infinite series occurring here are convergent when $|x| < 1$. Hence we have

$$r(n, k, s) = \frac{1}{2\pi i}\int_\Gamma \frac{\{f(x)\}^s}{x^{n+1}}\, dx$$

where Γ is a closed contour lying entirely within the unit circle and enclosing the point 0.

The study of this integral is intricate and difficult. Every point of the unit circle is a singularity of $f(x)$ of a very complicated kind and the circle itself is a

barrier beyond which the function does not exist. In his Oxford Inaugural Lecture Hardy has described as follows the fundamental idea employed in overcoming the difficulty:

Among the continuous mass of singularities which covers up the circle, it is possible to pick out a class which to a certain extent dominates the rest. These special singularities are those associated with the *rational* points of the circle, that is to say, the points

$$x = e^{2p\pi i/q},$$

where p/q is a rational fraction in its lowest terms. . . . In the neighbourhood of these points the behaviour of the function is, sufficiently complex indeed, but simpler than elsewhere. The function has, to put the matter in a rough and popular way, a general tendency to become large in the neighbourhood of the unit circle, but this tendency is most pronounced near these particular points. They are not only the *simplest* but also the *heaviest* singularities; their weight is greatest when the denominator q is smallest, decreases as q increases, and (as a physicist would say) becomes infinitely small when q is infinitely large. There is, therefore, at any rate, the hope that we may be able to isolate the contributions of each of these selected points, and obtain, by adding them together, a series which may give a genuine approximation to our coefficient.

While this shows the point of departure of Hardy and Littlewood it can give no indication of the brilliance with which they have carried the program through to the results now to be stated.

19. *The results of Hardy and Littlewood.* With respect to the magnitude of $G_1(k)$ Hardy and Littlewood have proved the following theorem: If $k=3$ or $k \geqq 5$ then almost all positive integers are sums of

$$(\tfrac{1}{2}k - 1)2^{k-1} + 3$$

non-negative integral kth powers, and in particular of 5 cubes, 27 fifth powers, and 67 sixth powers; almost all positive integers are sums of 15 biquadrates, and $G_1(4) = 15$.

With respect to $G(k)$ they have the theorem: If $k > 3$ then all large positive integers are sums of

$$(\tfrac{1}{2}k - 1)2^{k-1} + k + 5 + \left[\frac{(k-2)\log 2 - \log k + \log(k-2)}{\log k - \log(k-1)}\right]$$

non-negative integral kth powers, where $[\alpha]$ denotes the integral part of α; and in particular of 19 biquadrates, 41 fifth powers, and 87 sixth powers.

The case of biquadrates is particularly interesting. Since $79(=4.2^4+15.1^4)$ actually requires 19 biquadrates, so that $g(4) \geqq 19$, the problem of determining the exact value of $g(4)$ is thus reduced (in theory at any rate) to a problem of computation.

From the work of Hardy and Littlewood, of Landau and of Wieferich it is known that

$$g(3) = 9, G(3) \leqq 8, \text{ and } G_1(3) \leqq 5.$$

It is known also that

$$\limsup_{k=\infty} g(k)/(k2^{k-1}) \leqq$$

It has also been proved that

$$G(k) \geqq G_1(k) \geqq k+1 \text{ if } k \geqq 2.$$

A lower bound to $g(k)$ is given in §17.

Lack of space forbids our giving Hardy and Littlewood's results concerning the asymptotic character of the number $r(n, k, s)$ of ways of representing n as a sum of s kth powers.

VI. A QUESTION RELATED TO WARING'S PROBLEM[1]

20. *Multiplicative domains numerically defined.* Separate the prime numbers into k classes $C_1, C_2, \cdots, C_k$ and associate with the class C_i a positive integer α_i for $i=1,2,\cdots,k$. Denote by D the multiplicative domain composed of 0, 1, the α_ith power of every prime in C_i $(i=1, 2, \cdots, k)$ and all the integers which may be obtained from these generators by multiplication (with repetition of factors permissible). If μ is the least common multiple of $\alpha_1, \alpha_2, \cdots, \alpha_k$, then D contains the μth power of every positive integer. It also contains other integers unless $\alpha_1=\alpha_2=\cdots=\alpha_k$.

Since every positive integer is representable as a sum of $\nu(\mu)$ μth powers of non-negative integers, where $\nu(\mu)$ depends on μ alone, it follows that for every domain D there exists an integer $N(D)$, not greater than $\nu(\mu)$, such that every positive integer is representable as a sum of $N(D)$ integers belonging to D. The problem of determining $N(D)$, and other related integers suggested by the known facts concerning Waring's problem, is perhaps of little interest in the general case. But there is a special class of domains D for which the problem appears to be not devoid of interest.

21. *The domains D_m.* Let m be an integer greater than unity. Let D_m denote the domain D which has the following generators: 0, 1, every prime factor of m, and the τth power of every prime of the form $mx+\alpha$ where α ranges over the integers less than m and prime to m, and where τ $(=\tau_\alpha)$ is the exponent to which α belongs modulo m, so that $t=\tau$ is the least positive integer t such that $\alpha^t \equiv 1$ mod m. For such a domain D_m we propose the following questions (suggested by Waring's problem). What is the least integer $g(m)$ such that every positive integer is a sum of $g(m)$ integers belonging to D_m? What is the least integer $G(m)$ such that all but a finite number of positive integers are each expressible as a sum of $G(m)$ integers belonging to D_m? What is the least integer $G_1(m)$ such that nearly all positive integers are each expressible as a sum of $G_1(m)$ integers belonging to D_m? It is obvious that $g(m) \geqq G(m) \geqq G_1(m)$.

In the paper referred to I have established the following results concerning the values of these numbers:

I. $g(2)=G(2)=G_1(2)=1$.
II. $g(3)=G(3)=G_1(3)=2$.
III. $g(4)=G(4)=G_1(4)=2$.

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[1] *Quarterly Journal of Mathematics* 2 (1931): 59–67.

IV. $g(6)=G(6)=G_1(6)=2$.
V. $g(8)=G(8)=G_1(8)=3$.
VI. $g(12)=3$, $G(12)=2$ or 3, $G_1(12)=2$ or 3.
VII. $g(24)=G(24)=G_1(24)=3$.

In no other cases have I succeeded in finding the values of $g(m)$, $G(m)$ and $G_1(m)$. These cases (except for the trivial case $m=2$) are characterized by the fact that they involve just those values of m which are such that 2 is the greatest exponent modulo m of any number prime to m, whence it follows that in these cases D_m contains every square. It is this fact which renders the treatment of these cases easy. I am unable to determine whether 2 or 3 is the correct value of either $G(12)$ or $G_1(12)$.

The remaining domains D_m which contain every fourth power are those for which $m=5$, 10, 15, 16, 20, 30, 40, 48, 60, 80, 120, 240. These seem to be the most promising ones for a further next investigation. It is easy to show that $G_1(p)\geqq p-1$ when p is any prime number, whence $G_1(5)\geqq 4$. There is some empirical evidence for believing that $g(5)=5$. More remarkable is the fact that every number up to 600 is a sum of two numbers in D_{10}; this raises the (as yet unanswered) question whether $g(10)=G(10)=G_1(10)=2$. Further empirical facts are given in the paper mentioned.

VII. GOLDBACH'S CONJECTURE

22. *The correspondence of Goldbach and Euler.* On June 7, 1742, Goldbach wrote to Euler that he considered it not without value that one should call attention to propositions which are very probable even though one is unable to give an actual demonstration and he hazarded the conjecture that every positive integer which is a sum of two primes is also a sum of as many primes as one wishes (unity being reckoned among the primes) up to a number of summands equal to the number itself. In his reply of June 30, 1742, Euler remarked that the foregoing proposition is easily proved if one grants a conjecture previously communicated to Euler by Goldbach, namely, that every even positive integer is a sum of two primes. Thus Euler definitely states that Goldbach had communicated to him the proposition (now known as Goldbach's conjecture) that every even number is a sum of two primes. But I do not know of any extant letter in which Goldbach actually states the theorem, though he did say (in his letter of June 7, 1742) that it appears that every number greater than two is an aggregate of three primes. Concerning the proposition that every even number is a sum of two primes, Euler remarked that he considered it a quite certain theorem although he was not able to demonstrate it.

On Nov. 18, 1752, Goldbach submitted to Euler the empirical proposition that every odd number is of the form $p+2a^2$ where p is a prime and a is a positive integer or zero. On Dec. 16, 1752, Euler replied that he had verified this theorem for all odd numbers up to 1000 and asks if the number of representations in the given form increases as the odd number itself increases. After some

further discussion, he adds: "Therefore I believe this theorem, although I would not swear to it." It is well that he would not swear to it if he is concerned not to perjure himself, since the theorem fails for $5777 = 53.109$.

23. *Early history of the Goldbach conjecture.* Descartes stated that every even number is a sum of 1, 2 or 3 primes. Waring says that every even number is a sum of two primes and every odd number is either a prime or the sum of three primes. Lagrange stated some related empirical propositions. A. de Polignac conjectured that every even number is the difference of two primes and in fact that it is a difference of two consecutive primes in an infinite number of ways. Many other empirical theorems have been stated which might be associated with these, but we shall not take time to give them.

Desboves (*Nouv. Ann. Math.* 14 (1855): 281–4) formulated the following conjecture (verified up to 10,000): Every even number except 2 is the sum of two prime numbers in at least two ways; and, when the even number is the double of an odd number, it is always simultaneously the sum of two primes of the form $4n+1$ and the sum of two primes of the form $4n-1$. The theorem as thus formulated has contacts with other propositions. It implies that between n and $2n$ $(n>6)$ there are always at least two prime numbers. That every positive integer is a sum of four squares is implied by the theorem and the fact that a prime of the form $4n+1$ is a sum of two squares.

The problem has been considered also by Sylvester, Stäckel, Merlin and Brun in papers which set forth conjectured or otherwise insecure or incomplete results. For an analysis of them see the first paper by Hardy and Littlewood referred to in our §24.

In 1912 the state of Goldbach's conjecture and related matters was still such that Landau, in his address before the Fifth International Congress at Cambridge, pronounced inaccessible (unangreifbar) in the state of science at that time the questions: Is every even integer greater than 2 the sum of two primes? Is there an infinite number of prime-pairs whose difference is 2? In the remainder of this paper we shall see something of the extent to which such questions have since become accessible.

24. *The work of Hardy and Littlewood.* In two path-breaking memoirs (*Acta Math.* 44 (1922): 1–70, *Proc. Lond. Math. Soc.* (2) 22 (1923): 46–56) on the expression of a number as a sum of primes, Hardy and Littlewood attack the named problem with the aid of their new transcendental method (already partially described in our earlier sections), show that it can be brought into contact with the recognized methods of the analytic theory of numbers and say therefore that the problem is not "unangreifbar." They do not solve the problem; they do not even prove that every number is the sum of not more than a million or any other number of primes. In fact, as they themselves say, all the results of their memoirs (except certain incidental ones), so far as they are novel, depend upon the truth of a hypothesis which they assume but can not establish.

Have they thus rendered accessible such questions as those which Landau pronounced inaccessible in 1912? In his *Vorlesungen über Zahlentheorie*, 1927,

Landau (Vol. I, p. 183) replies that history has found him partly right and partly wrong, but that he finds himself quite right. What Hardy and Littlewood did in 1922 and 1923 leaves him unqualifiedly right in what he said about the state of science in 1912. But Landau holds that he was still right in 1927 since Hardy and Littlewood, though they have proved everything which they claimed to prove, have nevertheless done nothing more than to show the consequences (remarkable enough indeed) of a hypothesis which really consists of an infinite number of assumptions. (To guard against a misunderstanding of this friendly and good-natured controversy I need only remind you that Landau, in common with all who have read the Hardy-Littlewood memoirs, has a very high appreciation of their importance.)

In order to state the main results of Hardy and Littlewood it is necessary to employ the functions $L(s, \chi)$ defined by the series

$$L(s, \chi) = \sum_{a=1}^{\infty} \frac{\chi(a)}{a^s},$$

one such function being formed for each positive integer k and each character $\chi(a)$ modulo k. Here s is the complex variable $\sigma+it$ where σ and t are real. The unproved hypothesis relates to the distribution of the zeros of each of the infinitude of functions $L(s, \chi)$.

The main Hardy-Littlewood theorems (together with their hypothesis) may now be stated as follows:

If no root of any whatever $L(s, \chi)$ belongs to the half-plane $\sigma > \frac{1}{2}$ (and indeed if there exists an absolute constant C less than $3/4$ such that no root of any whatever $L(s, \chi)$ belongs to the half-plane $\sigma > C$), then

I. Every sufficiently large odd number is a sum of three odd primes;

II. Nearly every even positive number is a sum of two odd primes;

III. The number $N(n)$ of ways of representing the odd number n $(n > 1)$ as a sum of three odd primes is of the form

$$N(n) = \frac{n^2}{2\log^3 n}\left\{ \prod_{p}\left(1 + \frac{1}{(p-1)^3}\right)\cdot \prod_{p\mid n}\left(1 - \frac{1}{p^2 - 3p+3}\right) + \lambda(n)\right\}$$

where $\lim_{n=\infty}\lambda(n) = 0$ and the first product is taken for all odd primes p while the second is taken for the odd prime divisors of n.

The character of the investigation is indicated by these three fundamental theorems; but, taken alone, they do not give a clear indication of the richness of the results. There is a treatment of each of the following problems: the representation of integers as sums of r primes; the existence of an infinitude of prime pairs whose difference in each case is any preassigned even number k; generalizations of the last problem; the existence of an infinitude of primes of the form m^2+1; representation of a large non-square number as the sum of a prime and a square; representation of a large number as the sum of a prime and the double

of a square; and many others. In much of this work conjectures only are formulated but they are conjectures which come into close relation with the general methods of the memoir.

VIII. SCHNIRELMANN'S THEOREMS

25. *Character of Schnirelmann's methods.* Landau, who has given an exposition of Schnirelmann's theorems with some simplifications (Göttingen Nachrichten, 1930), says that the memoir from which the theorems are taken contains one of the greatest advances in number theory which he (Landau) has lived to see. The theorems themselves belong to the range of ideas of Goldbach's conjecture. But here there is no question of rendering the problem accessible. Important theorems are actually demonstrated.

But the method is not abstruse. Landau says of his own whole exposition (in all essential elements going back to Schnirelmann, as Landau emphasizes) that it could have been written a hundred years ago and can be understood by a reader who knows no differential or integral calculus (to say nothing of functions of a complex variable) and is acquainted only with the exponential function and logarithms.

A fundamental lemma by aid of which the theorems are easily proved is itself established by methods which Brun has used, methods which rest in the last analysis on ideas analogous to those involved in the so-called sieve of Eratosthenes. Besides this a few of the simple results from the analytic theory of prime numbers are needed.

26. *Schnirelmann's theorems.* With the help of algebraic and number-theoretic lemmas due essentially to Brun and by aid of the sieve-process, Landau's exposition passes by way of a proof of the interesting theorem, that the number $f(y)$ of representations of the positive integer y as a sum of two primes has the property that

$$f(y) = O\left(\frac{y}{\log^2 y} \prod_{p|y} \left(1 + \frac{1}{p}\right)\right)$$

where p runs over the prime factors of y, to the demonstration of the following two theorems due to Schnirelmann:

I. If $N(x)$ is the number of integers y less than x each of which is a sum of two primes, then

$$\limsup_{x=\infty} N(x)/x > 0.$$

II. There exists an integer P such that every integer x greater than unity is a sum of at most P primes.

THE PRACTICAL EVALUATION OF RESULTANTS

By T. A. PIERCE, The University of Nebraska

The purpose of the present note is to give a practical method of evaluating the resultant of two equations. The method is particularly effective when the degree of one of the equations is high while that of the other is low.

Use will be made of certain results in the theory of matrices. Let A be a square matrix and let $|A-\lambda I|=f(\lambda)=0$ be the characteristic equation of A. Then if $g(\lambda)=0$ be another equation, the resultant $R(f, g)$ of $f(\lambda)=0$ and $g(\lambda)=0$ is known (Frobenius, Journal für Mathematik, Vol. 84, p. 11) to be

$$R(f, g) = | g(A) | . \tag{1}$$

Furthermore if B is the matrix which has $g(\lambda)=0$ as its characteristic equation then

$$R(g, f) = | f(B) | . \tag{2}$$

The matrix A which has the given equation

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0 \tag{3}$$

as its characteristic equation is

$$\begin{pmatrix} -a_1, & -a_2, & \cdots, & -a_{n-1}, & -a_n \\ 1, & 0, & \cdots, & 0, & 0 \\ 0, & 1, & \cdots, & 0, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & \cdots, & 1, & 0 \end{pmatrix}$$

This is readily proved by forming the determinant of this matrix after having subtracted λ from each element of the principal diagonal and expanding this determinant and each resulting determinant according to the elements of their last columns. Similarly the matrix B which has

$$g(\lambda) = \lambda^m + b_1\lambda^{m-1} + \cdots + b_m = 0 \tag{4}$$

as its characteristic equation may be written down.

Since the left members of (1) and (2) differ at most by sign in obtaining the resultant of f and g it is only necessary to calculate that right member of (1) or (2) which is the simpler. It is this consideration of (2) in conjunction with (1) that makes our method practical. The matrices A and B are of order n and m respectively and $g(A)$ and $f(B)$ are matrices of orders n and m. If $m<n$ we compute $|f(B)|$. In forming

$$f(B) = B^n + a_1B^{n-1} + \cdots + a_{n-1}B + a_nI$$

where I is the unit matrix of order m we first reduce this expression by means of

$$g(B) = B^m + b_1B^{m-1} + \cdots + b_{m-1}B + b_mI = 0, \tag{5}$$

since B satisfies its characteristic equation. Every power of B higher than the

$(m-1)$ st in $f(B)$ may be eliminated by using (5). In order to calculate these powers of B some recursion formulas are needed. Let

$$B^r = \alpha_r B^{m-1} + \beta_r B^{m-2} + \cdots + \lambda_r I \tag{6}$$

where $r \geqq m$, then

$$\begin{aligned} B^{r+1} &= \alpha_r B^m + \beta_r B^{m-1} + \cdots + \lambda_r B \\ &= (\beta_r - b_1\alpha_r)B^{m-1} + (\gamma_r - b_2\alpha_r)B^{m-2} + \cdots + (-b_m\alpha_r)I \text{ by } (5) \\ &= \alpha_{r+1}B^{m-1} + \beta_{r+1}B^{m-2} + \cdots + \lambda_{r+1}I \end{aligned}$$

hence $\alpha_{r+1} = \beta_r - b_1\alpha_r$, $\beta_{r+1} = \gamma_r - b_2\alpha_r$, $\cdots$, $\lambda_{r+1} = -b_m\alpha_r$. These are the recursion formulas desired. It is now a simple process to calculate $f(B)$ and evaluate the determinant of the resulting matrix.

As an example we will find the resultant of

$$\begin{aligned} f(\lambda) &= \lambda^{12} + \lambda^7 - \lambda^4 - 2\lambda^3 + 3\lambda + 10 = 0 \\ g(\lambda) &= \lambda^4 - \lambda^2 + 2\lambda - 3 = 0. \end{aligned}$$

The dialytic method of Sylvester would give a determinant of order 16. We have

$$B = \begin{Vmatrix} 0, & 1, & -2, & 3 \\ 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \end{Vmatrix}$$

and must first find $f(B)$. To calculate B^{12} and B^7 in terms of B^3, B^2, B, and I we use the recursion formulas

$$\alpha_{r+1} = \beta_r, \quad \beta_{r+1} = \gamma_r + \alpha_r, \quad \gamma_{r+1} = \delta_r - 2\alpha_r, \quad \delta_{r+1} = 3\alpha_r$$

with the initial values $(\alpha_4, \beta_4, \gamma_4, \delta_4) \equiv (0, 1, -2, 3)$. Hence

r	4	5	6	7	8	9	10	11	12
α_r	0	1	−2	4	−4	11	−18	31	−52
β_r	1	−2	4	−4	11	−18	31	−52	100
γ_r	−2	3	−2	7	−14	20	−34	69	−116
δ_r	3	0	3	−6	12	−12	33	−54	93.

Thus $B^{12} = -52B^3 + 100B^2 - 116B + 93I$ and $B^7 = 4B^3 - 4B^2 + 7B - 6I$. Substituting these expressions in $f(B)$ we find $f(B) = -50B^3 + 95B^2 - 104B + 94I$ and this is the matrix

$$\begin{Vmatrix} 289, & -494, & 593, & -462 \\ -154, & 289, & -340, & 285 \\ 95, & -154, & 194, & -150 \\ -50, & 95, & -104, & 94 \end{Vmatrix}.$$

The value of the desired resultant is the determinant of this matrix and is 584549.

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

A METHOD OF HANDLING CERTAIN ELEMENTARY INTEGRALS[1]

By OTTO DUNKEL, Washington University

In texts on the calculus the integrals

$$(1)\qquad \int (x^2 + 2Ax + B)^{-1/2}dx, \qquad\qquad (2)\qquad \int (x^2 + 2Ax + B)^{1/2}dx,$$

are usually derived from standard types in which $A=0$. The integration of these standard types is carried out in a variety of ways, and some of the methods involve troublesome computations with the substitutions used. The method which will be here given appears to be simpler in its computations, and it applies just as easily to the general forms above as to the simpler standard types. In both cases (1) and (2) the saving of labor results from the use of the principle called "composition and division" in elementary algebra.

We set

$$(3)\qquad x^2 + 2Ax + B = u^2, \quad (x + A)dx = udu;$$

and there results

$$(4)\qquad \frac{dx}{u} = \frac{du}{x + A} = \frac{d(x + A + u)}{x + A + u},$$

where the last expression in the equality is obtained first by "composition and division," and then by noting that $d(x+u)=d(x+A+u)$. We may now set down immediately the integral of (1)

$$(5)\qquad \int \frac{dx}{u} = \log\,[x + A + (x^2 + 2Ax + B)^{1/2}],$$

where the arbitrary constant is omitted for brevity of writing. This method has been seen in one text, but only for the case $A=0$ in (1), and there was no extension of the method to (2) in any form. The writer cannot recall the author of the text.

For (2) we use the same transformation (3), and we multiply both sides of the first equation in (4) by u^2 obtaining

$$(6)\qquad udx = \frac{u^2}{x + A}\,du = \frac{x^2 + 2Ax + B}{x + A}\,du.$$

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[1] Read before the sixteenth annual meeting of the Association held at New Orleans, Dec. 31, 1931.

By division there results

$$\begin{aligned} udx &= (x + A)du + \frac{B - A^2}{x + A} du, \\ &= \tfrac{1}{2}\left[udx + (x + A)du + \frac{B - A^2}{x + A} du \right], \end{aligned} \tag{7}$$

where the last expression may be seen to result by composition and division, or more simply otherwise. The first two terms in the brackets reduce to $d\{u(x+A)\}$, and the integral of the last term is obvious from (4) and (5). Thus we have for (2)

$$\begin{aligned} \int udx = \tfrac{1}{2}(x + A)(x^2 + 2Ax + B)^{1/2} \\ + \tfrac{1}{2}(B - A^2) \log \left[x + A + (x^2 + 2Ax + B)^{1/2} \right]. \end{aligned} \tag{8}$$

It will be observed that the methods above are equivalent in a way to the substitution given in some texts

$$(x^2 + 2Ax + B)^{1/2} = u - x. \tag{9}$$

This substitution may be applied to both (1) and (2), but the steps of the process require tedious algebraic and differential computations. In type (1) there is a cancellation of complicated terms and the final result is easily integrated. In the method above the use of "composition and division" avoids the labor wasted in computing terms which finally cancel. When (9) is applied to (2) there are no such serviceable cancellations, and the work becomes so involved that one is led to wish for a less laborious process.

The processes above are obviously inapplicable to (1) when the coefficient of x^2 is -1 instead of $+1$. But we can derive the result for the type corresponding to (2) if we have the integral for the one corresponding to (1). This is not of any practical importance in integration, but we mention it as an interesting fact. Thus

$$\int (B + 2Ax - x^2)^{-1/2} dx = \arcsin \left[(B + A^2)^{-1/2}(x - A) \right].$$

Here $B+2Ax-x^2=u^2$, $dx/u=du/(A-x)$, and our integral may be written $\int du/(A-x)$. For the other case we have

$$\begin{aligned} udx &= \frac{u^2}{A - x} du = \frac{B + 2Ax - x^2}{A - x} du = \left(x - A + \frac{B + A^2}{A - x} \right) du, \\ &= \tfrac{1}{2}\left[udx + (x - A)du + \frac{B + A^2}{A - x} du \right]. \end{aligned}$$

Hence

$$\begin{aligned} \int (B + 2Ax - x^2)^{1/2} dx = \tfrac{1}{2}(x - A)(B + 2Ax - x^2)^{1/2} \\ + \tfrac{1}{2}(B + A^2) \arcsin \left[(B + A^2)^{-1/2}(x - A) \right]. \end{aligned}$$

Expression for a Determinant in Terms of Five Minors

By E. B. Stouffer, University of Kansas

In 1881 Muir[1] announced two theorems on determinants which he named the Law of Complementaries and the Law of Extensible Minors. These theorems make it possible to write a whole series of relationships among the minors of determinants as soon as a single such relationship is given. In particular, from the expansion of a second order determinant there can be obtained by means of the Law of Extensible Minors a simple expression for a determinant of any order in terms of *five* of its minors. In other words, the value of a determinant is known at once when the values of five properly selected minors are known, even though the order of the determinant is not known and the value of not a single element is known. This fact could hardly have escaped the attention of Muir but there appears to be no specific mention of it in the literature. Consequently, it seems worth while to restate briefly the two laws and to show their application by deriving at least this one interesting result.

Muir states the Law of Complementaries practically as follows: *To every general theorem which takes the form of an identical relationship among a number of the minors of a determinant (the determinant itself may be included as a minor) there corresponds another theorem derivable from the former by substituting for every minor its algebraic complement in the determinant, and then multiplying each term by such a power of the determinant as will make the relationship homogeneous in the elements.*

In order to prove this theorem for a determinant D it is only necessary to write the given relation for the adjoint of D and then replace each minor in this relationship by its equivalent in terms of the algebraic complement of the corresponding minor in D multiplied by the proper power of D.

Muir's statement of the Law of Extensible Minors is essentially as follows: *If any identical relationship be established among a number of minors of a general determinant (the determinant itself may be included as a minor), the minors being denoted by means of their principal diagonals, then a new relationship involving the minors of a determinant with k additional rows and columns is always obtainable by annexing the k new elements in the principal diagonal to the end of the diagonal of every minor occurring in the identity, and then multiplying each term by such a power of the principal minor of order k, consisting of new elements only, as will make the relationship homogeneous in the elements.*

The proof of this theorem consists merely in applying the Law of Complementaries twice, first to the given relationship, considering the minors as belonging to the original determinant, and then to this new relationship, considering the minors as belonging to the extended determinant.

Let us now apply the Law of Extensible Minors to the simplest possible identity

[1] Transactions of the Royal Society of Edinburgh, vol. 30, pp. 1–4. See also, Muir and Metzler, *A Treatise on the Theory of Determinants*, paragraphs 179 and 187.

$$(1)\qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = | a_{11}a_{22} | = a_{11}a_{22} - a_{12}a_{21}.$$

If we add enough rows and columns to make the order of the determinant n and continue to denote the new elements by a_{ij}, we have by the Law of Extensible Minors

$$(2)\qquad | a_{11}a_{22}a_{33} \cdots a_{nn} | = \frac{| a_{11}a_{33} \cdots a_{nn} | \cdot | a_{22}a_{33} \cdots a_{nn} | - | a_{12}a_{33} \cdots a_{nn} | \cdot | a_{21}a_{33} \cdots a_{nn} |}{| a_{33} \cdots a_{nn} |}.$$

We find thus that *a determinant of order n is a function of four minors of order $n-1$ and one of order $n-2$.*

The principal minor $|a_{11}a_{22}|$ has been used as the starting point in the demonstration but it is evident that any principal minor of the second order would serve. However, in applying the theorem to a given determinant it is necessary to observe the single limitation that the minor of order $n-2$ which appears in the denominator of (2) must not be zero. By interchange of rows or columns the determinant can always be put into such a form that the principal minor selected from any fixed $n-2$ rows is different from zero, unless the determinant itself vanishes identically.

Many similar but less simple relationships may be obtained. For example, we may start with a determinant of order three and obtain an expression for a determinant of any higher order in terms of seven minors, three of order $n-1$, three of order $n-2$ and one of order $n-3$. Likewise from the determinant of order three we may obtain an expression for a determinant of any higher order in terms of nine minors of order $n-2$ and one of order $n-3$.

In an earlier paper[1] the author has shown that the Law of Extensible Minors may be used to prove that any determinant can be expressed in terms of fourteen *principal minors.*

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Applications of the Absolute Differential Calculus. By A. J. McConnell. London, Blackie and Son, 1931. xii+318 pages. 20s.

This excellent work is essentially a textbook on the tensor calculus written primarily for the advanced undergraduate and having as its outstanding char-

[1] This Monthly, vol. 35 (1928), pp. 18–21.

acteristic an abundance of illustrative material. A second distinguishing feature is the large collection of problems, 879 in all, of which forty-one are solved and the remainder left to the student. Some of the latter set may be solved without recourse to paper and pencil while others, although not difficult, furnish considerable additional theory. This book is well written, the notation is modern, and the general plan, which we shall now sketch briefly, is admirable.

Part One is a detailed and elementary exposition of tensor algebra including a treatment of determinants by means of Kronecker deltas and the allied systems. In this and the succeeding section, "Algebraic Geometry," linear transformations are employed almost exclusively.

Under "Algebraic Geometry" we find the application of the preceding theory to cones, conics, and quadrics. Incidentally much of Gibbs' vector analysis appears in somewhat different dress.

The third of the four major subdivisions, "Differential Geometry" is perhaps the most inclusive of all. Here the emphasis has shifted from the linear to the general transformation and from the algebraic to the differential properties of tensors. Some of the topics treated are covariant differentiation, parallel displacement, the Frenet formulas, the Riemann-Christoffel tensor, Gaussian curvature, the first, second, and third ground forms, geodesics, and the principal curvatures.

The last section is devoted to mechanics, electricity, and special relativity. The chapters on mechanics, "Dynamics of a Particle," "Dynamics of Rigid Bodies," and "Mechanics of Continuous Media" presuppose but little preliminary knowledge, whereas the discussion of electricity, for the most part, is not expository, being essentially a tensor reformulation of the fundamental equations. The author has attacked special relativity from the tensor point of view and thereby provided an excellent, though not comprehensive, introduction to the general theory.

Apparently this book contains but few errors. We noted one or two slight misstatements and a half dozen harmless misprints. In addition, there occur, in a few places, what might be considered as errors of omission. For example, in the derivation of the differential equations of the extremal curves no reference is made to the fundamental lemma.

H. V. CRAIG

Debunking Science. By E. T. Bell, University of Washington Chapbooks, #44, 1930, 40 pages.

Scientists, like other men, are constantly overestimating what they know. They repeatedly believe they have proved and disproved more than they have. When a scientific hypothesis works well in mentally organizing a few experimental results and fails only for experiments which are reported in small type or in obscure journals, it becomes a law of nature. A slightly changed cosmology, or a new biological discovery may lead to atheism; while the discovery of a very penetrating ray or an application of a non-definite quadratic form to physics

seems ample justification for concluding that there is a God and that he is a pure mathematician.

It is pleasant, therefore, to review even such a short booklet as this, warning scientists that their group self-respect should prevent them of all people from being overcredulous.

The book is divided into three parts. It lacks a table of contents. I, therefore, furnish it with the following:

Part I. Stop Being Childish and Dishonest—pp. 7–17. (This part is a pleasant vituperation against credulity and is very good reading, as is always true with whole-hearted scolding when there is a vein of humor running through it. See Stubb's exhortations to his crew in Moby Dick. Both the irony and the pleasantry are well done.)

Science as a pseudo religion and religion as a pseudo science. As Disko Troop said, "Keep things separate," 7; The flapper as the sculptured goddess of Debunkiana, 9; Bunk in science more disgusting than elsewhere, 11; Blessed be the doubters, 14; Science worth saving, 16.

Part II. Even Mathematics Needs Debunking—pp. 17–24. (This part is a little more solemn. Toe treading isn't so much fun among friends. This chapter, however, gives a very good account of the highly optimistic attitude of pre-Brouwer Mathematics.)

Real start of modern debunking in mathematics dates from 1908, 17; Popular expositors aren't aware of it yet, 17; What is proof and what is nonsense is not yet settled, 18; The pleasant pre-Brouwer world, 19; Not even mathematics has reached certainty, 23.

Part III. Description of the Mathematical Controversy—pp. 24–40. (This is really a very good non-technical hint of what is going on in mathematics. It is about all that can accurately be said to the non-mathematician. Though well written, it will never be included in the "World's Greatest Humor.")

Kronecker started it, 24; The tender-minded and tough-minded, 25; A paradox, 27; Berkeley wins, 29; Cantor and Dedekind, 29; Kronecker, 30; Brouwer, 32; Russell's Principia stillborn, 34; Hilbert, 36; Peroration, and final hymn without benediction, 40.

Part IV. Not written. (Should be technical discussions for the Transactions.)

Part V. Not written. (By extrapolation it would be so solemn that it should be buried in an Academy publication.)

Really the pamphlet is worth reading, and the author's point of view needs emphasis. It is a little too bad that the author had such a good time writing Part I that he couldn't keep mad for the next two parts.

MARK H. INGRAHAM

Enciclopedia delle Matematiche Elementari a cura di L. Berzolari, G. Vivanti e D. Gigli. Volume 1, part 2. Milan, Ulrico Hoepli, 1932. XVI+611 pages.

The fact that there is now no modern mathematical encyclopedia in the English language is perhaps sufficient evidence of the magnitude of such an un-

dertaking even if it is restricted to the elements of our subject. Part I of the present volume appeared in 1930 and was reviewed in this MONTHLY, volume 37 (1930), page 378, by Professor A. A. Bennett. The second and somewhat larger part of this volume is dated 1932 and completes the treatment of the subject of analysis in a little over a thousand pages, while the *Encyklopädie der mathematischen Wissenchaften* devotes a little over five thousand pages to the same subject. The treatment in the present volume is however more extensive than that given of the same subject in the *Encyklopädie der Elementar-Mathematik* by Weber and Wellstein.

The thirteen main headings of this second part, with the number of pages devoted to each of these subjects and the authors of the articles, are as follows: combinatory calculus (9), L. Berzolari; elements of the theory of groups (51), L. Berzolari; determinants (30), L. Berzolari; linear equations (13), L. Berzolari; linear substitutions and linear, bilinear and quadratic forms (28), L. Berzolari; rational functions of one or more variables (37), O. Nicoletti; general properties of algebraic equations (59), O. Nicoletti; equations of the second, third and fourth degree and other particular algebraic equations, systems of algebraic equations of elementary type (57), E. G. Togliatti; methods for the discussion of problems of the second degree and remarks on some of the third and fourth degree (63), R. Marcolongo; limits, series, continued fractions and infinite products (45), G. Vitali; elements of infinitesimal analysis (101), G. Vivanti, relations between the theory of aggregates and elementary mathematics (11), G. Vivanti; the analytic functions from an elementary point of view (29), S. Pincherle.

The names of the authors of these various articles are a sufficient guarantee of the high standards maintained in the preparation of this very useful work, which was under consideration for more than twenty years but was greatly delayed by the world war and its adverse consequences. From the titles noted in the preceding paragraph it is clear that the term elementary mathematics is used here with a much wider meaning than the one found, for instances, in the *New International Encyclopedia*, 1923, under the term "Mathematics," where it is stated that "the period of the development of elementary mathematics closes with the seventeenth century." Such a subject as the theory of groups, in particular, was practically unknown in 1700 but its elements are developed in the present encyclopedia in an unusually clear and comprehensive manner, together with a large number of references to sources.

One of the most valuable features of this encyclopedia is its large number of references to the earliest developments, and hence it is very useful to those who are mainly interested in the history of elementary mathematics. Its object is to give proofs of a number of the most fundamental theorems and simply to give references to the places where proofs of others can be easily found, with an emphasis on the earliest ones. In a few cases one misses here the most recent historical developments relating to the subjects which are discussed. For instance, no references appear therein to the recent discoveries relating to the early

partial solutions of quadratic equations by the ancient Babylonians, which were published by O. Neugebauer and others in the recently inaugurated periodical entitled *Quellen und Studien zur Geschichte der Mathematik*. In fact, this periodical is not included in the lists which appear, together with their abbreviations, near the beginning of each of the two parts of the volume under consideration.

The author index of the present part includes about 800 different names. In view of the emphasis on the earliest developments one would not expect that many of these are those of American mathematicians except when the subjects treated are modern as in the case of the theory of groups. The number of the different names of American mathematicians therein is however more than forty. An international spirit is also exhibited by the fact that the largest numbers of references under individual names appear under those of the two non-Italian mathematicians, A. L. Cauchy and L. Euler. Unfortunately the volume does not contain a subject index, but it is to be hoped that such an index will appear in a later volume of this work. The arrangement of the material is topical under the main headings noted above, and the treatment aims to develop only the more elementary concepts, and hence it is especially adapted to the needs of those who desire to restrict their attention for the time being to the most fundamental ideas involved in a particular subject relating to elementary mathematics.

According to the announced plan the completed work will be composed of three volumes. Volume II is to be devoted to geometry while volume III is expected to treat a variety of different subjects including the history and the teaching of mathematics. According to the original plan these two subjects were also to be treated in the closing volume of the *Encyklopädie der mathematischen Wissenschaften* but up to the present time nothing thereon has appeared as a part of this work. Hence the Italian encyclopedia under consideration has no modern forerunner along these lines and it will be interesting to see what advances will be made along these difficult and backward lines of mathematical development where permanent progress seems to be most uncertain. In view of the remarkable recent advances made in geometry by Italian mathematicians the volume on geometry will be awaited with high expectations.

On page 43 it is stated that the natural numbers form a group when they are combined according to multiplication while on page 33 it is noted that these numbers do not constitute a group if we assume as a postulate with respect to infinite groups a very useful property of finite groups. It seems to the reviewer very unfortunate that the term group when it relates to an infinite number of elements is not fully defined in the present work, for it would be very desirable to establish uniformity of usage along this line in the modern mathematical literature. On page 31 it is stated that the octic group is generated by three distinct transpositions which is obviously impossible. The number of such oversights is, however, small; the work as a whole can be heartily recommended and it should do much to secure higher standards as regards a knowledge of elementary

mathematics. In view of the fact that the mathematical language is so largely international those who know very little Italian will be able to consult this work without much language trouble and by doing so they may be able to improve their knowledge of a language which has become essential to all who desire to keep in touch with the most important modern mathematical developments.

G. A. MILLER

Intermediate Calculus. By Percey F. Smith and William R. Longley. Boston, Ginn and Company, 1931. xiii+457 pages. $3.20.

There are a few colleges and universities in this country in which some or all of the entering students have completed the study of trigonometry in their preparatory schools. For such, a brief course in analytics followed by one in the calculus is the usual procedure. Or the course may be built around the calculus with the analytical geometry developed as needed. Thus at Yale University we find written for this particular purpose Longley and Wilson's *An Introduction to The Calculus*. It is not desirable to follow such a course with the ordinary text on the elementary calculus, hence Smith and Longley have prepared this book entitled *Intermediate Calculus*. In reviewing it we should keep in mind, first the contents of Longley and Wilson, since the two books together constitute a unit; secondly, we should remember that in 1929 Smith and Longley published a revision of Granville's Calculus, with which we can compare the present work.

Following a collection of formulas in Chapter I, there are three review chapters—one of fourteen pages mainly on the analytical geometry of conic sections, one of eighteen pages on the differential calculus of algebraic functions and one of thirty-eight pages on the integral calculus of algebraic functions. Since the technique of differentiation and integration was learned in the previous course, these fifty-six pages, except for a few definitions, are devoted exclusively to problems in maxima and minima, curve tracing, rates, areas, volumes of solids of revolution, moments of inertia, work, and fluid pressure. The trapezoidal rule and Simpson's rule for the approximate evaluation of the definite integral are given. Here in compact form the student has recalled to him all that is absolutely essential for his study of this book. It is well done; the only slight criticism would be the use of dy/dx as the symbol for the derivative.

The transcendental functions are introduced in Chapters V and VI, and treated fully as new material with applications to curve tracing, simple harmonic motion, damped and forced vibration. In Chapters VII and VIII are the usual methods of integrating expressions which do not come directly under the standard forms. Throughout these four chapters are numerous exercises which enable the student to develop the technique of differentiation and integration of transcendal functions.

In Longley and Wilson there was no discussion of polar coordinates and but brief mention of parametric equations. Consequently these topics must be developed from the very beginning; curves must be plotted, equations of cycloidal

curves must be derived, and the application of the calculus pointed out. Various graphic methods as well as Newton's method of calculating the approximate value of the roots of algebraic and transcental equations are discussed. A chapter on empirical equations fits a straight line to the given points by the *method of averages*. Six other two-constant equations, which can be handled similarly to that of the straight line, are treated together with one three-constant curve, the general parabola $y=a+bx+cx^2$. While the mathematical derivation of the *method of least squares* is beyond the scope of the book, one regrets that in a mathematical text-book this method could not have been stated and used instead of the *method of averages*.

The chapter on curvature is practically the same, text and exercises, as the one in Granville, Smith and Longley. The paragraphs on transformation of derivatives and on involutes have been omitted, and more added on the tangential and normal components of the acceleration vector in curvilinear motion. The same is true for the chapter on ordinary differential equations. The discussion of linear differential equations of the nth order with constant coefficients is not in the later book. The exercises are the same; in some y'' and y' are used instead of d^2y/dx^2 and dy/dx respectively. The only multiple integrals treated are the double integrals in the final chapter of the book; this again is precisely as found in the Smith and Longley revision of Granville. The chapters on series and on partial differential equations differ in arrangement of subject matter and in problems from those in the other book. The material is condensed, especially that on partial differential equations, and some of the applications omitted. A chapter on solid analytical geometry is inserted before that on partial differential equations. In the chapter on series the reviewer feels that it should be stated that an infinite series represents the function only when the remainder approaches zero, as the number of terms increases without limit.

The dimensions of the book make it more convenient to use than the 1929 revision of the Granville text. The figures are good, the printing well done. A short table of integrals has been placed at the end of the book. Since in so many colleges it is necessary to teach trigonometry and even algebra in the first year, many teachers will be prevented from utilizing this as a text-book.

J. R. MUSSELMAN

Elements of the Theory of Integers. By Joseph Bowden: Revised Edition. Published by the Author, Garden City, N. Y., 1931. 268 pages. $2.50.

This is substantially a reprint of the earlier (1903) edition, save for some changes in the last chapter. The work forsakes tradition in many superficial ways. Such spelling as "giv," "ar," "hav," "ther," "orderd," first suggest an attitude that one meets throughout. There are correspondingly new terms employed, and new symbols for old terms. Probably few will recognize the relations, "sul," "mul," "kor," the inverted Pi for "is prime to" and many others.

The work deals essentially with the rational integers and their elementary properties, with emphasis upon the logical aspects. It is only in the last chapter, that dealing with linear congruences, that any mathematical relations entirely new to the high school pupil are touched upon; but the extensive use of axioms, starting even with axioms for the formal treatment of hypothetical propositions and the explicit discussion of the Aristotelian syllogism, will be unfamiliar (and perhaps unwelcome). Probably the most interesting item from the young reader's point of view is the explicit use of single symbols for the numbers, 10, 11, $\cdots$, 16, thus liberating one from the accidental dependence upon the decadic notation appropriate to the decimal system, although little is made of this opportunity.

The book has been employed as the text in a course for first year students and is apparently intended for this purpose, so should probably be considered with the purpose in mind. Indeed except for notation there are no new concepts suggested that would recommend the work for advanced students. There are no exercises! The references are numerous but may be viewed as rather in the nature of acknowledgment of sources than as suggestions for the encouragement of youthful readers. Since no numbers other than the traditional ones of elementary arithmetic are suggested, the reader is likely to exclaim, "Why all this talk about prefactor, postfactor, etc. etc." No proofs of consistency of independence are offered so that the discussion of these concepts although verbally clear must remain almost without significance. The choice of logical axioms (resulting in a system neither sufficient nor independent) does not encourage further study.

With the wealth of mathematics available which is of importance and adapted to arouse the interest of the immature freshman, one may question whether a course based upon this text may not do harm in many cases by strengthening the prejudice of the student to the effect that beyond high school topics there is no mathematics worthy of attention.

The work is in general careful and accurate and the author is well-informed upon traditional matters. Whether the project is psychologically adapted to the training of the young is the only question at issue. With essentially the same commendable purpose in mind a markedly contrasting work may be mentioned, contrasting as to the type of readers addressed, the choice of material, the style, the contacts with other fields, and particularly the success of the venture, namely H. Beck: Einführung in die Axiomatik der Mathematik, which for its purpose one may recommend unreservedly.

A. A. BENNETT

A LETTER TO THE EDITOR

DEAR SIR:

I was surprised to read in the December number of the MONTHLY in the review of Smail's algebra, "Here is something really new: a textbook which does

not present the subject matter of college algebra as a sequence of disconnected topics, etc." Evidently the reviewer, who claims more for the book than its author, does not take his Ecclesiastes seriously for there I read: "Is there anything whereof it may be said, See, this is new? It hath been already of old time, which was before us." In support of the Preacher, I rise to remark that fifteen years ago Wilczynski wrote and Slaught edited a college algebra,[1] conceived in the same spirit and executed on the same general pattern as Smail's, and containing some of the topics which the reviewer noted as omitted in the latter book. I don't know how widely Wilczynski's book was adopted—though we used it at Washington for several years—but it seems strange that a work of two such prominent mathematicians should have been overlooked.

Do mathematics and pedagogy move so fast that a book is swallowed up in oblivion in 15 years? The Preacher scores again when he observes: "One generation passeth away and another generation cometh . . . There is no remembrance of former things." Perhaps the answer is "Of making many books there is no end." At any rate it will be interesting to see which shall prevail—the reviewer's hope that Smail's book will prove popular and successful, or the Preacher's prophecy, given another fifteen years, "neither shall there be any remembrance of things that are to come with those that shall come after."

Incidently, I may add that we are adopting Smail's book.

Sincerely,

R. M. Winger

University of Washington,
January 6, 1932.

PROBLEMS AND SOLUTIONS

Edited *by* B. F. Finkel, Otto Dunkel *and* H. L. Olson

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3535. *Proposed by J. M. Feld, Brooklyn College of the City of New York.*

The pedal curve p with respect to a point P of a given curve c is defined as the locus of the feet of the normals from P to the tangents of c, and c is known

[1] Reviewed in the Monthly, Vol. 24 (1917), p. 230.

as the negative pedal of p. Prove that the negative pedals of circles and straight lines with respect to a point P, not on the circles or straight lines, are respectively central conics and parabolas having a focus at P. The parabolas are tangent to their pedals and the central conics are doubly tangent to theirs.

3536. *Proposed by Martin Rosenman, Brooklyn, New York.*

Consider fractions of the form 1/2, 1/3, 1/4, 1/5, $\cdots$. We seek to determine which n of these fractions (repetitions allowed) give a sum as near unity as possible but actually less than it. Thus for $n=3$, we have $1/2+1/3+1/7=41/42$. Prove or disprove that, in general, the first n of the fractions in the series $1/2+1/3+1/7+1/43+1/1807\cdots$ give the desired result; in which series each denominator exceeds by 1 the product of all preceding.

3537. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

The letters H, V, and P designate a system of mutually perpendicular coordinate planes, and the letters α, β, and γ the angles which a plane Q makes with H, V, and P respectively. If none of these angles exceeds 90°, what are the extreme limits of the sum $\alpha+\beta+\gamma$?

3538. *Proposed by Eugene M. Berry, Lynchburg College.*

Solve the differential equation

$$ldy=F(x)\sin(x-y)dx,$$

where l is a constant and $F(x)$ is any function of x.

3539. *Proposed by R. E. Gaines, University of Richmond.*

A slender rod of length $2a$ rests on a circular table of radius r, $r>a$. What are the probabilities that neither end, one end, or both ends, will project over the edge of the table?

3540. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Consider the four spheres having for diameters the four lines joining the ends of two diameters, chosen arbitrarily, in two given circles of a given sphere. Prove that the radical center of the four spheres is fixed.

3541. *Proposed by J. B. Reynolds, Lehigh University.*

A rod of length $2a$ standing on a rough level plane of coefficient of friction μ falls from a vertical position. At what angle with the vertical will the rod begin to slip at the bottom?

3542. *Proposed by W. E. Buker, Leetsdale, Pennsylvania.*

It is known that $3^2+4^2=5^2$ and $3^3+4^3+5^3=6^3$. Find other integral values of a, r, and l, if they exist, such that

$$a^r+(a+1)^r+(a+2)^r+\cdots+(a+l)^r=(a+l+1)^r$$

SOLUTIONS

3408 [1930, 38]. *Proposed by J. V. Uspensky.*

Show that the integral

$$V_n = \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{x_1 + x_2 + \cdots + x_n} dx_1 dx_2 \cdots dx_n$$

converges to the limit $2/3$ when n increases indefinitely and that the product $n(V_n - 2/3)$ remains bounded.

Solution by the Proposer.

By the use of the formula

$$(1) \qquad (x_1 + x_2 + \cdots + x_n)^{-1} = \int_0^\infty e^{-t(x_1+x_2+\cdots+x_n)} dt,$$

the integral denoted by V_n may be reduced to

$$(2) \qquad V_n = n \int_0^\infty [\psi(t)]^{n-1} \phi(t) dt, \quad \psi(t) = (1 - e^{-t})t^{-1}, \quad \phi(t) = \int_0^1 e^{-t\xi} \xi^2 d\xi.$$

The whole question is now reduced to finding an asymptotic expression for the integral V_n in (2). To this end it is to be observed that $\psi(t)$ is a function of t which decreases from 1 to 0 as t varies from 0 to ∞. Hence the equation

$$(3) \qquad \psi(t) = (1 - e^{-t})t^{-1} = 1 - y$$

has one and only one solution in t for each value of y, $0 < y < 1$. This solution can be expressed as a power series in y

$$(4) \qquad t = 2y + \frac{4}{3} y^2 + \cdots,$$

which is convergent for $y \leqq c$, c being any positive number less than unity. For the chosen value of c, we suppose that t has the value a. Hence

$$(5) \qquad dt = [2 + y\theta(y)]dy,$$

where $|\theta(y)| < K$, K being a suitably chosen constant for the interval $0 \leqq y \leqq c$, and where $0 \leqq t \leqq a$.

It will now be convenient to split the integral in (2) into two parts as follows:

$$(6) \qquad I_1 = \int_a^\infty [\psi(t)]^{n-1} \phi(t) dt, \quad I_2 = \int_0^a [\psi(t)]^{n-1} \phi(t) dt,$$

and we consider first I_1. We have

$$\phi(t) < \int_0^1 e^{-t\xi}d\xi = \psi(t) \leqq \psi(a) = h < 1.$$

Hence

$$I_1 < \int_a^\infty [\psi(t)]^n\, dt < h^{n-2}\int_a^\infty t^{-2}dt = h^{n-2}a^{-1},$$

$$I_1 = \beta h^{n-2}a^{-1},\quad 0 < \beta < 1. \tag{7}$$

For the part I_2 we observe that we may write

$$\phi(t) = \tfrac{1}{3} - t\int_0^1 e^{-\theta\xi}\xi^3 d\xi = \tfrac{1}{3} - \omega t,\quad 0 < \omega < \tfrac{1}{4}.$$

Hence

$$I_2 = \tfrac{1}{3}\int_0^a [\psi(t)]^{n-1}dt - \omega^1\int_0^a [\psi(t)]^{n-1}tdt,\quad 0 < \omega^1 < \tfrac{1}{4}.$$

We now make a change of variable by means of (3) and (5), and for brevity we write

$$f(y) = [2 + \frac{4}{3}y + \cdots][2 + \theta(y)].$$

Then

$$\begin{aligned} I_2 &= \tfrac{1}{3}\int_0^c (1-y)^{n-1}[2 + y\theta(y)]dy - \omega^1\int_0^c (1-y)^{n-1}yf(y)dy \\ &= \tfrac{2}{3}\int_0^c (1-y)^{n-1}dy + H\int_0^c (1-y)^{n-1}ydy \\ &= \tfrac{2}{3}n^{-1} - \tfrac{2}{3}(1-c)^n n^{-1} + Hn^{-1}(n+1)^{-1}[1 - (nc+1)(1-c)^n]. \end{aligned}$$

Combining the results and introducing certain simplifications we have

$$V_n = \tfrac{2}{3} - \tfrac{2}{3}(1-c)^n + \frac{\alpha}{n} + \frac{\beta h^{n-2}}{a};$$

and from this expression for V_n results the statement in the problem.

Also solved by Elbert F. Cox.

3460 [1930, 508]. *Proposed by B. F. Kimball, University of New Hampshire.*

Show that the integral:

$$V_n = \int_0^1\int_0^1\cdots\int_0^1 \frac{x_1{}^r + x_2{}^r + \cdots + x_n{}^r}{x_1 + x_2 + \cdots + x_n}dx_1dx_2\cdots dx_n$$

converges to the limit $2(r+1)^{-1}$ when n increases without limit, and that the product $n[V_n - 2(r+1)^{-1}]$ remains bounded; r to be any real number greater than -1. A generalization of 3408 [1930, 38].

A Note by Otto Dunkel: The solution of this problem follows easily along the lines given in the interesting solution of 3408. Since here the integrand is in-

finite when r is negative, and since an additional integration will be introduced with an infinite limit and the order of the integration will then be changed, it will be shown that the integral has a finite value which is unaltered by these changes. The given integral may be divided into n integrals of which the ith integral is the limit of

$$I_\epsilon = \int_0^1 \cdots \int_\epsilon^1 \cdots \int_0^1 \frac{x_i^r}{x_1 + x_2 + \cdots + x_n} dx_1 dx_2 \cdots dx_n,$$

as ϵ approaches zero, where ϵ is the lower limit of x_i. Since the integrand for I_ϵ is continuous we may change the order of integration and write

$$I_\epsilon = \int_\epsilon^1 \int_0^1 \cdots \int_0^1 \int_0^\infty \xi^r e^{-t(x_1+x_2+\cdots x_{n-1}+\xi)} dt dx_1 \cdots dx_{n-1} d\xi.$$

The integral with respect to t is split into two parts, the first of which has the limits 0 and l, and the second part has the limits l and ∞. For the second part we have

$$\int_l^\infty e^{-t(x_1+x_2+\cdots+\xi}dt < \int_l^\infty e^{-t\epsilon}dt = \epsilon^{-1}e^{-l\epsilon}.$$

For a given ϵ we can find a number L so that for an arbitrarily small η we have $\epsilon^{-1}e^{-l\epsilon} < (1+r)\eta$ when $l \geqq L$. There are now two parts for I_ϵ, and in the first part we may change the order so that we have

$$I_\epsilon = \int_0^l \int_\epsilon^1 \int_0^1 \cdots \int_0^1 \xi^r e^{-t(x_1+x_2+\cdots+\xi)} dx_1 dx_2 \cdots dx_{n-1} d\xi dt + \eta', \eta' < \eta, l \geqq L.$$

Thus

$$I_\epsilon = \int_0^\infty \left[\int_\epsilon^1 \xi^r e^{-t\xi} d\xi\right]\left[\int_0^1 e^{-tx}dx\right]^{n-1} dt.$$

The fact that I_ϵ approaches a limit I for $\epsilon \to 0$ results from the rest of the proof. Thus

$$V_n = nI, \quad I = \int_0^\infty \phi(t)[\psi(t)]^{n-1}dt,$$

$$\phi(t) = \int_0^1 \xi^r e^{-t\xi} d\xi, \quad \psi(t) = (1 - e^{-t})t^{-1}.$$

For the case where r is not negative the proof in 3408 requires merely a change of the exponent 2 to r. For the case of r negative, $1+r>0$, we introduce a trifling change in the part I_1. Here we have $\phi(t) \leqq (1+r)^{-1}$. Then in the same notation we have

$$I_1 = \int_a^\infty \phi(t)[\psi(t)]^{n-1}dt < (1+r)^{-1}\int_a^\infty [\psi(t)]^{n-1}dt$$

$$< [\psi(a)]^{n-3}(1+r)^{-1}\int_a^\infty t^{-2}dt = h^{n-3}(1+r)^{-1}a^{-1}$$

$$I_1 = \beta(1+r)^{-1}a^{-1}h^{n-3}, \quad \beta < 1.$$

The rest of the proof follows as in 3408.

3480 [1931, 170]. *Proposed by G. W. Wishard, Norwood, Ohio..*

Prove the following theorem: *If 1 be annexed to any triangular number in the nonary scale of notation, the result will be another triangular number.* Thus, 1, 11 111, 1111, etc. *ad infinitum* are triangular numbers in the nonary system.

Solution by Helen A. Merrill, Wellesley College.

To annex 1 to any number in the nonary scale of notation is to multiply it by 9 and add 1. Any triangular number is of the form $\frac{1}{2}n(n+1)$. Then $9\cdot\frac{1}{2}n(n+1)+1=\frac{1}{2}(3n+1)(3n+2)$, a triangular number. Q. E. D.

If a triangular number written to the base 3 has 01 annexed to it, the resulting number is triangular. Similarly, if a triangular number written to the base 25 has 3 annexed, or if written to the base 5 has 03 annexed, the resulting number is triangular. Or, more generally, if a triangular number written to the base $(2k+1)^2$ has the digit $\frac{1}{2}k(k+1)$ annexed; or, written to the base $2k+1$, $k\leqq 3$, has the digits 0 and $\frac{1}{2}k(k+1)$ annexed; or written to the base $2k+1$, $k>3$, has the two digits representing the number $\frac{1}{2}k(k+1)$ annexed; the resulting number is triangular.

Also solved by H. T. R. Aude, J. E. Burnam, Mannis Charosh, W. R. Church, Ralph Deutsch, L. S. Johnston, Emma T. Lehmer, A. Pelletier, R. C. Staley, Emory P. Starke, and the Proposer.

3485. [1931, 171]. *Proposed by J. M. Feld, New York, N. Y.*

Let the sides of triangle $A_1A_2A_3$ be trisected. A_{ij} is the trisection point on a_j nearer A_i. Let P_i be the intersection of $A_{ij}A_j$ and $A_{ik}A_k$ and let Q_i be the intersection $A_{kj}A_j$ and $A_{jk}A_k$. Prove:

(1) The triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ are homothetic to $A_1A_2A_3$ and the common homothetic center is the centroid of $A_1A_2A_3$.

(2) $$P_1P_2/A_1A_2 = \tfrac{1}{4},\ Q_1Q_2/A_1A_2 = \tfrac{1}{5}.$$

(3) If C is the intersection of $A_{23}A_3$ and $A_{12}A_2$, and D is the intersection of $A_{13}A_3$ and $A_{32}A_2$, CD is parallel to A_2A_3.

Solution by J. R. Musselman, Western Reserve University.

Given any triangle $A_1A_2A_3$; choose the axes of the coordinate system so that the coordinates of the vertices shall be A_1: $(0, b)$; A_2: $(-a, 0)$; A_3: $(c, 0)$. Easy

calculation will furnish the coordinates of the points P_i and Q_i as follows:

$$P_1: [(c-a)/4, 2b/4]; \quad Q_1: [2(c-a)/5, b/5]$$
$$P_2: [(c-2a)/4, b/4]; \quad Q_2: [(2c-a)/5, 2b/5]$$
$$P_3: [(2c-a)/4, b/4]; \quad Q_3: [(c-2a)/5, 2b/5].$$

It is now an easy matter to prove that the lines P_iQ_i are the medians of the triangle $A_1A_2A_3$ and therefore (1) that $P_1P_2P_3$ and $Q_1Q_2Q_3$ are homothetic with the centroid G as the center of similitude. Also (2) one can prove that the sides of the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ are parallel respectively to the sides of $A_1A_2A_3$, and a calculation of their lengths proves that

$$P_1P_2/A_1A_2 = \tfrac{1}{4}, \; Q_1Q_2/A_1A_2 = \tfrac{1}{5}.$$

It is worth while noticing that the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ are fourfold perspective; the four centers of perspection are the three vertices $A_1A_2A_3$ and the centroid G. Here we have a simple example of two fourfold perspective triangles.

(3) The trisectors meet in six other points whose coordinates are

$$C_1: [(c-4a)/7, 2b/7]; \quad D_1: [(4c-a)/7, 2b/7]$$
$$C_2: [(2c-a)/7, 4b/7]; \quad D_2\cdot [(c-2a)/7, 4b/7]$$
$$C_3: [2(2c-a)/7, b/7]; \quad D_3: [2(c-2a)/7, b/7].$$

Hence, we see that the three lines C_iD_i are parallel to A_2A_3 and equal in length respectively to 3/7, 1/7, 2/7 of A_2A_3; that the three lines C_2D_3, C_3D_1 and C_1D_2 are parallel and equal respectively to 3/7, 1/7, 2/7 of A_1A_2; and that the three lines C_3D_2, C_1D_3, C_2D_1 are parallel and equal respectively to 3/7, 1/7, 2/7 of A_1A_3. The triangles $C_1C_2C_3$ and $D_1D_2D_3$ are threefold perspective, the three axes of perspection are the medians of the triangle $A_1A_2A_3$ which meet at G. The three centers of perspection are on the line at infinity. The six points C_1, C_2, C_3, and D_1, D_2, D_3 lie on an ellipse whose center is G.

A Note by the Editors: All of these results may be deduced from the similar triangles of the figure.

Also solved by J. W. Clawson, Ralph Deutsch, L. S. Johnston, A. Pelletier, F. Underwood, and the Proposer.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

INTERNATIONAL CONGRESS OF MATHEMATICIANS ZURICH, SWITZERLAND, SEPT. 4–12, 1932

The International Congress of Mathematicians will be held in the Federal Institute of Technology at Zurich September 4–12 next. The scientific part of

the program will consist of a large number of general lectures and section meetings. The lectures are intended to provide a complete survey of the present situation in mathematics. Eminent lecturers have, of course, been secured.

The section meetings are intended for briefer discussions of the results of new investigations. The following sections are planned: 1. algebra and theory of numbers; 2. analysis; 3. geometry; 4. probability and insurance mathematics; 5. astronomy; 6. mechanics and mathematical physics; 7. mathematical-technical science; 8. philosophy; 9. history; 10. pedagogy.

In addition to these scientific meetings, social gatherings and excursions to different points of interest in Switzerland, are being planned, so that the mathematicians' sojourn at Zurich will be made pleasant and interesting in every respect. Mathematicians from all over the world are cordially invited to participate in this event.

The Edison medal for 1931 has been awarded to E. W. Rice, Jr., of the General Electric Company, "for his contributions to the development of electrical systems and apparatus and his encouragement of scientific research in industry."

Professor Warren Weaver, Chairman of the Department of Mathematics at the University of Wisconsin, was appointed Director of the Division of Natural Sciences of the Rockefeller Foundation, beginning February 1, 1932.

Professor T. R. Hollcroft, of Wells College, delivered an address entitled *Algebraic curves and surfaces with assigned singularities* before the mathematical section of the University of Durham Philosophical Society, at Armstrong College, Newcastle, on November 27, 1931.

Professor J. W. Young, of Dartmouth College, a charter member and recent President of the Association, died at his home in Hanover, N. H., on February 17, 1932.

The following courses in mathematics are announced for the summer 1932:

University of Chicago, first term, June 20 to July 22; second term, July 25 to August 26. In addition to the regular courses in differential and integral calculus, the following advanced courses will be offered: By Professor H. E. Slaught: Theory of definite integrals. By Professor L. E. Dickson: Theory of numbers; Algebras and their arithmetics. By Professor E. P. Lane: Analytic mechanics; Surfaces and congruences. By Professor L. M. Graves: Solid analytic geometry; Theory of functions of a complex variable. By Professor W. T. Reid: Differential equations. By Professor Wilhelm Blaschke: Introduction to differential geometry; Topological questions of differential geometry. By Professor G. Bol: Galois Theory of equations. By Professor Walter Bartky: Partial differential equations in mathematical physics; Modern theories of differential equations II.

Columbia University, July 6 to August 14. In addition to courses in trigo-

nometry, solid geometry, analytic geometry, calculus, and methods of teaching secondary mathematics, the following advanced courses are offered: By Professor E. Kasner: Introduction to modern mathematics; Differential geometry. By Professor W. B. Fite: Functions of a complex variable. By Professor J. F. Ritt: Differential equations. By Professor R. G. Archibald: Theory of numbers.

Cornell University, July 11 to August 19. In addition to the usual elementary work, the following advanced courses will be offered: By Professor Virgil Snyder: Algebraic plane curves; Teachers' Course. By Professor F. R. Sharpe: Projective geometry. By Professor W. A. Hurwitz: Advanced calculus. By Professor W. B. Carver: Theory of numbers. By Professor D. C. Gillespie: Functions of a complex variable. Reading and research work will be directed by Professors J. I. Hutchinson, Virgil Snyder, F. R. Sharpe, W. A. Hurwitz, W. B. Carver, D. C. Gillespie, C. F. Craig.

University of Illinois, June 20 to August 13. In addition to the usual courses in college algebra, trigonometry, analytic geometry, and calculus, the following advanced courses are offered: By Professor A. B. Coble: Elliptic functions; Introduction to higher algebra. By Professor Arnold Emch: Geometric transformations; Geometry. By Associate Professor A. R. Crathorne: Analysis; Calculus of variations. By Assistant Professor H. R. Brahana: Algebra. By Assistant Professor H. W. Bailey: Fundamental concepts of mathematics. By Dr. L. L. Steimley: Introduction to higher analysis. By Dr. V. A. Hoersch: Introduction to higher geometry.

University of Iowa, first term, June 13 to July 21. In addition to courses in college algebra, trigonometry, analytic geometry and calculus, the following subjects are offered: By Miss Ruth Lane: Subject-matter and teaching of mathematics. By Professor Reilly: Quadrature and cubature formulas; Seminar in interpolation. By Professor Dines: Linear inequalities; Seminar in analysis. By Associate Professor Wylie: Celestial mechanics; Astronomy; Mathematics of finance. By Associate Professor Woods: Elliptic integrals; Modern geometry. By Assistant Professor Ward: Differential equations; Tensor analysis. By Dr. Robinson: Advanced calculus. By Mr. Fischer: Statistics. By the Staff: Reading and Research. Second term, July 25 to August 25. By Professor Chittenden: Advanced calculus; Theory of sets of points; Seminar in topology. By Professor Nordgaard: The history of mathematics; Theory of equations. By Dr. Conkwright: Differential equations; Theory of numbers. By Dr. Craig: Theory of probability. By the Staff: Reading and Research.

Johns Hopkins University, June 27 to August 5. By Professor F. D. Murnaghan: College algebra; Analytic geometry; Tensor analysis.

University of Kansas, June 8 to August 3. In addition to courses in trigonometry, algebra, and calculus, the following advanced courses are offered: By Professor Smith: Modern analytical geometry; Higher plane curves; Seminar. By Professor Jordan: Analytical mechanics. By Professor Wheeler: Higher algebra.

Massachusetts Institute of Technology, first period, June 14 to July 16. Calculus and differential equations, covering the prescribed work of the first two

years. Second period, July 27 to September 7. Courses in the first period repeated. August 8 to September 10. Courses in algebra, solid geometry and trigonometry, in preparation for fall entrance examinations in those subjects. June 14 to July 5. Course in theoretical aeronautics. July 6 to July 26. Course in aeronautics continued. June 14 to July 26. Advanced calculus.

University of Michigan, June 27 to August 19. In addition to courses in algebra, trigonometry, analytic geometry, elementary calculus, statistics, and finance, the following advanced courses will be offered: By Professor C. J. Coe: Topics in calculus. By Professor Norman H. Anning: Differential equations; Solid analytic geometry. By Professor W. O. Menge: Higher algebra. By Professor C. C. Craig: Theory of probability. By Professor W. D. Baten: Finite differences. By Professor Peter Field: Analytic projective geometry; Vector analysis. By Professor L. A. Hopkins: Analytic mechanics. By Professor W. B. Ford: Advanced Calculus; Infinite series, with special reference to Fourier Series. By Professor L. C. Karpinski: Teaching of algebra; History of geometry. By Mr. Dushnik: Empirical formulas. By Professor R. L. Wilder: Topology; Introduction to the foundations of mathematics. By Professor T. H. Hildebrandt: Theory of functions of a complex variable; Calculus of variations. By Professor G. Y. Rainich: Algebraic theory; Differential geometry. By Professor H. C. Carver: Mathematical theory of statistics. By Professor V. C. Poor: Applied mathematics—engineering problems. By Hildebrandt, Rainich, and others: Seminar in pure mathematics.

University of Minnesota, first term, June 15 to July 23. In addition to the usual elementary work the following courses will be offered: By Professor Dunham Jackson: History of ancient and modern mathematics; Frequency curves and surfaces. By Assistant Professor Gladys Gibbens: Differential equations. By Professors Jackson, Underhill, and Gibbens: Reading in advanced mathematics. Second term, July 25 to August 27. By Associate Professor Anthony L. Underhill: Elliptic integrals. By Professors Underhill and Carlson: Reading in advanced mathematics.

Northwestern University, June 20 to August 13. In addition to courses in trigonometry, college algebra, analytic geometry, differential and integral calculus, the following advanced courses will be offered: By Dr. H. L. Garabedian: Introduction to modern geometry. By Professor D. R. Curtiss: Advanced Calculus. By Professor H. A. Simmons: The theory of numbers.

University of Pennsylvania, July 5 to August 13. In addition to the elementary courses, the following advanced courses will be offered: By Professor G. H. Hallett: Infinite series. By Professor M. J. Babb: Differential equations. By Professor D. N. Lehmer of the University of California: Theory of numbers; Modern analytic geometry. By Professor J. R. Kline: Theory of functions of a complex variable. By Professor J. A. Shohat: Definite integrals.

University of Pittsburgh, June 13 to August 19. In addition to the undergraduate courses the following more advanced courses will be offered: By Professor K. D. Swartzel: Teaching of mathematics; Functions of a complex va-

riable. By Professor F. A. Foraker: Modern synthetic geometry; Projective geometry. By Professor J. S. Taylor: Advanced calculus; Functions of a real variable. By Associate Professor Culver: Differential equations; Theory of equations, or Theory of groups. Other graduate courses will be offered should demand warrant.

Syracuse University, June 27 to August 5. In addition to the regular courses in plane and solid geometry, intermediate and advanced algebra, trigonometry, analytic geometry, and differential and integral calculus, the following courses will be offered: By Professor May N. Harwood: Differential equations. By Professor I. S. Carroll: Teaching of mathematics. By Professor F. F. Decker: Introduction to theory of groups or History of Mathematics. By Professor A. D. Campbell: Functions of a real variable or Vector Analysis.

University of Texas, first term, June 7 to July 18; second term, July 20 to August 29. In addition to freshman courses the first term, the following courses are offered: First term. By Professor R. L. Moore: Foundations of geometry; Theory of sets. By Professor E. L. Dodd: Mathematical statistics; Analytic Functions. By Professor H. J. Ettlinger: Ruler and compass constructions; Fourier series. By Professor H. S. Vandiver: Number theory; Finite groups. By Professor P. M. Batchelder: Teaching problems in mathematics; Difference equations. By Professor Mary E. Decherd: Calculus. By Professors C. M. Cleveland and H. V. Craig: Advanced calculus. Second term. By Professor A. E. Cooper: Advanced calculus; Continuous groups. By Professor R. N. Haskell: Calculus; Potential theory. By Mr. E. C. Klipple: Differential equations; Non-Euclidean geometry.

University of Vermont. The following courses are offered by Professors Bullard, Butterfield, Millington: Courses in algebra, plane trigonometry, solid geometry, differential and integral calculus, differential equations, astronomy and history of mathematics.

University of Wisconsin, six weeks session, June 27 to August 5. By Professor T. Bennett: Differential equations; College geometry. By Dr. M. L. Hartung: The content of secondary mathematics; Teaching of mathematics. By Professor H. W. March: Advanced calculus. By Professor E. B. Skinner: Theory of equations; Finite groups. By Professor I. S. Sokolnikoff: Differential calculus; Vector analysis. Special nine weeks session for graduates, June 27 to August 26. By Professor R. E. Langer: The Lie theory of differential equations; Fourier's series. By Professor H. W. March: Theory of potential.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Sixteenth Summer Meeting of the Association, Los Angeles, Calif., Aug. 29-30, 1932.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1932 and reported to the Secretary.

ILLINOIS, Urbana, May 6-7.
INDIANA, Indianapolis, May 6-7.
IOWA, Cedar Falls, April 29-30.
KANSAS.
KENTUCKY.
LOUISIANA-MISSISSIPPI, Oxford, Miss., March 11-12.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May.
MICHIGAN, Ann Arbor, March 19.
MINNESOTA.
MISSOURI.
NEBRASKA.
OHIO, Columbus, Ohio, April 7.
PHILADELPHIA, Philadelphia, Pa., Nov. 26.
ROCKY MOUNTAIN, Laramie, Wyo., April 15-16.
SOUTHEASTERN, Gainesville, Fla., Mar. 18-19.
SOUTHERN CALIFORNIA, San Diego, March 26.
TEXAS, Austin, Jan. 30.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

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VOLUME XXXIX, 1932

NUMBER 4, APRIL

PUBLISHED BY THE ASSOCIATION

MENASHA, WIS., AND ITHACA, N.Y.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members

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NEW MATHEMATICAL PERIODICALS

By RAYMOND CLARE ARCHIBALD, Brown University

In this MONTHLY last October (p. 436–439) I gave a list of 32 periodicals which had been started during the five years 1926–1930 and which contained mathematical material of importance. Eight more periodicals are now to be added to this list, namely three from Japan, two from Germany, one from India, one from Italy, and one from Russia. The distribution of the 40 new periodicals is then as follows: Roumania (7), Italy (6), Germany (4), Japan (4), Poland (4), Argentina (3), United States (3), India (2), Russia (2), Czechoslovakia (1), England (1), Holland (1), Hungary (1), and Switzerland (1). Since seven of the eight periodicals now mentioned are research publications, at least 23 of the 40 may be so characterized. The titles of the new periodicals and additional notes concerning five of those already listed are given below.

2. *Bulletin de Mathématiques et de Physiques, Pures et Appliquées de l'Ecole Polytechnique de Bucarest*, v. 2, 1930–31, 246 p.

4. *Mathematica*, Kolozsvar [Cluj], v. 3, 1930, 157 p.; v. 4, 1930, 16+196 p.

Volume 3 is devoted to the papers and proceedings of the first Rumanian congress of mathematics held at Cluj, 9–12 May 1929. Although dated 1930 it was really not published till 1931, after volume 4. Many of the articles in *Mathematica* have been previously published in *Societatea de Stiinte din Cluj, Buletinul*.

14. *Mathesis Polska*, Warsaw, v. 5, 1930, 8+218 p.; v. 6, nos. 1–8, January–October 1931, p. 1–170.

As a supplement of each number of volume 6 appears a number of *Uranja*, volume 10, journal of the Polish Society of Friends of Astronomy.

16. *Studia Mathematica*, Lemberg, v. 2, 1930, 4+251 p.; v. 3, 1931, 4+247 p.

31. *Journal de la Société Physico-Mathématique de Leningrade*.

According to an article published in Berlin the Soviet government abolished further publication of this journal after volume 2.

33. *Blätter für Versicherungs-Mathematik und verwandte Gebiete*, (published by the Deutscher Verein für Versicherungs-Wissenschaft, Berlin, as supplement to its *Zeitschrift für die gesammte Versicherungs-Wissenschaften*), v. 1, July 1928–October 1930, 4+445 p.; v. 2, parts 1–4, 1931, p. 1–167.

Quarterly, in German, ten numbers to a volume. Contains well-edited articles, obituary notices, and reviews of current literature. A number of the articles are of considerable mathematical interest. In the issue for April 1931 there is a German-English, English-German dictionary of technical terms used in life insurance.

34. *Ingenieur-Archiv*, Berlin, v. 1, December 1929–December 1930, 4+648 p.; v. 2, March 1931–February 1932, 4+674 p.

A periodical with many articles which the all-round mathematician cannot afford to neglect.

35. *The India Physico-Mathematical Journal*, Dacca, v. 1, 1930, 36 p.; v. 2, no. 1, March 1931, 14 p.

Contains ten articles in English.

36. *Giornale dell' Istituto Italiano degli Attuari*, Rome, v. 1, 1930, 4, 236 p.; v. 2, 1931, 558 p.

In Italian. Contains many valuable articles by distinguished mathematicians.

37. *Journal of Science of the Hiroshima University, Series A (Mathematics, Physics, Chemistry)*, Hiroshima, Japan, v. 1, nos. 1–2, December 1930, p. 1–157.

Eleven articles (seven mathematical) in English and German.

38. *Journal of the Faculty of Science, Hokkaido Imperial University, Series I. Mathematics*, Sapporo, Japan, v. 1, nos. 1–2, September 1930–August 1931, p. 1–273.

Contains four articles in German.

39. *Science Reports of the Tokyo Bunrika Daigaku*, [Tokyo University of Literature and Science], *Section A* [$-B$], Koishikawa, Tokyo, v. 1, nos. 1–14, May 1930–December 1931, p. 1–165.

Contains fourteen articles (five mathematical) in English.

40. *Nauchno-Issledovatel'skiĭ Institut Matematiki i Mekhaniki pri pervom Moskovskom Gosudarstvennom Universitete*, [Research Institute of Mathematics and Mechanics of the First State Moscow University], [Reports], [nos. 1–7, 1926–1930], 24+55+91+27+68+63+20 p.

This appears to be one of a series of publications in continuation of Moscow, Universitet, Otdiel fiziko-matematicheskiĭ, *Uchenyia Zapiski* [Moscow, University, Faculty of physics and mathematics, Learned Memoirs] of which 29 volumes appeared, 1880–1926.

All seven numbers are in Russian but three have a synopsis in German. Two of the important numbers without synopsis are A. Khintchin's Fundamental Laws of the Theory of Probability (1927) and, L. Lusternik and L. Schnirelmann's Topological Methods in the Calculus of Variations (1930) listed as separate publications. A third number (1930) is mainly devoted to the life and work of Luigi Bianchi.

Among new periodicals of 1931 we may note the following:

I. *Sphinx, Revue Mensuelle des Questions Récréatives*, Brussels, v. 1, April–December 1931, 4+200 p.

Practically a continuation, under the editorship of M. Kraitchik, of the department "Récréations mathématiques" which he edited in *L'Echiquier* for several years. Articles and questions and notes relating to the theory of numbers naturally bulk large.

II. *Zentralblatt für Mathematik und ihre Grenzgebiete (reine und angewandte Mathematik, theoretische Physik, Astrophysik, Geophysik)*, Berlin, v. 1, April-September 1931, 5+466 p.; v. 2, nos. 1–6, 1931, p. 1–376.

Bibliographic abstract journal in English, French, German, and Italian.

III. *The Science Reports of National Tsing Hua University*, Peiping, China, *Series A, Mathematical and Physical Sciences*, v. 1, nos. 1–2, April and July 1931, p. 1–90.

The numbers contain eight articles in English; five of them are mathematical.

A new mathematical periodical under the editorship of Doctor Ginsburg is soon to be published by Jeshiva College, New York City. In it the history of mathematics and bibliography are to receive special attention.

January 26, 1932.

A DISCUSSION OF THE GENERAL LINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER, $d^2y/dx^2+P\,dy/dx+Qy=R$, IN WHICH P, Q, AND R ARE FUNCTIONS OF x.

By ROLAND SCHAFFERT, University of Cincinnati

Various methods have been devised to obtain solutions of special forms of the above equation, and many of these special forms lead to interesting results which are of practical significance.

In the first part of the present paper several forms of this equation, heretofore not discussed in the various methods for obtaining solutions of special forms, are integrated.

The second part contains a discussion of the general equation, in which it is shown that its solution may be made to depend upon the solution of a differential equation of the first order.

Part I

If we substitute $y = e^{\int a\,dx}$, where a is a constant, in the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0,$$

we get

$$e^{\int a\,dx}(a^2 + Pa + Q) = 0.$$

From this we obtain

$$a = \tfrac{1}{2}[-P \pm (P^2 - 4Q)^{1/2}].$$

In the equation,

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qv = R, \tag{1}$$

write

$$y = ue^{\frac{1}{2}\int[-P\pm(P^2-4Q)1/2]dx} \tag{2}$$

where u is a function of x to be determined. Equation (1) then becomes

$$(3)\qquad \frac{d^2u}{dx^2} + (P^2 - 4Q)^{1/2}\frac{du}{dx} + \frac{1}{2}\left[-\frac{dP}{dx} \pm \frac{d}{dx}(P^2 - 4Q)^{1/2}\right]u = Re^{-\frac{1}{2}\int[-P\pm(P^2-4Q)^{1/2}]dx}.$$

Now, if

$$(4)\qquad -\frac{1}{2}\frac{dP}{dx} \pm \frac{1}{2}\frac{d}{dx}(P^2 - 4Q)^{1/2} = 0,$$

equation (3) can at once be solved.

Equation (4) may be written $dP = \pm d(P^2-4Q)^{1/2}$ and therefore

$$P = \pm(P^2-4Q)^{1/2} \pm A.$$

Solving for Q, we get $-\frac{1}{4}A^2 \pm \frac{1}{2}AP$. Letting $\frac{1}{2}A = n$, we have $Q = -n^2 \pm nP$. Substituting this value of Q in equation (3), we have

$$\frac{d^2u}{dx^2} \pm (P \mp 2n)\frac{du}{dx} = Re^{-\frac{1}{2}\int[-P\pm(P\pm 2n)]dx}.$$

The first integral of this is

$$\frac{du}{dx} = Be^{-\int\pm(P\mp 2n)dx} + e^{-\int\pm(P\mp 2n)dx}\int Re^{\frac{1}{2}\int[P\pm(P\mp 2n)]dx}dx$$

and therefore

$$u = C + B\int e^{-\int\pm(P\mp 2n)dx}dx + \int\left[e^{-\int\pm(P\mp 2n)dx}\int Re^{\frac{1}{2}\int[P\pm(P\mp 2n)]dx}dx\right]dx.$$

Write this value of u in equation (2), and we get

$$y = Ce^{\frac{1}{2}\int[-P(P\mp 2n)]dx} + Be^{\frac{1}{2}\int[-P\pm(P\mp 2n)]dx}\int e^{-\int\pm(P\mp 2n)dx}dx + e^{\frac{1}{2}\int[-P\pm(P\mp 2n)]dx}\int\left[e^{-\int\pm(P\mp 2n)dx}\int Re^{\frac{1}{2}\int[P\pm(P\mp 2n)]dx}dx\right]dx,$$

which is a general solution of equation (1), where

$$-\frac{1}{2}\frac{dP}{dx} \pm \frac{1}{2}\frac{d}{dx}(P^2 - 4Q)^{1/2} = 0.$$

The form to which this may readily be applied is

$$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} - n^2y \pm nf(x)y = R(x).$$

Example 1:

$$\frac{d^2y}{dt^2} + t\frac{dy}{dt} - n^2y + nty = \sin(pt).$$

The solution of this is

$$y = Ce^{-nt} + B\frac{e^{nt-\frac{1}{2}t^2}}{2n-t} + e^{-nt}\int\left[e^{2n-\frac{1}{2}t^2}\int \sin(pt)e^{\frac{1}{2}t^2-nt}dt\right]dt.$$

Part II

In the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \tag{1}$$

let $Q = t+v$, where t and v are functions of x.
Equation (1) becomes

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + (t+v)y = R. \tag{3}$$

If we put

$$= ue^{\frac{1}{2}\int[-P\pm(P^2-4v)^{1/2}]dx}, \tag{4}$$

equation (3) becomes

$$\frac{d^2u}{dx^2} \pm (P^2-4v)^{1/2}\frac{du}{dx} + \frac{1}{2}\left[-\frac{dP}{dx} \pm \frac{d}{dx}(P^2-4v)^{1/2} + t\right]u = Re^{-\frac{1}{2}\int[-P\pm(P^2-4v)^{1/2}]dx}. \tag{5}$$

Thus far, the only limitation placed upon t and v is that their sum must be equal to Q. Hence we may choose any relation between them. Let this relation be

$$t = \frac{1}{2}\frac{dP}{dx} \mp \frac{1}{2}\frac{d}{dx}(P^2-4v)^{1/2}. \tag{6}$$

The third term on the left-hand side then vanishes, and equation (5) becomes

$$\frac{d^2u}{dx^2} \pm (P^2-4v)^{1/2}\frac{du}{dx} = Re^{-\frac{1}{2}\int[-P\pm(P^2-4v)^{1/2}]dx};$$

or, putting $(P^2-4v)^{\frac{1}{2}} = \theta$, this becomes

$$\frac{d^2u}{dx^2} \pm \theta\frac{du}{dx} = Re^{-\frac{1}{2}\int(-P\pm\theta)dx}. \tag{7}$$

This being an equation of the linear form in du/dx, when solved, it is

$$\frac{du}{dx} = Ae^{\pm\int\theta dx} + e^{\pm\int\theta dx}\int Re^{\frac{1}{2}\int(P\pm\theta)dx}dx,$$

It follows that

$$u = B + A\int e^{\mp\int\theta dx}dx + \int\left[e^{\pm\theta dx}Re^{\frac{1}{2}\int(P\pm\theta)dx}dx\right]dx;$$

so that

$$ye^{-\frac{1}{2}\int(-P\pm\theta)dx} = B + A\int e^{\mp\int\theta dx}dx + \int\left[e^{\mp\int\theta dx}Re^{\frac{1}{2}\int(P\pm\theta)dx}dx\right]dx$$

or

$$(8)\quad y = Be^{\frac{1}{2}\int(-P\pm\theta)dx} + Ae^{\frac{1}{2}\int(-P\pm\theta)dx}\int e^{\mp\int\theta dx}dx + e^{\frac{1}{2}\int(-P\pm\theta)dx}\int\left[e^{\mp\theta\int dx}\int Re^{\frac{1}{2}\int(P\pm\theta)dx}dx\right]dx.$$

It remains, then, to get a value for θ.
By combining equation (2) and equation (6), we get

$$\frac{1}{2}\frac{dP}{dx} \mp \frac{1}{2}\frac{d}{dx}(P^2 - 4v)^{1/2} = Q - v;$$

or

$$\frac{1}{2}\frac{dP}{dx} \mp \frac{1}{2}\frac{d\theta}{dx} = Q - \tfrac{1}{4}P^2 + \tfrac{1}{4}\theta^2,$$

so that[1]

$$(9)\qquad \pm\frac{d\theta}{dx} - \tfrac{1}{2}\theta^2 = \left(2Q - \tfrac{1}{2}P^2 - \frac{dP}{dx}\right).$$

Any solution whatever of this equation, giving a value for θ, permits a general solution of equation (1).
For example, if $Q = dP/dx$, a solution of equation (9) in this case is $\theta = P$. It is then evident that differential equations of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + \frac{dP}{dx}y = R$$

can be solved.

[1] Observe that the right-hand side of this expression is $2I$, where I is the invariant as defined in Forsyth's *Differential Equations*, Sixth edition, p. 105.

Example 2:

$$\frac{d^2y}{dx^2} + x^n \frac{dy}{dx} + nx^{n-1}y = \sin px.$$

We have here

$$\pm \frac{d\theta}{dx} - \frac{\theta^2}{2} = nx^{n-1} - \frac{x^{2n}}{2},$$

so that $\theta = x^n$. The primitive, then, is given by

$$y = Be^{-x^{n+1}/n+1} + Ae^{-x^{n+1}/n+1}\int e^{x^{n+1}/n+1}dx + e^{-x^{n+1}/n+1}\int\left[e^{x^{n+1}/n+1}\int \sin pxdx\right]dx.$$

Also, if $Q = \frac{1}{4}P^2 + \frac{1}{2}dP/dx$ equation (9) becomes $d\theta/dx - \frac{1}{2}\theta^2 = 0$.
Its solution is $\theta = \pm 2/x$.
It is thus seen that equations of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + \left(\frac{P^2}{4} + \frac{dP}{dx}\right)y = R$$

may be solved.

Example 3:

$$\frac{d^2y}{dx^2} + x^n \frac{dy}{dx} + \left(\frac{x^{2n}}{4} + nx^{n-1}\right)y = f(x).$$

Since $\theta = \pm 2/x$ the primitive of this is

$$y = Be^{\frac{1}{2}\int(2x^{-1}-x^n)dx} + Ae^{\frac{1}{2}\int(2x^{-1}-x^n)dx}\int e^{-\int 2x^{-1}dx}dx$$
$$+ e^{\frac{1}{2}\int(2x^{-1}-x^n)dx}\int\left[e^{-\int 2x^{-1}dx}\int f(x)e^{\frac{1}{2}\int(2x^{-1}+x^n)dx}dx\right]dx.$$

The equation for θ may in some cases become integrable by suitable substitutions. For example, if we let $\theta = -(1/z)(dz/dx)$, equation (9) becomes[1]

$$(d^2z/dx^2) + 4Iz = 0, \tag{10}$$

where $2I = 2Q - \frac{1}{2}P^2 - (dP/dx)$.

Evidently, if solutions of equation (10) are possible, we get values for θ satisfying the general solution of the original equation. For example, if $2I =$ a constant $= n$, then

$$z = e^{\pm ix\sqrt{2n}} \text{ and } \theta = i\sqrt{2n}.$$

In this case, $Q = \frac{1}{2}n + \frac{1}{4}P^2 + \frac{1}{2}dP/dx$. Hence the equations of the form

$$d^2y/dx^2 + Pdy/dx + [\tfrac{1}{2}n + \tfrac{1}{4}P^2 + \tfrac{1}{2}dP/dx]y = f(x)$$

can be solved.

[1] This is the normal form given by Forsyth, p. 105.

Example 4:

$$\frac{d^2y}{dx^2} + x^m \frac{dy}{dx} + [\tfrac{1}{2}n + \tfrac{1}{4}x + \tfrac{1}{2}mx^{m-1}]y = f(x).$$

Since $\theta = i\sqrt{(2n)}$ the primitive is

$$y = Be^{\frac{1}{2}\int[-x^m \pm i\sqrt{(2n)}]dx} + Ae^{\frac{1}{2}\int[-x^m \pm i\sqrt{(2n)}]dx}\int e^{\mp\int i\sqrt{(2n)}dx}dx$$

$$+ e^{\frac{1}{2}\int[-x^m \pm i\sqrt{(2n)}]dx}\int\left[e^{\mp\int i\sqrt{(2n)}dx}\int f(x)e^{\frac{1}{2}\int[x^m \pm i\sqrt{(2n)}]dx}dx\right]dx.$$

In general, if any solution whatever of

$$\pm (d\theta/dx) - \tfrac{1}{2}\theta^2 = 2Q - \tfrac{1}{2}P^2 - (dP/dx)$$

can be obtained, the general solution of the original differential equation is at once possible.

A Supplementary Note by Harris Hancock

The differential equation

$$\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = 0$$

may be written symbolically $f(D)y = 0$. (See Forsyth, *Differential Equations*, 4th Edition, page 66.) In this equation write $y = e^{kx}u(x)$, where k is a constant and $u(x)$ a function of x to be determined; and use the theorem given by Forsyth, p. 58. It is seen that

$$(1)\qquad \begin{aligned} f(D)y &= f(D)\{e^{kx}\cdot u(x)\} = e^{kx}f(D+k)\{u(x)\} \\ &= e^{kx}\left[f(k) + \frac{f'(k)}{1!}D + \cdots + \frac{f^n(k)}{n!}D^n\right]\{u(x)\}, \end{aligned}$$

the expansion terminating, as f is a polynomial of finite degree n.

If the equation $f(k) = 0$ has only simple roots, we give constant values to $u(x)$. If there are double roots, so that $f'(k) = 0$,we take $u(x) = A_1x + A$; if there is a triple root, so that $f''(k)$ is also zero, we take $u(x) = A_2x + A_1x + A$, etc. It is thus seen that in all cases the right-hand side of (1) is made zero. The complementary function may be thus determined without requiring the arbitrary constants, A_1 and A, say, to "become infinite in such a way" that the coefficients of the powers of x remain finite.

The same method is applicable to solutions of equations of the form

$$x^n\frac{d^ny}{dx^n} + A_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + A_{n-1}x\frac{dy}{dx} + A_ny = 0.$$

The above methods of solution were found by my colleague, Professor Louis Brand, when he was a student in one of my classes in the University of Cincinnati.

ON THE ISODYNAMIC POINTS OF FOUR SPHERES

By NATHAN ALTSHILLER-COURT, University of Oklahoma

1. Theorem. *The two inverses, with respect to two given spheres (circles), of a point on their sphere (circle) of similitude are symmetrical with respect to the radical plane (axis) of the given spheres (circles).*

Let P, Q, be the inverse points of the point W with respect to two given spheres (A), (B), having A, B for centers, and a, b for squares of their radii. We have then

$$AP\cdot AW = a,\quad BQ\cdot BW = b.$$

If, further, W is a point on the sphere of similitude of (A), (B), we have also

$$WA^2 : WB^2 = a:b,$$

hence

$$AP:BQ = AW:BW.$$

This proportion shows that the lines PQ, AB are parallel.

The pairs of points W, P; W, Q, being inverse with respect to (A), (B), any sphere (O) passing through W, P, Q, is orthogonal to both (A) and (B), hence the plane (R) through the center O of (O) and perpendicular to AB is the radical plane of the spheres (A), (B), and since PQ is parallel to AB the plane (R) is also perpendicular to PQ. But the plane through O perpendicular to PQ is clearly the mediator (i.e., the perpendicular bisecting plane) of the segment PQ, hence the proposition.

2. *Two imaginary spheres (A), (B), with real centers A, B, but the squares a, b of whose radii are negative, have a real sphere of similitude, namely the locus of the point W such that*

$$WA^2 : WB^2 = a:b.$$

The above proposition is valid for two such spheres, for in the proof no use is made of points on either of the two given spheres.

3. If the point W (1) describes the sphere of similitude (S) of the given spheres (A), (B), we have the theorem: *If the sphere of similitude of two given spheres is inverted with respect to the given spheres as spheres of inversion, the two inverse spheres thus obtained are equal and symmetrical with respect to the radical plane of the given spheres.*

It may further be observed that the centers of the two inverses (S_a), (S_b) of (S) with respect to (A), (B) lie on the line of centers of (A), (B). For (S) is coaxal with (A), (B), and (S_a) is coaxal with (A) and (S). Similarly for (S_b).

4. Definitions. Four given spheres (A), (B), (C), (D), taken in pairs have six spheres of similitude. These six spheres have two points W, W' in common, real or conjugate imaginary.[1] The points W, W' are the limiting points of the

[1] J. L. Coolidge. A treatise on the circle and the sphere, p. 244, Theorem 55. Oxford, 1916.

coaxal pencil of spheres determined by the orthogonal sphere (R) of the four given spheres and the sphere (O) passing through the centers A, B, C, D, of these four spheres. The points W, W' are thus perfectly determined, independently of the six spheres of similitude, and may be considered as two points connected directly with the four given spheres. For want of a better term the points W, W' may be called the "isodynamic points" of the four given spheres.

The radical plane of the two spheres (R), (O) is sometimes referred to as the "Newtonian plane" of the spheres (A), (B), (C), (D). This plane contains the centers of the six spheres of similitude of the given spheres.[1]

In what follows it is assumed that the isodynamic points are real, which implies that the four given spheres are either all real or all imaginary.

5. Theorem. *The four inverses, with respect to four given spheres, of an isodynamic point of these spheres, determine a sphere concentric with the orthogonal sphere of the given spheres.*

With the same notations as in (4), let P, Q, be the inverses of W with respect to (A), (B). The lines PQ, AB, are parallel (1), hence the mid-point L of PQ and the mid-point M of the line of centers AB of (A), (B), are collinear with W, and we have

$$WP:WA = WL:WM.$$

We observe also that the mediators of the segments PQ, AB are parallel.

The mediator of AB contains the center O of the sphere (O) (4), and the mediator of PQ coincides with the radical plane of the spheres (A), (B) (1), and therefore contains the center R of the sphere (R) (4). Furthermore the points R, O, are collinear with W. Hence OM is parallel to RL, and we have

$$WL:WM = WR:WO$$

hence

$$WP:WA = WR:WO.$$

This proportion shows that the lines OA, RP are parallel, and we have

$$RP:OA = WR:WO.$$

The last three terms of this proportion do not depend upon the sphere (A) considered, hence RP is constant, which proves the proposition.

6. Let (R') denote the sphere determined by the four inverses of W with respect to the spheres (A), (B), (C), (D). It follows from the proof of the preceding proposition (5) that the point W divides the line of centers OR of the two spheres (O), (R') in the ratio of the radii of these spheres, hence the isodynamic point W is a center of similitude of the two spheres (O) and (R').

[1] Mathesis, 1926, pp. 467–8.

7. The last two propositions (5, 6) have their analogues in relation to three circles in the plane. The reader may find it of interest to formulate these propositions. Their proofs differ but little from those given above.

8. Let a be the radius of the sphere (A) (5). The points W, P being inverse with respect to (A), we have

$$AP\cdot AW = a^2,$$

and from the similar triangles WOA, WRP we have

$$AW:AP = OW:OR$$

hence

$$AW^2:a^2 = OW:OR. \tag{a}$$

Since the right hand side of this proportion does not depend upon the sphere (A) considered we have here a proof of the known proposition: *The distances of an isodynamic point of four spheres from the centers of these spheres bear a constant ratio to the radii of the respective spheres.* Furthermore the proportion (a) furnishes a geometric meaning for the constant involved in this proposition.

9. Applying Stewart's theorem to the triangle WOA (8), we have, both in magnitude and in sign,

$$OW\cdot AR^2 + WR\cdot OA^2 + RO\cdot AW^2 + OW\cdot WR\cdot RC = 0.$$

Let a, r be the radii of the spheres (A), (O) (5), and s the square of the radius of the orthogonal sphere (R) of the four given spheres (A), (B), (C), (D), (5).

The two spheres (A), (R), being orthogonal, we have

$$AR^2 = a^2 + s.$$

Further, if we put (8)

$$AW^2:a^2 = OW:OR = 1:k$$

we have

$$WR = (k-1)OW,\ RO = -k\cdot OW,\ RO\cdot AW^2 = -OW\cdot a^2.$$

Substituting these values in the Stewart formula above, we obtain, after simplification,

$$s = (1-k)(r^2 - k\cdot OW^2). \tag{b}$$

Thus if the positions of the three points R, O, W, are known, and either s or r^2 is given, the other radius is determined.

10. The foregoing propositions yield interesting resulst when applied to the tetrahedron. By way of introduction, I shall state here, for the convenience of

the reader, the following propositions, since they can hardly be assumed to be "well known," nor can they be dismissed with a ready reference. They are due to Gaspar Monge (1746–1818).[1]

(a). Given a tetrahedron $(T)=ABCD$, if through the three pairs of opposite edges of (T) we pass three pairs of parallel planes we obtain a parallelepiped (P) which is said to be "circumscribed" about (T). Two opposite edges of (T) are two non-parallel diagonals of a pair of opposite faces of (P).

(b). Let A', B', C', D', be the vertices of (P) diagonally opposite A, B, C, D, respectively. The diagonal AA' of (P) meets the face BCD of (T) in the centroid G_a of the triangle BCD, and $AG_a=\frac{2}{3}AA'$. Thus the four diagonals of (P) contain the medians AG_a, BG_b, CG_c, DG_d, of (T), and the centroid G of (T) is the point common to the diagonals of (P).

(c). The six planes perpendicular to the edges of a tetrahedron (T) and passing through the mid-points of the respectively opposite edges have a point M in common, the "Monge point" of the tetrahedron (T). The six planes themselves may be referred to as the "Monge planes" of (T). The Monge point is the symmetric of the circumcenter O of (T) with respect to the centroid G of (T).

(d). The tetrahedron (T') determined by the points A', B', C', D' (10b) is also inscribed in (P). The two tetrahedrons (T), (T') may be said to be "twin" to each other. They are symmetrical with respect to their common centroid G, so that the Monge point M of (T) is the circumcenter of (T'), and the Monge planes of (T) are the mediators of the edges of (T').

11. Consider now the four spheres (A), (B), (C), (D), having for diameters the medians AG_a, BG_b, CG_c, DG_d, of the tetrahedron $(T)=ABCD$. The distance of the centroid G of (T) from the center of (A) is equal to $\frac{1}{4}AG_a$, according to Commadino's theorem, and similarly for the other three spheres. Thus the distances of G from the centers of the four spheres considered are proportional to the radii of these spheres, hence G is an isodynamic point of the four spheres (A), (B), (C), (D) (8).

The inverse of G with respect to the sphere (A) coincides with the harmonic conjugate X of G with respect to A, G_a. Now since

$$AG:GG_a = 3:1,$$

we have

$$AG_a:G_aX = 2:1,$$

i.e., X coincides with the vertex A' of the tetrahedron (T') twin to (T) (10).

Thus the four inverses of G with respect to (A), (B), (C), (D), are the vertices of (T'), hence the sphere determined by these four inverses is identical with the circumsphere of (T'). Now the circumcenter of (T') is the Monge point M of (T) (10), hence (5): *The radical center of the four spheres having for diameters the medians of a tetrahedron coincides with the Monge point of the tetrahedron.*

[1] Correspondence sur l'Ecole Polytechnique, vol. II.

12. The line of centers of the spheres (A), (B), (11) is parallel to the edge AB of (T), hence the radical plane (F) of these two spheres is perpendicular to AB. But (F) also passes through the Monge point M of (T) (11), hence (F) is a Monge plane of (T) (10c), and therefore bisects the edge CD opposite to AB on (T). This property of the radical plane of the two spheres (A), (B), has been pointed out by Prof. R. W. Genese.[1]

13. The centers of the four spheres (A), (B), (C), (D) (11) are the homothetic points of the vertices A, B, C, D, of the tetrahedron (T), the point G being the homothetic center, and the homothetic ratio being $1:3$; hence the sphere (O') determined by these four centers is the homothetic of the circumsphere (O) of (T). Thus the center O' of (O') lies on the Euler line MGO of (T), and we have $GO' : GO = 1:3$; and further if r, r' are the radii of (O), (O'), we have $r':r = 1:3$.

The square of the radius of the orthogonal sphere (M) of the four spheres considered may now be determined by the formula (b) of (9). The ratio $OW:OR$ is here to be replaced by $O'G:O'M = 1:4$, hence $k = 4$, and (b) gives for the square of the radius of (M)

$$s = (1 - 4)(r'^2 - 4.0'G^2).$$

In order to have s in terms of the elements of (T), we may write

$$s = (1 - 4)(r^2:9 - 4\cdot OG^2:9) = \tfrac{1}{3}(4\cdot OG^2 - r^2) = \tfrac{1}{3}(MO^2 - r^2).$$

Now the last parenthesis represents the power of M with respect to (O), hence: *The square of the radius of the orthogonal sphere of the four spheres having for diameters the four medians of a tetrahedron is equal to one third of the power of the Monge point of the tetrahedron with respect to the circumsphere of the tetrahedron.*

13a. The value of s may be obtained directly, without the use of formula (b). Let H be the other isodynamic point, besides G, of the four spheres (A), (B), (C), (D) (13). The two points are thus inverse with respect to both (O') and (M), hence

$$O'G\cdot O'H = r'^2, \quad MG\cdot MH = s$$

Now

$$MG = -3O'G, \quad MH = MO' + O'H = -4O'G + O'H,$$

hence

$$s = 12O'G^2 - 3r'^2.$$

From this formula and the relations

$$MO = -2\cdot OG = -6\cdot O'G, \quad r = 3r'$$

we obtain the required formula (13).

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[1] Educational Times, Reprints, vol. VI (1918), p. 57, Q. 18677.

14. In a given sphere (O) an infinite number of tetrahedrons (T) may be inscribed having for centroid a given point G, taken inside the sphere (O). All these tetrahedrons will have the same Monge point M (10c). Now the power of the fixed point M with respect to the fixed sphere (O) is obviously fixed, hence the sphere (M) (13) is fixed. It is also clear that the sphere (O') (13) is fixed. Hence: *If a tetrahedron varies so that its circumsphere and its centroid remain fixed, the spheres having for diameters the medians of the tetrahedron are orthogonal to a fixed sphere and have their centers on another fixed sphere.*

It may be observed that the sphere (M) will be real, null, or imaginary, according as the distance of the centroid G from the center O of (O) is greater than, equal to, or smaller than half the radius of (O).

15. Suppose the median AG_a of the tetrahedron $(T)=ABCD$ be divided by the point A' in the given ratio $GA':GA=u$. Let (A') be the sphere with A' as center, and $a'=v.GA$ as radius, where v is a given number. Let (B'), (C'), (D'), be the analogous spheres relative to the vertices B, C, D, of (T).

As before (11) here again the distances of the centroid G of (T) to the center of the four spheres considered are proportional to the radii of these spheres, hence G is an isodynamic point of these spheres. Furthermore the circumsphere (O) of (T) and the sphere (O') determined by the centers of the four spheres (A'), (B'), (C'), (D'), are homothetic with respect to G, hence the center O' of (O') is collinear with O and G, i.e., O' lies on the Euler line OG of (T). Now the radical center R of the four given spheres is collinear with G and O', hence: *If the medians of a tetrahedron are divided in the same ratio and the points of division are taken for centers of four spheres whose radii are proportional to the respective medians, the radical center of these four spheres lies on the Euler line of the tetrahedron.*

The special case, when the points of division coincide with the vertices of (T) and the radii are equal to the medians, has already been noticed.[1]

16. The exact position of R(15) with reference to G and O may be found by the formula (a) established before (8). We have, in the present case,

$$GA'^2:(v\cdot GA)^2 = O'G:O'R, \text{ and } GA' = u\cdot GA;$$

hence

$$O'G:O'R = u^2:v^2.$$

On the other hand we have, from the homothetic ratio of the spheres (O'), (O),

$$GO':GO = GA':GA = u,$$

hence

$$O'G = u\cdot OG$$

[1] Hoffmann's Zeitschrift, vol. 27 (1896), p. 589, Q. 1456.

and

$$O'G:O'R = O'G:(O'G + GR) = u\cdot OG:(u\cdot OG + GR) = u^2:v^2$$

hence

$$OR:OG = (v^2 - u^2 + u):u. \qquad (f)$$

This is the required formula, in magnitude and in sign.

If A' coincides with the mid-point of AG_a, and the radius of (A') is equal to one half of AG_a, we have $u = 1/3$, $v = 2/3$, and the formula (f) gives

$$OR:OG = 2$$

i.e., R coincides with the Monge point of (T), as has been found already (11).

17. The value of the square of the radius of the orthogonal sphere (R) of the four spheres (A'), (B'), (C'), (D') (15) may be found by the formula (b), derived before (9). The value of $1:k$ of that formula is in the present case equal to $u^2:v^2$. If r, r' are the radii of the spheres (O), (O'), we have

$$r' = ur, \text{ and } O'G = u.OG.$$

Thus the formula (b) gives for the square of the radius of (R)

$$s = (u^2 - v^2)(u^2r^2 - v^2\cdot OG^2):u^2.$$

For the special case when $u = 1/3$, $v = 2/3$ this formula gives

$$s = \tfrac{1}{3}(4\cdot OG^2 - r^2)$$

as was to be expected (13).

TWO PROBLEMS IN POTENTIAL THEORY

By THORNTON C. FRY, Bell Telephone Laboratories

1. *Introductory Remarks.*

"Electron guns" are electrical devices that produce a beam of electrons, all traveling as nearly as possible in the same direction. They are generally constructed on a very simple principle. The source of electrons is enclosed by a metal surface, in which a small hole is cut. When this surface is maintained at a potential higher than that of the source, the electrons are drawn toward it. Some, however, pass through the hole by virtue of their own inertia. Under suitable conditions, all the particles in this "beam" will move with substantially the same speed and in substantially the same direction.

The study of the potential field set up in such an electron gun therefore requires only the solution of Laplace's equation, with two known equipotential surfaces as boundary conditions. In this sense it belongs to the simplest type of Potential Theory problems; but the shape of the electrodes, and in particular the presence of the holes, removes it from the class of the purely trivial.

The two particular cases treated in this paper have been chosen largely because they require such radically different methods of attack. It is interesting to note that, in spite of the similarity of the problems when expressed in physical terms, neither of the methods can be used conveniently, if at all, in the solution of both problems.

2. *The Potential Field about a Slotted Cylinder.*

The first problem is that of a fine cylindrical wire placed on the axis of a slotted cylinder. In order to simplify the solution, we shall regard the wire as infinitely thin[1] (a mathematical line), and shall call the radius of the cylinder r. For convenience we shall also choose the axis of the cylinder as one coordinate axis; we know by symmetry that this coordinate will not appear in our solution. Another coordinate axis (called u) will be passed through the center of the slot, and of course the third (called v) will be normal to both. The cross-section of the system thus has the appearance shown in Fig. 1.

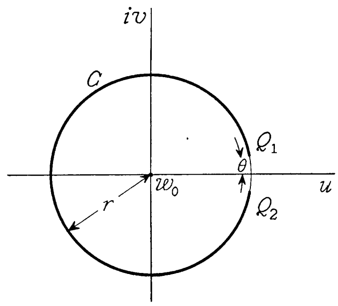

FIG. 1

Our problem, then, is to satisfy the differential equation

$$\nabla^2\phi = 0 \tag{1}$$

in two dimensions, subject to the boundary conditions, first, that[2]

$$\phi = 0, \text{ along curve } C, \tag{2}$$

and second, that ϕ shall have a logarithmic[3] infinity at the origin, which we may write

$$\lim_{(u^2+v^2)^{1/2} \doteq 0}\left[\frac{\phi(u, v)}{\log (u^2 + v^2)^{1/2}}\right] = k, \tag{3}$$

k being a finite constant, not zero.

[1] After the solution has been obtained it will be found that the equipotential surfaces in the neighborhood of this line differ but little from concentric cylinders of circular cross-section, so that the solution is an excellent approximation to the case of a wire of finite dimensions inside a cylinder of much larger radius.

[2] To give ϕ any other value Φ along this curve, it is only necessary to add the constant Φ to the solution which satisfies (2), since if (1) and (3) are satisfied by ϕ they are also satisfied by $\phi+\Phi$.

[3] The potential field produced by a charge distributed along an infinitely thin wire is known to have a logarithmic singularity. See O. D. Kellogg, *Foundations of Potential Theory*, p. 62.

These conditions are obvious enough. There is another which might be overlooked. It is, that the potential shall *not* become infinite at infinite distance from the cylinder. To violate this condition would be equivalent to supposing that the slotted cylinder C was surrounded by another (infinitely) large conductor on which additional charge was placed.

Since either the real or imaginary part of any analytic function satisfies (1), the problem may also be stated as follows: to find an analytic function $f(w)$, where $w=u+iv$, of which the real part vanishes along C and becomes logarithmically infinite at w_0. Moreover, since singularities can only occur in the presence of charge,[1] $f(w)$ must be regular throughout all space, except perhaps on C and at w_0.

Such a function can easily be found by the use of certain simple conformal transformations.

We begin with the variable $\zeta=\xi+i\eta$ and write down the analytic function

$$f = \tfrac{1}{2}k \log \left[(\zeta - \zeta_0)/(\zeta - \bar{\zeta}_0)\right], \tag{4}$$

in which ζ_0 is an arbitrary complex number and $\bar{\zeta}_0$ its conjugate. (See Fig. 2). For the time being k may also be regarded as an arbitrary constant, though we

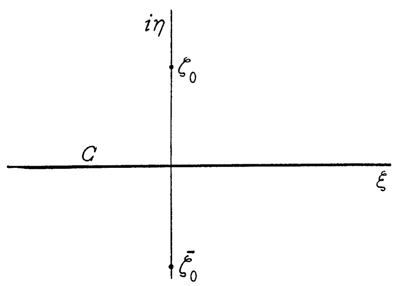

FIG. 2

shall see later on that it has been so chosen as to conform to the use of the same letter in equation (3).

It is readily seen that the real part of this function f vanishes along the entire real axis, and becomes logarithmically infinite at the point[2] ζ_0. In other words, it possesses at these places just the properties which our desired function must have on the curve C and at the point w_0 of Fig. 1; and our purpose is to construct from it a new function which will have these properties where desired.

The transformation

$$z = (\zeta^2 + a^2)^{1/2} \tag{5}$$

maps the ζ-plane on the z-plane in such a way that the real axis of the former corresponds to that part of the real z-axis for which $|x|>a$. (See Fig. 3). The

[1] See O. D. Kellogg, loc. cit., p. 139.

[2] It also becomes logarithmically singular at $\bar{\zeta}_0$, but this will cause us no difficulty, for reasons which will appear later.

remainder of the real z-axis corresponds to imaginary values of ζ; and $z = \pm a$ are branch points. The arbitrary point $\bar{\zeta}_0$ goes into an arbitrary point z_0 of Fig. 3.

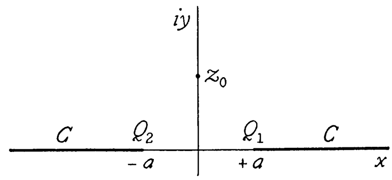

FIG. 3

Since every possible value of z may be obtained by letting ζ vary over the upper half of its plane, the physical z-plane may be said to correspond to only that part of the ζ-plane. As $\bar{}_0$ is thus excluded, the undesirable singularity of (4) has now disappeared.[1]

It follows then, that the function

$$f = \tfrac{1}{2}k \log \frac{(z^2 - a^2)^{1/2} - \zeta_0}{(z^2 - a^2)^{1/2} - \bar{\zeta}_0} \tag{6}$$

which we obtain by expressing (4) in terms of z, must have a real part which vanishes along the part C of the real z-axis, and which becomes logarithmically infinite at z_0.

Finally, we use the bilinear transformation

$$w = r[(i - z)/(i + z)]. \tag{7}$$

which carries all the straight lines of the z-plane into circles in the w-plane. In particular, the real axis of Fig. 3 is carried into the circle of radius r in Fig. 1; and the portion of the axis which was C in Fig. 3 therefore becomes the heavy circular arc of Fig. 1. The point z_0 of Fig. 3 also goes into some point w_0 of Fig. 1.[2]

It follows, then, that the real part of the function[3]

$$f = \tfrac{1}{2}k \log \frac{\left[-a^2 - \left(\dfrac{w - r}{w + r}\right)^2\right]^{1/2} - \zeta_0}{\left[-a^2 - \left(\dfrac{w - r}{w + r}\right)^2\right]^{1/2} - \bar{\zeta}_0}, \tag{8}$$

[1] In other words, (5) defines a multiple-valued transformation, only one branch of which is of physical significance. The singularity at ζ_0 has passed into the non-physical branch.

[2] Whether or not w_0 is at the center of the circle depends, of course, on the choice of the (as yet) arbitrary point ζ_0. In constructing the figures, however, we have placed all the arbitrary points in the positions which our subsequent discussion shows they should occupy.

[3] Since we are to use only ζ's which lie above the real axis, the radicals in (6) and (8) must all be chosen with their imaginary part positive. This leads to the rule given later on, that the *real* part of the radicals in (9), (11) and (14) must be positive.

which we obtain by rewriting (6) in terms of w, vanishes along the arc C and becomes logarithmically singular at w_0.

We must choose a and ζ_0 in such a way that $w_0=0$, and also so that the angular distance between the points Q_1 and Q_2 of Fig. 1 is 2θ. But if we give w_0 the value zero—that is, if we place the singularity at the center of the circle—we obtain from (7)

$$z_0 = i,$$

and hence also from (5)

$$\zeta_0 = i(a^2 + 1)^{1/2}. \tag{9}$$

These equations fix the positions of the points ζ_0 and z_0 in Figs. 2 and 3.

Similarly, the points Q_1 and Q_2 in Fig. 3 correspond to $z=\pm a$, and hence by (7) the corresponding values of w are

$$w = r[(i \mp a)/(i \pm a)].$$

But to conform to the notation of Fig. 1, we must write this as

$$w = re^{\pm i\theta};$$

and upon comparing these equations we readily obtain the relation

$$a = \tan \tfrac{1}{2}\theta. \tag{10}$$

Finally, upon substituting (9) and (10) in (8) we get the result

$$f = \tfrac{1}{2}k \log \frac{\left[\tan^2 \tfrac{1}{2}\theta + \left(\dfrac{w-r}{w+r}\right)^2\right]^{1/2} - \sec \tfrac{1}{2}\theta}{\left[\tan^2 \tfrac{1}{2}\theta + \left(\dfrac{w-r}{w+r}\right)^2\right]^{1/2} + \sec \tfrac{1}{2}\theta}, \tag{11}$$

in which, as noted in the footnote accompanying equation (8), the real parts of the radicals are to be taken positive.

This function satisfies the differential equation (1) and the boundary value (2). It also has a logarithmic singularity at the origin. But we are still not sure that the k in (11) has really been so chosen that (3) is satisfied, nor that the function is regular at infinity. To decide these points we must now investigate the behavior of (11) when w is very small and when w is very large.

We readily find that, in the former case

$$f \doteq \tfrac{1}{2}k[\log (w/r) + \pi i + \log \cos^2 \tfrac{1}{2}\theta], \tag{12}$$

and in the latter,

$$f \doteq \tfrac{1}{2}k[- \log (w/r) + \pi i + \log \cos^2 \tfrac{1}{2}\theta]. \tag{13}$$

We first notice that the term $-\frac{1}{2}k \log (w/r)$ in (13) becomes infinite with w. This violates the requirement of regularity at infinity. But as this term is itself a solution of Laplace's equation, which becomes infinite at $w=0$, and the real part of which vanishes over C, both (1) and (2) will be satisfied by $f+\frac{1}{2}k \log(w/r)$ which is regular at infinity, as well as by f itself. Moreover, we easily see from (12) that in the neighborhood of w_0

$$[f + \tfrac{1}{2}k \log (w/r)] \doteq k \log (u^2 + v^2)^{1/2},$$

the remaining terms on the right remaining finite as u and v vanish. Hence (3) also is satisfied by the real part of the function $f+\frac{1}{2}k \log (w/r)$. Thus the solution of our problem is the real part of

$$\phi = \tfrac{1}{2}k\left[\log (w/r) + \log \frac{\left[\tan^2 \frac{1}{2}\theta + \left(\dfrac{w-r}{w+r}\right)^2\right]^{1/2} - \sec \frac{1}{2}\theta}{\left[\tan^2 \frac{1}{2}\theta + \left(\dfrac{w-r}{w+r}\right)^2\right]^{1/2} + \sec \frac{1}{2}\theta}\right]. \tag{14}$$

This formula (14) is in suitable form for computation when we wish to know the exact potential at any given point (u, v). If, however, we wish to locate an equipotential surface—that is, the locus of all points at which the potential has a given value—it turns out to be more convenient to put (14) in another form. For this purpose, we shall write $s=w/r$ and $t=-\phi/k$, and then solve for s in terms of t. The result is

$$s = \frac{e^{-t} + i \sec \frac{1}{2}\theta}{e^{t} - i \sec \frac{1}{2}\theta}. \tag{15}$$

If we now assign to t a succession of values all having *the same real part*, and plot the results in the complex s-plane, we will obtain the locus of points at which the potential has the assigned value.

Fig. 4 was produced in this way. It shows the equipotential surfaces about a cylinder, the slit in which is of angular width 20°.

It should be said in explanation of this figure that the heavy circular arc represents the split cylinder, and that the figures attached to the curves are eighteen times[1] the real parts of t to which they correspond. The curves marked 0.01, 0.02, $\cdots$, 0.5, after passing the edge of the slit, lie so close to the inner surface of the cylinder that it is impossible to distinguish between them on the scale of the drawing. These curves, together with those marked 1, 2 and 5, were actually computed from formula (15). The remaining curves were drawn in accordance with the usual logarithmic potential formula,[2] from which they differ so little that it would be impossible to detect the discrepancy on the scale of the drawing.

[1] This apparently arbitrary factor was introduced to afford more convenient comparison with the case where the electrode is a plane from which a slit has been cut. It will be noticed that just 1/18 of our cylinder has been cut away.

[2] That is, the number attached to any circle is eighteen times the negative of the Naperian logarithm of its radius.

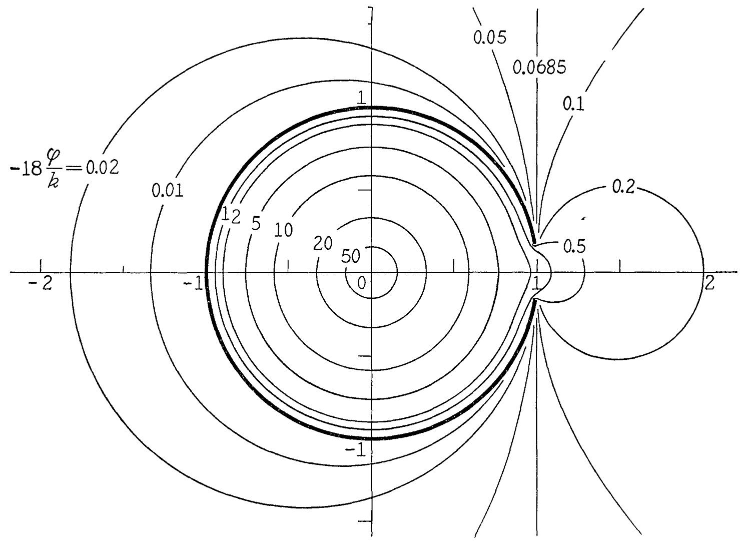

FIG. 4

Two things are immediately evident from this figure. One is that, although our solution has been obtained upon the assumption that the inner conductor is an *infinitely thin* wire, the result actually holds with a very high degree of accuracy, even when the inner cylinder is nearly as large as the outer one. For the field would be unchanged if the inner conductor had the same shape as any equipotential surface; and even so large an oval as that marked 5 differs but little from a circle.

The other is, that the stray field due to the slit in the cylinder is very small, even when the slit is as wide as that shown in the figure.

3. *The Potential Field Between Two Parallel Planes in One of Which There is a Circular Hole.*

The second problem deals with a pair of parallel planes, in one of which there is a circular hole. This problem can also be much simplified by assuming that the planes are infinitely far apart—an assumption which is in all respects analogous to that used in the first problem when we spoke of an "infinitely thin wire."

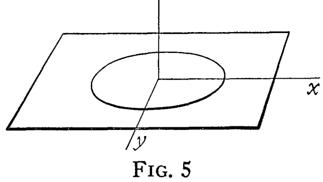

FIG. 5

Our problem may then be stated as that of satisfying Laplace's equation subject to the boundary conditions that: (1) ϕ shall vanish over the entire plane $z=0$ (see Fig. 5), except within a circle of radius r about the origin; (2) that the

gradient $\partial\phi/\partial z$ shall approach a well-defined value k as z becomes positively infinite; and (3) that $\partial\phi/\partial z$ shall approach zero as z becomes negatively infinite.[1]

The essential element in our method of attack upon this problem will be the choice of a suitable coordinate system. Specifically, we shall choose it in such a way that one of the coordinate surfaces occupies exactly that part of the plane $z=0$ which represents the physical electrode.

This can be done most easily by rotating the ellipses and hyperbolas of Fig. 6 about their minor axes, thus generating a set of mutually orthogonal ellipsoids and hyperboloids. In doing this, the hyperboloid $\mu=0$ degenerates into the desired physical electrode; while the ellipsoid $\lambda=0$ degenerates into the area of the hole.

It is easy to see that the curvilinear coordinates λ, μ, θ are related to the Cartesian coordinates x, y, z by the equations

$$
\begin{aligned}
x &= b \cosh \lambda \cos \mu \cos \theta, && 0 \leqq \theta \leqq 2\pi, \\
y &= b \cosh \lambda \cos \mu \sin \theta, && 0 \leqq \mu \leqq \pi/2, \\
z &= b \sinh \lambda \sin \mu, && -\infty \leqq \lambda \leqq \infty,
\end{aligned}
\tag{16}
$$

or what amounts to the same thing,

$$
\begin{aligned}
\tan\theta &= y/x, \\
r^2 &= x^2 + y^2 + z^2, \\
2b^2 \sinh^2 \lambda &= r^2 - b^2 + [(r^2 - b^2)^2 + 4b^2z^2]^{1/2}, \\
2b^2 \cosh^2 \lambda &= r^2 + b^2 + [(r^2 - b^2)^2 + 4b^2z^2]^{1/2}, \\
2b^2 \sin^2 \mu &= b^2 - r^2 + [(r^2 - b^2)^2 + 4b^2z^2]^{1/2}, \\
2b^2 \cos^2 \mu &= r^2 + b^2 - [(r^2 - b^2)^2 + 4b^2z^2]^{1/2};
\end{aligned}
\tag{17}
$$

b being the radius of the circular hole.

It is not difficult to establish the fact that in this coordinate system the Laplacian operator takes the form[2]

$$
\begin{aligned}
\nabla^2\phi = \frac{1}{b^2(\sinh^2\lambda + \sin^2\mu)\cosh\lambda\cos\mu}\Bigg[&\frac{\partial}{\partial\lambda}\cosh\lambda\cos\mu\frac{\partial\phi}{\partial\lambda} \\
&+ \frac{\partial}{\partial\mu}\cosh\lambda\cos\mu\frac{\partial\phi}{\partial\mu} + \frac{\partial}{\partial\theta}\frac{\sinh^2\lambda+\sin^2\mu}{\cosh\lambda\cos\mu}\frac{\partial\phi}{\partial\theta}\Bigg].
\end{aligned}
$$

However, since ϕ is independent of θ by symmetry, all derivatives with respect to θ drop out, and Laplace's equation takes the simpler form

[1] This last condition merely asserts that there shall be no third electrode (and hence no charge) *below* the xy-plane in Fig. 5.

[2] See Kellogg, loc. cit., p. 183.

$$\frac{1}{\cosh\lambda}\frac{\partial}{\partial\lambda}\cosh\lambda\frac{\partial\phi}{\partial\lambda}+\frac{1}{\cos\mu}\frac{\partial}{\partial\mu}\cos\mu\frac{\partial\phi}{\partial\mu}=0. \tag{18}$$

Our problem is therefore to solve this equation subject to the boundary conditions

$$\begin{aligned}\phi &= 0 \text{ on } \mu = 0;\\ d\phi/dz &= k \text{ at } z = +\infty;\\ d\phi/dz &= 0 \text{ at } z = -\infty.\end{aligned} \tag{19}$$

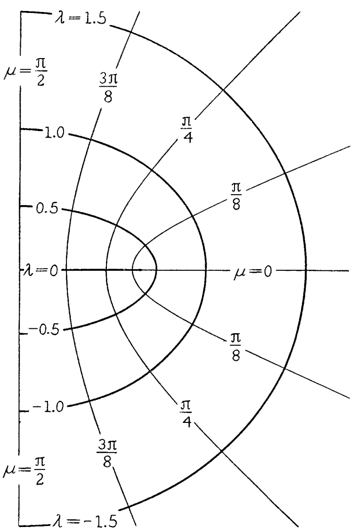

FIG. 6

Upon assuming that ϕ is the product of a function $\Lambda(\lambda)$ which does not depend upon μ and a function $M(\mu)$ which does not depend upon λ, we find that Λ and M must separately satisfy the equations

$$\begin{aligned}&\frac{d}{d\lambda}\cosh\lambda\frac{d\Lambda}{d\lambda}-m(m+1)\Lambda\cosh\lambda=0,\\ &\frac{d}{d\mu}\cos\mu\frac{dM}{d\mu}+m(m+1)M\cos\mu=0,\end{aligned} \tag{20}$$

in which m is an arbitrary constant. The last of these is the familiar equation for the surface spherical harmonic[1] of order m; and since the other obviously takes the same form if λ is replaced by $i\lambda$, its solution is a surface harmonic of a pure imaginary argument.

[1] See, for example, Byerly, *Fourier Series and Spherical Harmonics*, pp. 9 and 147. Our μ is related to Byerly's θ by the law $\mu=\frac{1}{2}\pi-\theta$.

We could, therefore, write down a very general solution of (18), building a series in which the terms were all composed of such spherical harmonics, but with various values of m. It turns out, however, that the conditions of our problem can be satisfied by means of just one such term, in which the harmonics are of order 1; so there is no need to use the series at all.[1]

On the other hand, we find by trial that in the case of the function $\Lambda(\lambda)$ we need, not only the well-known "function of the first kind," but also the less familiar "function of the second kind." Our solution, therefore, takes the form

$$\phi = [A \sinh \lambda + B(1 + \sinh \lambda \operatorname{gd} \lambda)] \sin \mu, \tag{21}$$

in which $\operatorname{gd} \lambda$, the "Gudermannian," is identical with $\tan^{-1}\sinh\lambda$, while A and B are constants whose values must be determined so as to satisfy our boundary conditions.[2]

We must now satisfy our boundary conditions (19). That the first is satisfied is at once obvious. To deal with the other two we confine ourselves to the axis $\mu = \pi/2$, along which, by (16), $z = b \sinh \lambda$. Along this line, then,

$$b\phi = Az + B[b + z \tan^{-1} (z/b)],$$

and hence

$$b\partial\phi/\partial z = A + B \tan^{-1}(z/b) + [Bbz/(b^2 + z^2)].$$

The last two of equations (19) now become

$$A + \tfrac{1}{2}\pi B = kb$$

and

$$A - \tfrac{1}{2}\pi B = 0;$$

whence $A = kb/2$ and $B = kb/\pi$. Inserting these values in (21) we get, as the final solution of our problem, subject to all its boundary conditions, the function

---

[1] In other words, if we followed the usual course of writing down a series with arbitrary co efficients, and then determining the coefficients so as to satisfy (19), we would find that all but two vanished. The remaining two would be the A and B of (21).

[2] Owing to the fact that the text-books say so little about spherical harmonics of the second kind, it may be well to prove that $1+\sinh \lambda \operatorname{gd} \lambda$ is really such a function. This can be done, of course, by direct substitution in (20); but it may be of more interest to derive it from first principles.

We know, to begin with, that $\sin \mu$ is the spherical harmonic of the first kind with a real argument, (see Byerly, loc. cit., p. 159), and is a solution of the second of equations (20). But since upon setting $\mu = i\lambda$, the second of equations (20) reduces to the first, it follows that $\sin i\lambda$, and hence $\sinh \lambda$ also, must be a solution of the first.

However, when one such solution is known the complete solution can always be found by variation of parameters. To do this, we suppose that

$$\Lambda(\lambda) = \alpha(\lambda) \sinh \lambda$$

is also a solution, $\alpha(\lambda)$ being as yet undefined. Substituting this in the first of equations (20) we get a solvable equation for $\alpha(\lambda)$, which leads us at once to $1+\sinh \lambda \operatorname{gd} \lambda$ for the function of the second kind with imaginary argument.

$$\phi = \tfrac{1}{2}kb\,[\sinh\lambda + 2\pi^{-1}(1 + \sinh\lambda\,\mathrm{gd}\,\lambda)]\sin\mu. \tag{22}$$

This formula is satisfactory when we wish to find the potential at any given point in space, since we need only find the values of λ and μ which correspond to this point from (17), and then substitute these values in (22). A better form for the plotting of equipotential surfaces, however, is easily obtained by introducing the parameter

$$\beta = \left[\frac{r^2 - b^2 + [(r^2 - b^2)^2 + 4b^2z^2]^{1/2}}{2b^2}\right]^{1/2},$$

which leads to the pair of parametric equations

$$\begin{aligned} z/b &= \frac{\pi\phi/kb}{\frac{1}{2}\pi + \tan^{-1}\beta + \beta^{-1}}, \\ r^2/b^2 &= (1 + \beta^2) - z^2b^{-2}\beta^{-2}. \end{aligned} \tag{23}$$

In using these equations to plot the equipotential surface corresponding to a given value of ϕ, the significance of β can be entirely forgotten: it may be assigned any convenient set of values and the coordinates of the corresponding points (z, r) computed.

Fig. 7 shows the equipotential surfaces for this case, the numbers attached to the curves being the values of $\pi\phi/kb$ to which they correspond.

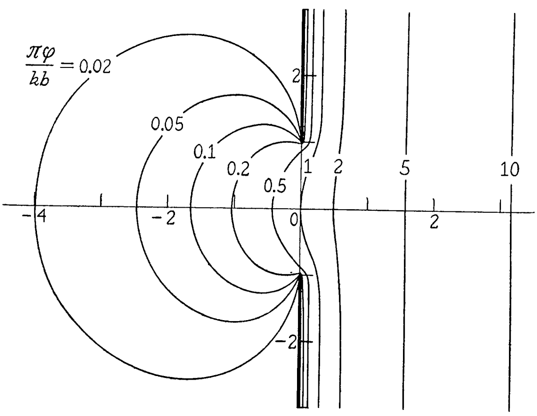

FIG. 7

We are now in a position to justify the use of the formulae (22) and (23) even when the distance between the electrodes is not infinite; for it is obvious that the curves marked 5 and 10 deviate from straight lines by amounts which cannot be detected on the scale of the drawing. Hence we may say, that if the distance between the planes is more than $\frac{3}{4}$ of the diameter of the hole, the solution (22) may be used with a very high degree of accuracy.

THE ROTATION-VECTOR IN ELEMENTARY DIFFERENTIAL GEOMETRY

By REV. M. F. EGAN, University College, Dublin

1. If $\boldsymbol{a}$ and $\boldsymbol{r}$ are vectors, the vector product $\boldsymbol{a}\times\boldsymbol{r}\,du$ represents the displacement of the point whose position-vector is $\boldsymbol{r}$, due to an infinitesimal rotation of amount $\boldsymbol{a}\,du$ about an axis through the origin directed along $\boldsymbol{a}$. We can call $\boldsymbol{a}$, or $\boldsymbol{a}\,du$, the *rotation-vector* for this rotation.

There is nothing new in this idea, but it simplifies a good deal of elementary differential geometry, and gives us some interesting results very easily.

2. Consider a curve in space, with torsion τ and curvature κ. If $\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{b}$ are unit vectors with a fixed origin, and congruent to the unit tangent, principal normal and binormal respectively; then corresponding to the translation ds along the curve, the tripod[1] $(\boldsymbol{tpb})$ has the rotations τds and κds about the tangent and binormal respectively; hence if $\boldsymbol{v}$ is a vector invariably connected with the tripod, we have

$$\frac{d\boldsymbol{v}}{ds} = (\tau\boldsymbol{t} + \kappa\boldsymbol{b}) \times \boldsymbol{v}.$$

Putting $\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{b}$ successively for $\boldsymbol{v}$, we get Frenet's equations.

3. Consider a curve on a surface, and let T be the unit tripod formed by the tangent $(\boldsymbol{t})$, the normal to the curve which is tangential to the surface (s), and the normal to the surface $(\boldsymbol{n})$. Let

$$\boldsymbol{a}ds = (\alpha\boldsymbol{t} + \beta\mathrm{s} + \gamma\boldsymbol{n})ds$$

be the rotation-vector corresponding to the translation $\boldsymbol{t}\,ds$. If $\boldsymbol{v}$ is a vector invariably attached to T, we have

$$\frac{d\boldsymbol{v}}{ds} = \boldsymbol{a} \times \boldsymbol{v}, \tag{3.1}$$

and putting $\boldsymbol{t}$, s, $\boldsymbol{n}$ successively for $\boldsymbol{v}$ we get three equations to determine α, β, γ.

Again, $\boldsymbol{p}$ being as before the principal normal of the curve, if θ is the angle $\boldsymbol{p}\rightarrow\boldsymbol{n}$, the rotation of the tripod T is the same as that of the tripod $(\boldsymbol{tpb})$, along with a rotation $d\theta$ about $\boldsymbol{t}$. We have therefore

$$\begin{aligned}\boldsymbol{a} &= \left(\tau + \frac{d\theta}{ds}\right)\boldsymbol{t} + \kappa\boldsymbol{b}\\ &= \left(\tau + \frac{d\theta}{ds}\right)\boldsymbol{t} - \kappa\mathrm{s}\cos\theta + \kappa\boldsymbol{n}\sin\theta,\end{aligned}$$

which gives α, β, γ as respectively the geodesic torsion, minus the normal curvature, and (plus) the geodesic curvature.

[1] "Tripod" seems preferable on all grounds to "trihedron" or "tri-vector." Blaschke (Differentialgeometrie, vol. I, 1930) uses "Dreibein."

4. Let $\alpha', \cdots, \boldsymbol{t}', \cdots, ds'$ refer to one of the parametric curves through a point on a surface S, and $\alpha'', \cdots, \boldsymbol{t}'', \cdots, ds''$ to the other. We denote $\partial/\partial s'$ and $\partial/\partial s''$ by the suffixes 1 and 2. Let ω be the angle $\boldsymbol{t}' \rightarrow \boldsymbol{t}''$. If $\boldsymbol{r}$ is the position-vector of the point on S, it is easily seen that

$$\boldsymbol{n}dS = \boldsymbol{r}_1 \times \boldsymbol{r}_2 ds'ds'' = \boldsymbol{n} \sin \omega ds'ds''.$$

Since $\boldsymbol{n}$ is also normal to the unit sphere N (locus of the extremity of $\boldsymbol{n}$), we have in like manner

$$\boldsymbol{n}dN = \boldsymbol{n}_1 \times \boldsymbol{n}_2 ds'ds''.$$

The Gaussian curvature K is dN/dS, we have therefore

$$\begin{aligned} \boldsymbol{n}K \sin \omega &= \boldsymbol{n}_1 \times \boldsymbol{n}_2 \\ &= (\boldsymbol{a}' \times \boldsymbol{n}) \times (\boldsymbol{a}'' \times \boldsymbol{n}) \\ &= (\boldsymbol{a}', \boldsymbol{a}'', \boldsymbol{n})\boldsymbol{n} \end{aligned}$$

whence

$$\left\{ \begin{aligned} K &= (\boldsymbol{a}', \boldsymbol{a}'', \boldsymbol{n}) \operatorname{cosec} \omega, \\ &= \alpha'\alpha'' + \beta'\beta'' + (\alpha'\beta'' - \alpha''\beta') \cot \omega. \end{aligned} \right. \tag{4.1}$$

We can find J (the sum of the principal normal curvatures) very easily in the same manner. If we consider the parallel surface S', locus of the point

$$\boldsymbol{r}' = \boldsymbol{r} + c\boldsymbol{n},$$

where c is a constant, we have

$$\begin{aligned} \boldsymbol{n}K \sin \omega = \boldsymbol{n}_1 \times \boldsymbol{n}_2 &= K'\boldsymbol{r}_1' \times \boldsymbol{r}_2' \\ &= K'(\boldsymbol{r}_1 + c\boldsymbol{n}_1) \times (\boldsymbol{r}_2 + c\boldsymbol{n}_2). \end{aligned}$$

If we consider the coefficient of c, it is clear that the value of

$$(\boldsymbol{n}_1 \times \boldsymbol{r}_2 + \boldsymbol{r}_1 \times \boldsymbol{n}_2) \operatorname{cosec} \omega,$$

i.e. of

$$\boldsymbol{n}\{\beta' + \beta'' + (\alpha' - \alpha'') \cot \omega\}$$

is independent of the choice of parametric curves. If we take the lines of curvature, the expression in brackets reduces to $\beta' + \beta''$, and since β is minus the normal curvature in the given direction, $\beta' + \beta''$ is $-J$: whence we get in the general case

$$-J = \beta' + \beta'' + (\alpha' - \alpha'') \cot \omega. \tag{4.2}$$

5. If the element of length on the surface is given by

$$ds^2 = e^2du^2 + 2eg \cos \omega du dv + g^2dv^2,$$

the elements of length

$$ds' = edu, \quad ds'' = gdv$$

along the parametric lines are not in general perfect differentials, and so we cannot consider s' and s'' as independent variables. From the identity

$$\int_C X du + Y dv = \iint \left(\frac{\partial Y}{\partial u} - \frac{\partial X}{\partial v}\right) du dv$$

we easily deduce the corresponding identity

$$\int_C P ds' + Q ds'' = \iint \operatorname{cosec} \omega (Q_1 + q''Q - P_2 - q'P) dS, \tag{5.1}$$

where

$$q' = \frac{1}{eg}\frac{\partial e}{\partial v}, \quad q'' = \frac{1}{eg}\frac{\partial g}{\partial u}. \tag{5.11}$$

(If the parametric lines are orthogonal, we have

$$q' = -\gamma', \quad q'' = \gamma''.)$$

From this we obtain the condition that $P\ ds' + Q\ ds''$ should be a perfect differential. We also have,[1] if

$$\theta_{12} = (\theta_1)_2,$$
$$\theta_{12} + q'\theta_1 = \theta_{21} + q''\theta_2. \tag{5.2}$$

6. If we express that

$$\boldsymbol{t}'ds' + \boldsymbol{t}''ds'' = d\boldsymbol{r}$$

is a perfect differential, we get

$$\boldsymbol{t}_1'' + q''\boldsymbol{t}'' = \boldsymbol{t}_2' + q'\boldsymbol{t}'.$$

Now the rotation of $\boldsymbol{t}''$, corresponding to the displacement $\boldsymbol{t}'\ ds'$, is equal to the corresponding rotation of the tripod T', along with a rotation $\omega_1 ds'$ about $\boldsymbol{n}$. Hence

$$\boldsymbol{t}_1'' = (\boldsymbol{a}' - \boldsymbol{n}\omega_1) \times \boldsymbol{t}'',$$

and in like manner

$$\boldsymbol{t}_2' = (\boldsymbol{a}'' - \boldsymbol{n}\omega_2) \times \boldsymbol{t}'.$$

Substituting these values and remembering that

$$\boldsymbol{t}'' = \boldsymbol{t}' \cos \omega + \boldsymbol{s}' \sin \omega,$$
$$\boldsymbol{s}'' = \boldsymbol{n} \times \boldsymbol{t}'' = \boldsymbol{s}' \cos \omega - \boldsymbol{t}' \sin \omega,$$

we easily get the identities

$$\begin{cases} (\alpha' + \alpha'') \sin \omega = (\beta' - \beta'') \cos \omega, \\ -q' + q'' \cos \omega = (\omega_1 + \gamma') \sin \omega, \\ -q'' + q' \cos \omega = (\omega_2 - \gamma'') \sin \omega. \end{cases} \tag{6.1}$$

[1] Cf. Blaschke, *op. cit.* I, p. 125.

The second and third of these identities give a geometrical definition of q' and q''; for it is easy to see that the turning-rates (round $\boldsymbol{n}$) of $\boldsymbol{t}'$ as we go along $\boldsymbol{t}''$, and of $\boldsymbol{t}''$ as we go along $\boldsymbol{t}'$, are respectively $\omega_1+\gamma'$ and $\gamma''-\omega_2$, per unit of length. It follows that the necessary and sufficient conditions that the tangent to each parametric line should undergo a parallel transport (in the Levi-Civita sense) along the other parametric line, are

(6.2) $$q' = q'' = 0,$$

whence

$$\partial e/\partial v = 0, \quad \partial g/\partial u = 0,$$

from which we get at once the well-known theorem that the coordinate lines form a Tchebychef net.

The equation (5.11) shows that q' and q'' are first-order magnitudes, i.e., they depend only on the expression for ds^2. The same is true of ω, and therefore also, by (6.1), of γ' and γ''.

The first of the equations (6.1) shows that the geodesic torsions α' and α'' are equal and of opposite signs when the parametric curves are orthogonal, and also when the normal curvatures are equal. In the case of the asymptotic lines the normal curvatures are both zero, and the geodesic torsions are equal to the actual torsions. We thus get from (4.1) and (4.2) the familiar results about the torsion of an asymptotic line:

$$\tau^2 = -\kappa, \quad 2\tau\cot\omega = -J,$$

where ω is the angle between the asymptotic lines.

7. Let $\boldsymbol{t}, \cdots, T, \cdots, \alpha, ds$ refer to a curve (C); let ϕ be the angle $\boldsymbol{t}'\to\boldsymbol{t}$, and let

$$d\boldsymbol{r} = \boldsymbol{t}ds = \boldsymbol{t}'ds' + \boldsymbol{t}''ds''.$$

Then the rotation of the tripod T corresponding to the translation $d\boldsymbol{r}$ is the sum of (1) that of T' along $\boldsymbol{t}'ds'$, (2) that of T'' along $\boldsymbol{t}''ds''$, less a rotation $\omega_2 ds''$ about $\boldsymbol{n}$, and (3) a rotation $d\phi$ about $\boldsymbol{n}$. We therefore have

(7.1) $$\boldsymbol{a}ds = \boldsymbol{a}'ds' + (\boldsymbol{a}'' - \boldsymbol{n}\omega_2)ds'' + \boldsymbol{n}d\phi.$$

If we suppose the coordinate curves *orthogonal*, we have

$$ds' = ds\cos\phi, \quad ds'' = ds\sin\phi,$$

whence we get

(7.21) $$\begin{cases} \alpha = \alpha'\cos^2\phi + \alpha''\sin^2\phi + (\beta' - \beta'')\sin\phi\cos\phi \\ \quad = \alpha'(\cos^2\phi - \sin^2\phi) + (\beta' - \beta'')\sin\phi\cos\phi, \end{cases}$$

$$\text{since } \alpha' + \alpha'' = 0;$$

(7.22) $$\beta = (\alpha'' - \alpha')\sin\phi\cos\phi + \beta'\cos^2\phi + \beta''\sin^2\phi,$$

(7.23) $$\gamma = \gamma'\cos\phi + \gamma''\sin\phi + \frac{d\phi}{ds}\cdot$$

For oblique coordinates we have

$$ds/\sin\omega = ds'/\sin(\omega - \phi) = ds''/\sin\phi,$$

and there is no difficulty in writing down the corresponding identities.

These equations are useful in discussing lines of curvature, asymptotic lines, etc.

8. *The Mainardi-Codazzi equations.* If v' is a vector invariably attached to T', we have

$$dv' = a' \times v'ds' + b' \times v'ds'',$$

where

$$b' = a'' - n\omega_2.$$

Hence

$$(b' \times v')_1 + q''b' \times v' - (a' \times v')_2 - q'a' \times v' = 0.$$

Now

$$v_2' = b' \times v', \quad v_1' = a' \times v',$$

and

$$a' \times (b' \times v') - b' \times (a' \times v) = (a' \times b') \times v',$$

so that we get[1]

$$(b_1' + q''b' - a_2' - q'a' - a' \times b') \times v' = 0$$

or, since v' is arbitrary,

$$(8.1) \qquad b_1' + q''b' - a_2' - q'a' = a' \times b'.$$

Multiplying *scalarly* by v', and adding to each side

$$b' \cdot v_1' - a' \cdot v_2' = -2(a', b', v'),$$

we get

$$(8.2) \qquad (b' \cdot v')_1 + q''b' \cdot v' - (a' \cdot v')_2 - q'a' \cdot v' = -(a', b, v').$$

If we multiply both sides by cosec ω and integrate over the area bounded by a closed curve (C), we get, in virtue of (5.1),

$$(8.21) \qquad -\iint (a', a'' - n\omega_2, v') \operatorname{cosec} \omega dS = \int_C v' \cdot [a'ds' + (a'' - n\omega_2)ds'']$$

$$(8.22) \qquad = \int_C v' \cdot (ads - nd\phi),$$

by (7.1). First, write n for v'. We have

$$(a', a'' - n\omega_2, n) = (a', a'', n) = K \sin\omega$$

by (4.1); accordingly from (8.22) we get Gauss' identity

$$(8.31) \qquad -\iint K dS = \int_C \gamma ds - \int d\phi, = \int_C [\gamma' ds' + (\gamma'' - \omega_2)ds''],$$

[1] Cf. Campbell, *Differential Geometry*, p. 111.

while from (8.2) we get

(8.32) $$-K\sin\omega = \gamma_1'' + q''\gamma'' - \gamma_2' - q'\gamma' - \omega_{21} - q''\omega_1.$$

(The right-hand side is symmetrical, since

$$\omega_{21} + q''\omega_2 = \omega_{12} + q'\omega_1.)$$

Putting t' for v' in (8.2), and substituting for ω_1 and ω_2 their values from (6.1), we get

(8.33) $$\alpha_1''\cos\omega - \beta_1''\sin\omega - \alpha_2' = q'(\alpha' - \alpha'') + (q'\cot\omega - q''\operatorname{cosec}\omega)(\beta' - \beta'').$$

On substituting s', we find in the same way

(8.34) $$\alpha_1''\sin\omega + \beta_1''\cos\omega - \beta_2' = q'(\beta' - \beta'') - (q'\cot\omega - q''\operatorname{cosec}\omega)(\alpha' - \alpha'').$$

If the coördinate curves are orthogonal, the equations are simplified. The second and third of the Mainardi-Codazzi equations becomes

(8.331) $$\alpha_2' + \beta_1'' = \gamma'(\alpha' - \alpha'') + \gamma''(\beta' - \beta''),$$

(8.341) $$\alpha_1'' - \beta_2' = \gamma''(\alpha' - \alpha'') - \gamma'(\beta' - \beta'').$$

$$(\alpha'' = -\alpha').$$

9. As an example of the use of these equations, suppose that the coordinates form a Tchebychef net. We have by (6.2) and (6.1)

$$q' = q'' = 0,\ \gamma' = -\omega_1,\ \gamma'' = \omega_2.$$

Since q' and q'' vanish, we can take s' and s'' as independent variables; and ω_{12} and ω_{21} are identical. Hence (8.31) gives

$$K = -\omega_{12}\operatorname{cosec}\omega$$

and

$$\iint K dS = -\iint \frac{\partial^2\omega}{\partial s'\partial s''} ds' ds''.$$

Integrating this over a quadrilateral formed by two pairs of lines belonging to the net, we find that the excess over 2π of the sum of the angles of the quadrilateral is equal to minus the total curvature of the enclosed area. If K is constant we get Hazzidakis's theorem.[1]

Again, let us look for the surfaces whose asymptotic lines form a Tchebychef net. Taking these lines as coordinate curves, we have

$$\alpha'' = -\alpha',\ \beta' = \beta'' = q' = q'' = 0,$$

whence we get from (8.33–4)

$$\alpha_1' = \alpha_2' = 0,$$

and hence α' is constant, and $K(=-\alpha'^2)$ is also constant. Conversely, if K is

[1] Blaschke, *op. cit.* I, pp. 208–9.

constant, the condition is satisfied. Accordingly Hilbert's theorem, that the asymptotic lines on surfaces of constant curvature form a Tchebychef net, expresses a property peculiar to those surfaces.

10. As a final example let us examine with Bonnet the question of the applicability of two surfaces with correspondence of one or both families of asymptotic lines. If *both* families correspond, the geodesic curvatures γ', γ'' are the same for both surfaces, since they are first-order quantities; the normal curvatures are zero; also we have on each surface

$$\alpha' = -\alpha'' = \pm\sqrt{(-K)},$$

so that the geodesic torsions on one surface are either equal, or equal and opposite, to those on the other. Hence the two surfaces are either congruent or symmetrically equal.

If *one* family on each surface corresponds, take the family and its orthogonal trajectories as parametric lines. Excluding the case of symmetry, we can take

$$\alpha',\ \alpha'' = -\alpha',\ \beta' = 0,\ \gamma',\ \gamma''$$

to be the same on both surfaces. Let β and B be the values of β'' on the two surfaces. Then the equation (8.341) gives us, for the two surfaces,

$$\begin{aligned}\alpha_1'' &= \gamma''(\alpha' - \alpha'') + \gamma'\beta \\ &= \gamma''(\alpha' - \alpha'') + \gamma' B,\end{aligned}$$

and hence

$$\gamma'(\beta - B) = 0.$$

If $\beta = B$, the two surfaces are congruent; rejecting this, we get $\gamma' = 0$, hence the asymptotic lines which correspond are geodesics and are therefore straight lines. The surfaces are therefore ruled, and the generators correspond.

For ruled surfaces the equation (8.331) is easily integrated.[1] We can write

$$ds^2 = du^2 + g^2 dv^2$$

where

$$g^2 = (u - a)^2 + b^2,$$

a and b being functions of v. The equation in question becomes

$$\alpha_2' + \beta_1'' + \gamma''\beta'' = 0$$

or

$$\frac{1}{g}\frac{\partial \alpha'}{\partial v} + \frac{\partial \beta''}{\partial u} + \frac{1}{g}\frac{\partial g}{\partial u}\beta'' = 0,$$

i.e.

$$\frac{\partial \alpha'}{\partial v} + \frac{\partial}{\partial u}(g\beta'') = 0.$$

If ϕ is the angle which the tangent plane makes with the tangent plane at

[1] This was suggested to me by Mr. P. Gormley.

the central point on the generator, then α', being the rotation-rate of the tangent plane about the generator, is $\partial\phi/\partial u$; and we get

$$\frac{\partial}{\partial u}\left(\frac{\partial\phi}{\partial v} + g\beta''\right) = 0,$$

whence

$$\beta'' = \frac{1}{g}\left(V - \frac{\partial\phi}{\partial v}\right),$$

V being a function of v. Since β and B are both of this form, we can suppose the function V to vary continuously from one expression to the other, so that the deformation can be carried out continuously. Again, equation (4.2) shows us that in this case β'' is $-J$, whence we get

$$J = \frac{1}{g}\left(\frac{\partial\phi}{\partial v} - V\right), \quad \text{and} \quad K = -\alpha'^2 = -\left(\frac{\partial\phi}{\partial u}\right)^2;$$

and it is well known and easily proved geometrically that ϕ is given by

$$\tan\phi = \frac{u-a}{b}, \quad \sin\phi = \frac{u-a}{g}, \quad \cos\phi = \frac{b}{g}\cdot$$

11. Instead of the usual six fundamental magnitudes, namely the three coefficients of ds^2 and L, M, and N, where

$$Ldu^2 + 2Mdudv + Ndv^2 = \boldsymbol{n}\cdot d^2\boldsymbol{r} = -d\boldsymbol{n}\cdot d\boldsymbol{r},$$

we have used e, g, ω (coefficients of ds^2—see §5), and the six components of the two rotation-vectors. These nine are connected by the three relations (6.1), besides the Mainardi-Codazzi equations, so they are equivalent as far as number is concerned to the usual set. Again the rotation-components are easily found in terms of the usual magnitudes; and conversely we have

$$\begin{aligned} -d\boldsymbol{n}\cdot d\boldsymbol{r} &= -\left\{(\boldsymbol{a}'ds' + \boldsymbol{a}''ds'')\times\boldsymbol{n}\right\}\cdot(\boldsymbol{t}'ds' + \boldsymbol{t}''ds'')\\ &= -\beta'ds'^2 + \left\{(\alpha'-\alpha'')\sin\omega - (\beta'+\beta'')\cos\omega\right\}ds'ds'' - \beta''ds''^2,\end{aligned}$$

whence

$$\begin{aligned} L/e^2 &= -\beta', \quad N/g^2 = -\beta'',\\ M/eg &= \tfrac{1}{2}(\alpha'-\alpha'')\sin\omega - \tfrac{1}{2}(\beta'+\beta'')\cos\omega\\ &= \alpha'\sin\omega - \beta'\cos\omega\\ &= -\alpha''\sin\omega - \beta''\cos\omega.\end{aligned}$$

The second and third expressions for M are obtained by means of the first of the equations (6.1). They show that if M vanishes, the component in the tangent plane of the rotation-vector corresponding to a displacement along $\boldsymbol{t}'$ lies along $\boldsymbol{t}''$, and conversely; in other words, that the two directions are conjugate.

12. The method we have used breaks down if we take the null-lines on the surface as coordinate curves. Fortunately, it is particularly easy in this case to get the Mainardi-Codazzi equations by the usual method.

MIXED SYSTEMS OF LINEAR EQUATIONS AND INEQUALITIES

By HELEN M. SCHLAUCH, Hunter College

In papers entitled "Systems of Linear Inequalities," Professor L. L. Dines[1] has found a necessary and sufficient condition for the existence of solutions in systems of linear inequalities, and Professor W. B. Carver[2] has found a different kind of necessary and sufficient condition for the non-existence of such solutions.[3] These theorems naturally suggest a consideration of systems containing both linear equations and linear inequalities. It is the purpose of this paper to give certain necessary and sufficient conditions both for the non-existence and for the existence of solutions of such systems.

Consider the mixed system S, in n unknowns, composed of p equations and $m-p$ inequalities:

$$L_i(x) = 0;\ i = 1, 2, \cdots, p$$

and

$$L_i(x) > 0;\ i = p+1, p+2, \cdots, m$$

where $L_i(x)$ stands for the linear expression:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + b_i.$$

The b's may or may not all be zero.

Since theorems 1 and 2, concerning the non-existence of solutions, may be proved by a method analogous to that used by Professor Carver for pure systems of inequalities, they are merely stated here.

Let M denote the matrix of coefficients, $\|a_{ij}\|$, not including the b's. Then,

THEOREM 1: *If for a system S the rank of the matrix M is m, the system is consistent.*

THEOREM 2: *A necessary and sufficient condition that a given system S be inconsistent is that there should exist a set of $m-p+1$ constants, $k_{p+1}, k_{p+2}, \cdots, k_{m+1}$, such that*

$$\sum_{i=p+1}^{m} k_i L_i(x) + k_{m+1} = 0$$

for all values of the x's satisfying the p equations; at least one of the k's being positive, and none of them being negative.

Consider again the system S above. It is obvious that if the system S is consistent, a consistent system of inequalities, S', may be derived from it by adding each of the p equations in turn, multiplied by some constant k_i; $i=1, 2, \cdots, p$ (which may or may not be the same for all the equations) to some one of the $m-p$ inequalities, and by including in this derived system, S', these new in-

[1] Annals of Mathematics; vol. 20 (1919), p. 191.

[2] Annals of Mathematics; vol. 23 (1922), p. 212.

[3] See also two papers by Dines in the Annals of Mathematics; vol. 27 (1925), p. 57, and vol. 28 (1926), p. 41.

equalities and each of the $m-p$ inequalities of S. The inequality to which the multiples of the p equations are added may be chosen quite arbitrarily, and, without loss of generality, may be assumed to be the first. Likewise, the p k's may have any values whatsoever.

Such a derived system S' is of the form:

$$L_{p+1}(x) + k_i L_i(x) > 0;\ i = 1, 2, \cdots, p$$

and

$$L_i(x) > 0;\ i = p+1, p+2, \cdots, m.$$

System S' contains m inequalities all of which must be satisfied by any set of values of the x's which satisfy the original system S.

THEOREM 3: *A necessary and sufficient condition that a mixed system of linear equations and inequalities, such as system S, be satisfied by a set of values a_1, $a_2, \cdots, a_n$, or, briefly, by the point A, is that the derived system of inequalities S' be satisfied by this point A for all values of the k's.*

That the condition is necessary is immediately apparent from the above discussion.

To prove the condition sufficient:

If the system S' is satisfied by the point A for all values of the k's, then the $m-p$ inequalities of system S, since they also appear in S', will be satisfied by this point. It remains to show that the p equations of system S are also satisfied by A.

Suppose one of the equations of S, say the rth equation, were not satisfied by A. Then if this set of values were substituted in it, the resulting sum would be different from zero, that is: $L_r(A) \neq 0$. But since A satisfies the system S',

$$L_{p+1}(A) + k_r L_r(A) > 0 \text{ for all values of } k_r.$$

Or,

$$L_{p+1}(A) > -\ k_r L_r(A) \text{ for all values of } k_r.$$

But since $L_r(A) \neq 0$, it is possible to choose a value of k sufficiently large in absolute value so that

$$L_{p+1}(A) < -\ k_r L_r(A).$$

Therefore the hypothesis was false, and $L_r(A) = 0$. The same argument applies to each equation of system S, and hence the entire system S is satisfied by the point A.

DEFINITION: A system of p equations and $m-p$ inequalities will be called a *simple* system when the rank of the augmented matrix of the coefficients of the p equations is p.

THEOREM 4: *From the simple system S, of p equations and m-p inequalities, 2^p systems of inequalities may be derived by including the $m-p$ inequalities of system S and one of the 2^p possible combinations of the p inequalities found by re-*

placing the equality symbol $(=)$ *in each equation by one of the inequality symbols* $(>$ *or* $<)$. *A necessary and sufficient condition that the simple system* S *be consistent is that each derived system of inequalities be consistent.*

To prove the condition sufficient:

We are given that there exist 2^p points, $P_1, P_2, \cdots, P_{2^p}$, each giving positive signs for the last $m-p$ expressions of system S and each giving one of the possible combinations of plus and minus signs for the first p expressions when they are substituted in the expressions of system S. Half of these points will make the pth expression positive and half will make it negative. Both of these types will appear with every possible combination of signs of the remaining $p-1$ first expressions. We may then pair our points in groups of two; P_i will be paired with P_j if P_i makes the pth expression positive, P_j makes the pth expression negative, and they both result in the same combination of signs for the first $p-1$ expressions. There will be 2^{p-1} such pairs.

For example:

Some P_i gives the signs:

$$- - - \cdots - -, + + + \cdots +$$

a corresponding P_j gives the signs:

$$- - - \cdots - +, + + + \cdots + .$$

Let us define a point F between C and D as the set of numbers

$$(h_1c_1 + h_2d_1, h_1c_2 + h_2d_2, \cdots, h_1c_n + h_2d_n)$$

where h_1 and h_2 are any two positive numbers whose sum is unity. Since the expressions of system S are linear, we have:

$$L_i(F) = h_1L_i(C) + h_2L_i(D);\ i = 1, 2, \cdots, m.$$

It is then possible for each pair P_i and P_j above, to find a point Q_k; $k=1, 2, \cdots, 2^{p-1}$ making the pth expression $=0$ and the last $m-p$ expressions positive. The signs of the first $p-1$ expressions will be the same as those determined by P_i and P_j. We have, then, 2^{p-1} points making the last $m-p$ expressions >0, the pth expression $=0$, and the first $p-1$ have all possible combinations of the inequality symbols, $(>$ or $<)$.

If we continue this process p times, we are lead to a point R which, when substituted in system S gives:

$$0\,0\,0\,0 \cdots 0, + + + \cdots + .$$

That is, point R satisfies system S, and system S is consistent.

To prove the condition necessary:

We are given that system S is consistent. That is, we are given that there exists a point, Q, which satisfies all the conditions of system S. We wish to find 2^p points which will satisfy the 2^p derived systems described in the theorem.

Let us here define $L_i'(x)$ to be $L_i(x)-b_i$.

Now, consider the possible points of the form: $(q_1+y_1, q_2+y_2, \cdots, q_n+y_n)$ where the y's may have any values whatever.

When a point of this form is substituted in the linear expressions $L_i(x)$, it will give:

$$(A) \qquad L_i'(Y); \qquad i = 1, 2, \cdots, p$$

and

$$L_i(Q) + L_i'(Y); \quad i = p+1, p+2, \cdots, m.$$

If it is possible to find 2^p points of the form $(Q+Y)$ such that when they are substituted in the first p expressions of (A) they give, respectively, the 2^p sets of inequality symbols ($>$ or $<$) required by the theorem, and such that each of them gives

$$|L_i'(Y)| < L_i(Q); \; i = p+1, p+2, \cdots, m,$$

then each of the 2^p derived systems of inequalities described in the theorem will be consistent.

Now consider the set (B) of p non-homogeneous equations obtained by setting

$$(B) \qquad L_i'(x) = R_i; \; i = 1, 2, \cdots, p,$$

where each R_i is some constant.

The rank of the augmented matrix of coefficients of the p equations in system S is p; and since the complete system S is consistent, the rank of the unaugmented matrix of coefficients of the equations (i.e., not including the b's) must also be p.[1] But the unaugmented matrix of the coefficients of (B) is exactly this unaugmented matrix of the coefficients of the p equations of system S. There is therefore at least one point Y satisfying the equations of (B) for any values of the R_i's; $i=1, 2, \cdots, p$.

Then to obtain a point Y which will satisfy any one of the 2^p derived systems of the inequalities, set $R_i = \pm 1$; $i=1, 2, \cdots, p$, according as the ith expression in the particular derived system under consideration is to be $>$ or <0. There will exist a point M satisfying the equations of (B) with these values of the R's. Now choose a k such that $k>0$ and $k > |L_i'(M)|/L_i(Q)$; $i=p+1, p+2, \cdots, m$. Then also, $|L_i'(M)|/k < L_i(Q)$; $i=p+1, p+2, \cdots, m$. Divide the set of values composing the point M by k, giving a point Y of the type described above. This particular point Y is the desired point for, since $L_i'(X)$ is a homogeneous linear expression,

$$|L_i'(Y)| = |L_i'(M)|/k < L_i(Q); \; i = p+1, p+2, \cdots, m,$$

and $L_i(Y) = \pm 1/k$; $i=1, 2, \cdots, p$ according as the ith expression in the derived system under consideration is >0 or <0.

We can then find 2^p points $(Q+Y_i)$; $i=1, 2, \cdots, 2^p$, satisfying the 2^p particular derived systems required by the theorem.

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[1] Bôcher: Introduction to Higher Algebra, p. 46.

It may at first appear that the restriction of the theorem to *simple* systems might limit its application. However, even if we have a system S of p equations and $m-p$ inequalities which is not simple, theorem (4) will give a complete test. For, if we first examine the p equations and find that they are inconsistent, we do not need to examine system S further, for it is impossible for the complete system to be consistent if the equations are not themselves consistent. If, however, the p equations are consistent but have a matrix of coefficients whose rank is $r<p$, then any solution of the r equations whose coefficients enter into a non-vanishing r-rowed determinant of the matrix will be a solution of all p equations.[1] We may test the simple system composed of these r equations and the original $m-p$ inequalities by theorem (4). The results will then apply to the non-simple system S.

ALGEBRAIC CHARTS

By EDGAR DEHN, Columbia University

To find simple devices for doing long computations, this task has always held a peculiar charm; and mathematicians have often invented such devices before there was an urgent demand for them, thus anticipating the needs of advancing science.

Tools like logarithms and the slide rule have long since become indispensable in the outfit of every scientific worker. And now the demand is growing for those devices which nomography has to offer. Among such devices are charts for solving algebraic equations, both in one and several unknowns. In this article I confine myself to equations in one unknown, explaining two principles for constructing charts which I found accidentally over a year ago. That anything at all could be added to a subject so exhaustively treated by d'Ocagne merely shows that much remains to be done in this field.

1. The first principle applies to trinomial equations of the form

$$x^n + px + q = 0$$

with real roots, and consists in plotting the coefficients p and q as functions of two roots.

In case of the quadratic equation

$$x^2 - px + q = 0,$$

we have only two roots, x_1 and x_2, with the relations

$$x_1 + x_2 = p, \quad x_1x_2 = q.$$

The first one gives straight lines for parametric values of p, the other equilateral hyperbolas for parametric values of q, if we plot in the x_1x_2-plane. The coördi-

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[1] Bôcher, *loc. cit.*, p. 45.

nates of any point where the proper line p and curve q intersect are the roots of a given equation.

If we assume that p, q, x_1, x_2 are positive, we can put every quadratic equation either into the form just used, or else into the form

$$x^2 - px - q = 0$$

with the relations

$$x_1 - x_2 = p, \quad x_1x_2 = q$$

between the roots. These relations give the same hyperbolas but other lines; and plotting both pairs of relations in the first quadrant, we can solve there any given equation. We do not even need the whole quadrant: since we have symmetry about the 45-degree line through the origin, one half of the first quadrant suffices.

In case of the cubic equation

$$x^3 - px + q = 0,$$

we have the relations

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1x_2 + x_1x_3 + x_2x_3 &= -p \\ x_1x_2x_3 &= -q \end{aligned}$$

between the roots. Eliminating x_3, we obtain from these

$$\begin{aligned} x_1^2 + x_1x_2 + x_2^2 &= p \\ x_1^2x_2 + x_1x_2^2 &= q. \end{aligned}$$

The first relation gives ellipses for parametric values of p, and the other gives, for parametric values of q, cubic curves which consist of three branches each. Whenever the roots of a given equation are different, the proper curves p and q intersect in six points. The coördinates of any such point give two roots, of course not always the same roots, and the third root is minus the sum of the other two. The plane of the points thus is the plane of two roots, but not of any definite two roots; we might call it the (x_i, x_j)-plane.

Since any trinomial cubic can be changed so that two of its roots are positive, we need again the curves p and q in the first quadrant alone; and even one half of the quadrant will suffice on account of the same symmetry about the 45-degree line through the origin.

In case of the biquadratic equation

$$x^4 - px + q = 0,$$

we use

$$\sum x_i = \sum x_ix_j = 0$$

to eliminate x_3 and x_4, thus obtaining the relations

$$x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 = p$$
$$x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 = q.$$

Again we can plot in the plane of two indefinite roots, but a chart so constructed would be of little practical value since ridding a biquadratic equation of its terms in x^3 and x^2 is more difficult than solving it by other available means. And the corresponding difficulty for the quintic equation is increased.

Charts constructed on the principle explained seem to give results a great deal more accurate than those obtained by any of the previously known graphic methods. Even a small sized but well drawn chart could be made to yield four significant figures.

2. The second principle for constructing algebraic charts is partly anticipated by Lalanne.

Any equation

$$x^n + px + q = 0$$

is for a definite value of x a line in the pq-plane. Drawing such lines for parametric values of x, one obtains the chart of Lalanne. To solve a given equation, one locates the point (p, q) on the chart and finds or interpolates those lines which go through the point. The x's of those lines are the roots of the equation.

In case of the quadratic equation, two lines go through a point corresponding to an equation with real roots; in the case of the cubic equation, as many as three lines. This makes it difficult to locate the lines, and near their intersections it is almost impossible to do so. I therefore left out the lines entirely in my charts and kept only their envelopes given by

$$\begin{cases} x^n + px + q = 0 \\ nx^{n-1} + p = 0, \end{cases}$$

marking on the envelope each value of x where the line for that value touches the envelope. Numerically such a value determines the slope of the line and the envelope.

The roots of a given equation then are found by holding a ruler so that it passes through the point (p, q) and touches the envelope, the number marked at any contact point being a root. A point (p, q) located on the envelope itself gives a double root.

The envelope for the quadratic equation

$$x^2 + px + q = 0$$

is

$$p^2 = 4q,$$

and the envelope for the cubic equation

$$x^3 + px + q = 0$$

is

$$4p^3 + 27q^2 = 0.$$

While such charts are quickly drawn and very clear, every trinomial equation requiring just one curve, with regard to accuracy they are much inferior to the charts discussed in section 1: only two significant figures can be obtained on the average.

On the other hand, the principle may be used to construct charts for equations with four terms, for instance

$$x^n + px^2 + qx + r = 0.$$

Taking one literal coefficient as a parameter, we have

$$\begin{cases} x^n + px^2 + qx + r = 0 \\ \quad nx^{n-1} + 2px + q = 0 \end{cases}$$

to determine envelopes in the plane of the other two literal coefficients, and all such envelopes compose a chart in that plane. To find the roots of a given equation, one holds a ruler passing through the point of two coefficients and touching the envelope of the third. Any contact point gives a root as before, and the slope at the point is either the root itself or a power of the root.

Points of different envelopes which carry the same number and have the same slope are all on a straight line, say slope line; and drawing the slope lines for different numbers is the best way of marking the numbers.

The cubic equation

$$x^3 + px^2 + qx + r = 0$$

with p as a parameter has the envelopes

$$\begin{cases} q = -2px - 3x^2 \\ r = px^2 + 2x^3 \end{cases}$$

and the slope lines

$$qx + 2r = x^3.$$

Each envelope has a cusp, and the locus of cusps is the envelope

$$\begin{cases} q = 3x^2 \\ r = -x^3 \end{cases}$$

of the slope lines.

A point (q, r) on the envelope p gives in general a double root, but it gives a triple root on the locus of cusps.

The biquadratic equation

$$x^4 + px^2 + qx + r = 0$$

with q as a parameter has the envelopes

$$\begin{cases} p = -2x^2 - q/2x \\ r = x^4 - qx/2 \end{cases}$$

and the slope lines

$$px^2 - r + 3x^4 = 0.$$

Each envelope has a cusp, and the locus of cusps is the envelope

$$p^2 + 12r = 0$$

of the slope lines. A point (p, r) on the envelope q gives in general a double root, but two double roots differing in their sign on the envelope $q=0$ and a triple root on the locus of cusps.

Although the accuracy obtainable on the last charts does not go beyond two significant figures, even solutions with deficient accuracy should be useful if only for further approximation. And the charts give a clear insight into the way the roots change when the coefficients of an equation are being altered.

ON THE PERPENDICULARS FROM A POINT ON A CIRCLE TO THE SIDES OF A REGULAR CIRCUMSCRIBED N-GON

By H. GROSSMAN, De Witt Clinton High School, New York

THEOREM: *If a is the radius of the circle inscribed in a regular n-gon and p_i the length of the perpendicular from any point of the circle on the i-th side of the n-gon, then*

$$\sum_{i=1}^{n} \left(\frac{p_i}{a}\right)^r = n \sum_{j=0}^{[r/2]} \frac{1}{2^{2j}} \binom{r}{2j} \binom{2j}{j} \quad [r = 1, 2, 3, \cdots, n-1].$$

PROOF: We shall need first the following lemma:
If m, n, and k are integers, $n>1$, $m \not\equiv 0 \bmod n$, and ϕ is any real number, then

$$\sum_{k=0}^{n-1} \cos\left(\phi + \frac{mk}{n} 2\pi\right) = 0.$$

For, if z is a primitive nth root of unity, and w the complex number whose absolute value is 1 and whose amplitude is ϕ, and $R(u)$ indicates the real part of u,

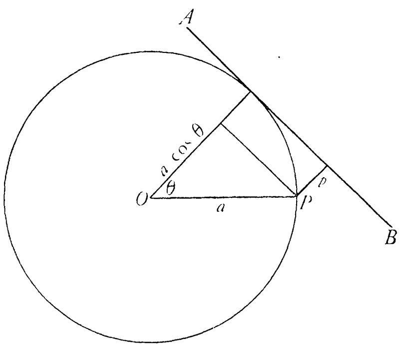
$$\begin{aligned}\sum_{k=0}^{n-1} \cos\left(\phi + \frac{mk}{n}2\pi\right) &= R\left(\sum_{k=0}^{n-1} wz^{mk}\right)\\ &= R\left[w\left(\frac{1-z^{mn}}{1-z^m}\right)\right]\\ &= R(0)\\ &= 0,\end{aligned}$$

since $z^{mn} = 1 \neq z^m$.

We shall employ also the trigonometric identities:[1]

$$2^{t-1}\cos^t\theta = \cos t\theta + \binom{t}{1}\cos(t-2)\theta + \binom{t}{2}\cos(t-4)\theta + \cdots + \binom{t}{\frac{t-1}{2}}\cos\theta \text{ if } t \text{ is odd;}$$

$$2^{t-1}\cos^t\theta = \cos t\theta + \binom{t}{1}\cos(t-2)\theta + \binom{t}{2}\cos(t-4)\theta + \cdots + \tfrac{1}{2}\binom{t}{\frac{t}{2}} \text{ if } t \text{ is even.}$$



Let O be the center of the circle, P be the point on the circle from which the n perpendiculars are dropped, and AB be that side of the circumscribed regular n-gon the perpendicular to which from O makes the smallest non-negative angle θ with OP.

[1] A Treatise on Plane and Spherical Trigonometry, J. B. Chauvenet, p. 142, §245.

Then

$$\sum\left(\frac{p_i}{a}\right)^r = \sum_{k=0}^{n-1}\left[1 - \cos\left(\theta + \frac{k}{n}2\pi\right)\right]^r$$

$$= \sum 1 - \binom{r}{1}\sum \cos\left(\theta + \frac{k}{n}2\pi\right) + \binom{r}{2}\sum \cos^2\left(\theta + \frac{k}{n}2\pi\right) - \cdots$$

$$+ (-1)^{r-1}\binom{r}{1}\sum \cos^{r-1}\left(\theta + \frac{k}{n}2\pi\right) + (-1)^r \sum \cos^r\left(\theta + \frac{k}{n}2\pi\right)$$

$$= \sum 1 - \binom{r}{1}\sum \cos\left(\theta + \frac{k}{n}2\pi\right) + \binom{r}{2}\sum \tfrac{1}{2}\left[\cos\left(2\theta + \frac{2k}{n}2\pi\right) + \tfrac{1}{2}\binom{2}{1}\right]$$

$$- \binom{r}{3}\sum \frac{1}{2^2}\left[\cos\left(3\theta + \frac{3k}{n}2\pi\right) + 3\cos\left(\theta + \frac{k}{n}2\pi\right)\right]$$

$$+ \binom{r}{4}\sum \frac{1}{2^3}\left[\cos\left(4\theta + \frac{4k}{n}2\pi\right) + 4\cos\left(2\theta + \frac{2k}{n}2\pi\right) + \tfrac{1}{2}\binom{4}{2}\right]$$

$$- \cdots$$

$$= \sum 1 + \binom{r}{2}\sum \tfrac{1}{2}\cdot\tfrac{1}{2}\binom{2}{1} + \binom{r}{4}\sum \frac{1}{2^3}\cdot\frac{1}{2}\binom{4}{2} + \cdots$$

$$= \sum_{k=0}^{n-1}\left[1 + \binom{r}{2}\frac{1}{2^2}\binom{2}{1} + \binom{r}{4}\frac{1}{2^4}\binom{4}{2} + \cdots\right]$$

$$= n\sum_{j=0}^{[r/2]} \frac{1}{2^{2j}}\binom{r}{2j}\binom{2j}{j} \qquad (r < n).$$

For this result, the restriction in the lemma that m is not a multiple of n makes it necessary in the last three steps of the proof that $r<n$ and so limits to $n-1$ the number of conditions to which the theorem subjects the n perpendiculars.

If we remove this limitation on r however and let $r=n$, the term

$$(-1)^n \sum \cos^n\left(\theta + \frac{k}{n}2\pi\right)$$

would give, after the trigonometric transformation, one non-vanishing term involving θ:

$$(-1)^n \sum \frac{1}{2^{n-1}}\cos(n\theta + k\cdot 2\pi) = \frac{(-1)^n n \cos n\theta}{2^{n-1}},$$

so that we may add the nth condition:

$$\sum\left(\frac{p_i}{a}\right)^n = n\left[\sum_{j=0}^{[n/2]}\frac{1}{2^{2j}}\binom{n}{2j}\binom{2j}{j} + \frac{(-1)^n \cos n\theta}{2^{n-1}}\right].$$

With appropriate changes the theorem may be extended to the case of a regular inscribed n-gon. In this case

$$\sum_{i=1}^{n}\left(\frac{p_i}{a}\right)^r$$

assumes the form

$$n\sum_{j=0}^{[r/2]}\frac{1}{2^{2j}}\binom{r}{2j}\binom{2j}{j}\left(\cos\frac{\pi}{n}\right)^{r-2j} \qquad [r = 1, 2, 3, \cdots, n-1].$$

It is necessary however, to make the following important reservation: for that one of the sides of the n-gon on whose minor arc the point P lies, the perpendicular p must be taken negative.

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE SOLUTION OF CUBIC EQUATIONS

By W. C. RISSELMAN, Arizona State Teachers College at Flagstaff

The purpose of this note is to give in symmetric form suitable for computational purposes the solution of cubic equations. Circular and hyperbolic functions will be used. Among the writers who have used hyperbolic functions in the solution of cubics are Grunert,[1] who obtained the results in this paper but did not use the fact that the period of the hyperbolic sine and the hyperbolic cosine is $2\pi i$; and Gleason,[2] who determined only the real root by hyperbolic functions and then gave a trigonometric determination of the other roots. Dickson[3] gives Case 1 below.

Consider the equation

$$y^3 + py + q = 0.$$

Let $D = -4p^3 - 27q^2$ denote its discriminant. Let $y = nz$. Then

$$z^3 + (p/n^2)z + q/n^3 = 0. \tag{1}$$

Case 1. $D > 0$.

Comparison of (1) with the identity

$$\cos^3 x - \tfrac{3}{4}\cos x - \tfrac{1}{4}\cos 3x = 0$$

gives

$$n = (-4p/3)^{1/2}, \quad \cos 3x = \frac{-q/2}{(-p^3/27)^{1/2}}.$$

[1] Archiv der Mathematik und Physik, vol. 38, (1862), pp. 48–76.

[2] Annals of Mathematics, ser. 2, vol. 13, pp. 120–122.

[3] First Course in the Theory of Equations, p. 49.

The roots are

$$y_1 = (-4p/3)^{1/2}\cos x,$$
$$y_2 = (-4p/3)^{1/2}\cos (x + 2\pi/3),$$
$$y_3 = (-4p/3)^{1/2}\cos (x + 4\pi/3).$$

Case 2. $D<0$, $p>0$.
Comparison of (1) with the identity

$$\sinh^3 x + \tfrac{3}{4}\sinh x - \tfrac{1}{4}\sinh 3x = 0$$

gives

$$n = (4p/3)^{1/2}, \quad \sinh 3x = \frac{-q/2}{(p^3/27)^{1/2}}.$$

The roots are

$$y_1 = (4p/3)^{1/2}\sinh x,$$
$$y_2 = (4p/3)^{1/2}\sinh (x + 2\pi i/3) = (4p/3)^{1/2}(-\sinh x + i\sqrt{3}\cosh x)/2,$$
$$y_3 = (4p/3)^{1/2}\sinh (x + 4\pi i/3) = (4p/3)^{1/2}(-\sinh x - i\sqrt{3}\cosh x)/2.$$

Case 3. $D<0$, $p<0$, $q<0$.
Comparison of (1) with the identity

$$\cosh^3 x - \tfrac{3}{4}\cosh x - \tfrac{1}{4}\cosh 3x = 0$$

gives

$$n = (-4p/3)^{1/2}, \quad \cosh 3x = \frac{-q/2}{(-p^3/27)^{1/2}}.$$

The roots are

$$y_1 = (-4p/3)^{1/2}\cosh x,$$
$$y_2 = (-4p/3)^{1/2}\cosh (x + 2\pi i/3) = (-4p/3)^{1/2}(-\cosh x + i\sqrt{3}\sinh x)/2,$$
$$y_3 = (-4p/3)^{1/2}\cosh (x + 4\pi i/3) = (-4p/3)^{1/2}(-\cosh x - i\sqrt{3}\sinh x)/2.$$

The case in which $D<0$, $p<0$, and $q>0$ may be brought under Case 3 since changing the sign of q merely changes the signs of the roots. The case in which $D=0$ is trivial.

A Trisection

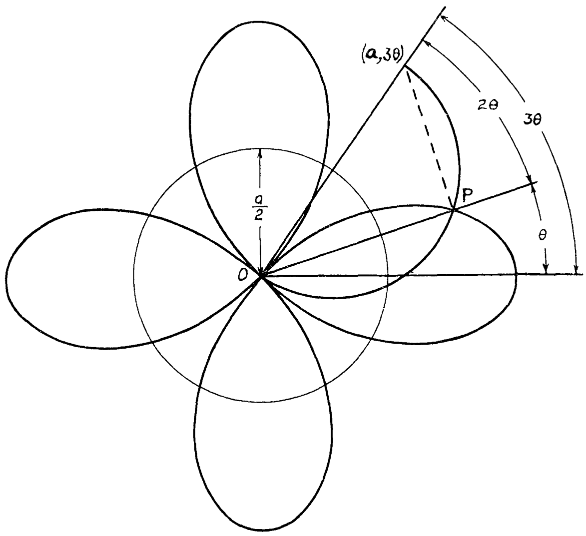
By R. K. Morley, Worcester Polytechnic Institute

That the four-leaved rose, $\rho = a\cos 2\theta$, is a very neat trisectrix does not seem to be generally known.

If a point $(a, 3\theta)$ be projected on the terminal side of θ at P, it is clear that the locus of P as θ changes is the curve $\rho = a\cos 2\theta$. Consequently if the given angle to be trisected is 3θ, a circle with center at $(a/2, 3\theta)$ will cut the rose on the terminal side of θ, the required one-third.

The circle in the figure with center at O and radius $a/2$ is convenient for locating the center $(a/2, 3\theta)$ for different values of 3θ.

Extension of the method to other sections is obvious; e.g. to finding fifth parts of angles by means of $\rho = a \cos 4\theta$.



RECENT PUBLICATIONS

Edited by Roger A. Johnson, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

W. R. Hamilton. Mathematical Papers. Vol. 1. Geometrical Optics. Cambridge, University Press, 1931. xxviii+534 pages.

As to the general plan of publishing Hamilton's Works we cannot improve the statement of the editors in their preface:—"This collected edition of the Mathematical Papers of Sir William Rowan Hamilton has been undertaken by the Royal Irish Academy with three main objects; first, as a tribute to the memory of Hamilton as a distinguished member and President of the Academy, and the greatest of Irish mathematicians; secondly to satisfy a need long felt by those to whom the published papers were difficult of access; and thirdly to publish for the first time those parts of the manuscript which have an historical or modern importance."

The present volume contains Hamilton's optical papers, beginning with his first published paper, *Theory of Systems of Rays. Part First,* presented to the Academy in 1827. In all there are fourteen previously published papers, the last

being dated 1841. They occupy 343 pages. In addition there are seven papers taken from his voluminous manuscripts and now published for the first time. These take up 115 pages. Besides these the volume contains:—The *Eloge on Hamilton* by his life long friend, the Very Rev. Charles Graves, read at the Academy directly after Hamilton's death; a photograph of Hamilton taken about 1845; also a photographic reproduction of a page of manuscript.

A notable feature of the present volume is the large number of notes added by the editors, Professors A. W. Conway and J. L. Synge, both of the University of Dublin. The editors have measured their duties by the severest standards; apparently they have read their way through this stupendous body of mathematical formulae, line by line. Errors have been noted and set right, and when the line of proof is obscure they have not only added necessary foot notes but have also appended a series of twenty-seven notes filling fifty-five pages. These notes are of great value; they are partly historical, partly explanatory, developing more fully the main text. In their last note, they have taken the trouble of applying Hamilton's ideas to compute the coefficients of aberration for an instrument of revolution. The reviewer has a large experience with Collected Works, be they English, German or French; he feels it a pleasant duty to state that he cannot recall any which have been edited with more meticulous care and felicity than the present volume.

Hamilton was born in 1805 and died in 1865. Already as a child he showed precocious ability. At seven he was studying Hebrew, at twelve he had a considerable knowledge of Sanskrit, Hindustani, Persian, and Arabic, as well as the classics and modern European languages. At sixteen he had mastered a great part of Newton's *Principia*, at seventeen he was studying Laplace's *Mécanique Céleste*. At eighteen he entered Trinity College, Dublin. The next year he presented his first paper "On Caustics" to the Royal Irish Academy, (1824). The Committee to whom it had been referred reported: "that the discussions included in the Memoir are of a nature so very abstract and the formulae so general as to require that the reasonings by which some of the conclusions have been obtained should be more fully developed." This paper found among Hamilton's manuscripts is now published for the first time. Remodelled and vastly extended, it was planned to form an immense memoir entitled *Theory of Systems of Rays. Part I, An ordinary System of reflected Rays. Part II, An ordinary System of refracted Rays. Part III, On extraordinary Systems, and Systems of Rays in general.* Part I was published in the Transactions of the Academy 1828. Part II is now published for the first time in the present volume. Part III has never appeared; it was perhaps never written. Instead Hamilton published three *Supplements* 1830–33.

These memoirs stamped Hamilton as one of the great mathematicians of his day. Brinkley, astronomer royal for Ireland early recognized his extraordinary abilities, for already in 1823 he is credited as saying "This young man, I do not say *will be*, but *is* the first mathematician of his age." This must have been a

pretty general opinion, for in 1827 while still an undergraduate he was made Andrews professor of astronomy in the university of Dublin.

Although Hamilton resided at the observatory Dunsink near Dublin for the rest of his life, it was not expected of him to become a great observer. Indeed the university had scant funds to replace the obsolete instruments by ones of modern date. They wished to assure him an honorable position which would permit him to develop his talents to their full extent. In this they amply succeeded.

In 1835 Hamilton was knighted, in 1837 he was elected president of the Dublin Academy. Already in 1827 Hamilton had noted the extension of his optical ideas to dynamics. His first papers on this matter appeared in the Transactions of the Royal Society of London 1834–35. In 1844 Hamilton discovered quaternions, and the last years of his life were chiefly spent in developing this new field.

Let us turn now for a moment to what is peculiar to Hamilton's treatment of optical problems. If we have given an optical instrument consisting of a number of lenses of known curvature and indices of refraction, placed in a certain position, we can by trigonometry trace the path of a ray through the instrument. By varying the ray, we find the rays issuing from a point do not in general meet at a point. Various defects in the image arise such as spherical aberration, coma, curvature of the field etc. The designer of an optical instrument endeavors to remove these various imperfections as far as possible by varying the number of the lenses, their curvature and position. The difficulties of this problem are immense, and it is of course of superlative importance to have certain general principles to guide the practical optician in his work of trial and error. Among the older authors writing from this point of view we may mention Airy and Coddington in England, Gauss and Seidel in Germany.

The point of view of Hamilton is quite different from that of these writers. He asks what one can say of any optical instrument quite disregarding its component parts. His methods seem to be more general and more flexible than any other known today.

Let a ray meet two planes in the points $P_0(x_0y_0z_0)$, $P_1(x_1y_1z_1)$ in an optical system formed of media of indices of refraction $\mu_1, \mu_2, \mu_3, \cdots$, the lengths of the paths in these media being $l_1, l_2, l_3, \cdots$. Then $V=\Sigma\mu l$, or more generally $\int\mu ds$ is called the optical length of the ray joining P_0 and P_1. It is a function of the coordinates $x_0, y_0, z_0, x_1, y_1, z_1$ of the terminal points. Then $V=V(x_0, y_0, z_0, x_1, y_1, z_1)$ is a characteristic function. If we keep P_0 fixed and vary P so that $V=$constant, we get the wave surface of light issuing from P_0. For from Fermat's principle the path of a ray of light joining P_0, P_1 is such that $\delta V=0$. From this we can infer that $l_0, m_0, n_0; l_1, m_1, n_1$ being the direction cosines of the ray at P_0, P_1 are given by

$$\frac{\partial V}{\partial x_0} = l_0,\ \frac{\partial V}{\partial y_0} = m_0,\ \frac{\partial V}{\partial z_0} = n_0\,,$$

and similar relations at P_1.

Instead of $x_0y_0z_0$; $x_1y_1z_1$ we may introduce other variables. For example Hamilton considers

$$T = l_1x_1 + m_1y_1 + n_1z_1 - l_0x_0 - m_0y_0 - n_0z_0 - V$$

regarding the end points P_0, P_1 fixed and the directions of the rays at these points as variable. As $l_0^2+m_0^2+n_0^2=1$, $l_1^2+m_1^2+n_1^2=1$, T is a function of four variables. For any instrument symmetric about an axis, T is a function of three variables

$$u_1=l_0^2+m_0^2, \quad u_1=l_1^2+m_1^2, \quad u_3=l_0l_1+m_0m_1.$$

If we develop T as a function of these variables, supposed small, and neglect small quantities of order higher than the second we get

$$T = T_0 + T_1 + T_2$$

where T_0 is a constant $T_1=a_1u_1+a_2u_2+a_3u_3$ and

$$T_2 = a_{11}u_1^2 + a_{22}u_2^2 + a_{33}u_3^2 + 2a_{12}u_1u_2 + 2a_{13}u_1u_3 + 2a_{23}u_2u_3 .$$

A third choice of variables is to suppose the initial position and final direction variable, leading Hamilton to a third characteristic function $W=l_1x_0+m_1y_0+n_1z_0-V$. These functions form the foundation of Hamilton's optical investigations and by their aid Hamilton discovered many of the results of later workers, for example those relating to the celebrated formulae of Seidel and Petzval.

Hamilton's optical methods lay unused and all but forgotten until quite recently. Thomson and Tait (1867), Maxwell (1875), M. Thiesen (1890), H. Bruns (1895) are the only writers the reviewer has noticed who used methods resting on a characteristic function prior to 1900. Strange to say, neither Thiesen nor Bruns seems to be aware of Hamilton's work.

With the turn of the century a marked change set in. F. Klein already in 1891 at a meeting of the *Naturforscher-Versammlung* had called attention to Hamilton's papers, and in his lectures in Göttingen, in the Summer Semester of that year he developed Mechanics as a kind of optics in an n-way space. In 1901 he called attention to the relation of Bruns's work to that of Hamilton. Klein says "I cannot close this short note without emphasizing the quite special importance of Hamilton's investigations in geometrical optics." The real source of Hamilton's profound dynamical theory lies in his optical views. Klein expresses the wish that Hamilton's optical papers may be made more easily accessible. Such a publication he declares would have not only historical value but would clarify and advance our ideas in many directions.

The first great advance, using Hamilton's optical methods, since Hamilton's last major paper *The Third Supplement* published in 1837 was made by K. Schwarzschild in his Untersuchungen zur geometrischen Optik in 1905, that is, after an interval of sixty-eight years. Contrary to the usual belief at that time, he showed how adaptable Hamilton's methods are to deduce practical formulae

in the design of optical instruments. As far as the reviewer is aware, Schwarzschild's paper has not been followed by others on the continent.

In England the situation is different. The last war made it clear to all that a great country could not allow itself to depend upon a foreign country for optical supplies. Immense efforts have been made to bring England back to the same relative level that it enjoyed in the days of Airy, Sir John Herschel, and Coddington. From all the reviewer can learn these efforts have been crowned with success. The feature most gratifying to him is their return to the method of Hamilton. We notice the splendid work of T. Smith[1] and G. C. Steward. It has commonly been thought that the methods of geometrical optics are useless in diffraction problems. Steward has shown[2] that the integral

$$\int e^{2\pi i(U+r)/\lambda} dS,$$

where U is a characteristic function and dS an element of surface may be used with marked success in treating aberration diffraction effects.

Not only have Hamilton's optical ideas been stimulating and fruitful today in the domain of pure optics but also in the new quantum mechanics. Schrödinger[3] who has been so successful in this field writes:—"Unfortunately this powerful and weighty circle of ideas of Hamilton has been robbed of its beautiful intuitive clothing in most of its modern presentations, as a superfluous scaffolding, in favor of a more colorless analytical treatment."

It is probably due to the new English school of theoretic and practical optics and the widespread interest in quantum mechanics that the Royal Irish Academy has incurred the heavy expense of publishing the Mathematical Papers of Sir William Rowan Hamilton. The moment is doubly opportune.

JAMES PIERPONT

Introduction to Higher Geometry. By W. C. Graustein. New York, The Macmillan Company, 1930. xv+486 pages.

The publication of this book would seem to be an event of major importance for the development of geometry in the United States. The qualifications of the author for writing such a book are eminent and well known. When a mathematician whose original researches of marked distinction in geometry have measurably advanced his science, and whose lectures in the classroom have become widely known for their clarity and thoroughness, announces a forthcoming volume from his pen, the result of his undertaking is to be awaited with impatience and received with enthusiasm. In this instance our expectations have been amply realized.

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[1] Cf. numerous papers "Transactions of the Optical Society" from volume 24 on.

[2] Trans. Roy. Soc. London, vol. 225, p. 131. Trans. Cambridge Phil. Soc., vol. 23, p. 236.

[3] Ann. d. Physik, vol. 79 (1926), p. 490.

This work of Graustein's seems to meet a definite need. Some of us who regularly teach *analytic projective geometry*, or *modern analytic geometry*, or *higher geometry*, or whatever else it may be called have been wishing for just such a book in English. Our students come to us with a knowledge of elementary metric analytic geometry in the plane and perhaps also in three-dimensional space. Some, but not all, of them have studied *synthetic projective geometry*. It is our task to prepare them for graduate work in *algebraic geometry*, *projective differential geometry*, and the like. In short, we are expected to lead them to some appreciation of developments in geometry as outlined by Klein in the Erlanger Programm. The book before us will be a very teachable text in these courses, and, moreover, will be just the thing to place in the hands of students who have not had the benefit of formal instruction in the subjects covered but who wish to read along these lines.

Examining the structure and contents of Graustein's book, we observe at once that it is built around the Erlanger Programm and contains nineteen chapters, of which only the last advances from geometry in the plane to geometry in space of three dimensions. It will not be necessary to attempt to outline the entire subject matter, but it does seem worth while to note the organization of the opening chapters. The first is entitled *Linear equations and linear dependence*, but it might properly have been called *Algebraic introduction*, since its purpose is to give the reader the essentials of higher algebra that he is going to need immediately. The second chapter is entitled *Geometrical introduction*, and may be passed over rapidly by readers who have studied synthetic projective geometry. In Chapter III homogeneous cartesian coordinates are introduced. In the next five chapters appear harmonic division, line coordinates, cross ratio, transformations, and metric geometry of the complex plane in natural and historical order. One-dimensional projective geometry is introduced in Chapter IX, and projective homogeneous coordinates in the plane are postponed to Chapter X.

The experience of the reviewer agrees with that of the author in dictating that it is most feasible pedagogically to base the treatment of our subject on the student's previous training in metric analytic geometry. The affine, projective, and other geometries should at first be presented to him couched in terms of his previous knowledge. Later on he can do what we might like to do in our first presentation, namely, organize each geometry on a postulational basis of its own, or at least he will then be in a position to appreciate a postulational presentation.

Experience differs as to details. For instance, the reviewer, after many trials and errors, has come to the conclusion that he can economically and effectively introduce general projective homogeneous point coordinates before attempting to consider homogeneous cartesian coordinates. The more general coordinate system seems to be more easily comprehended by the student than the more special cartesian system. The author, however, reverses this order, as has al-

ready been noted. Another instance of divergence of taste occurs in Chapter XVI. The reviewer is in the habit of calling the system of conics tangent to four given lines a *range* of conics, just as he calls the system of conics through four given points a *pencil* of conics, thus preserving the analogy between a *range* of points on a straight line and the dual *pencil* of lines through a point. The author does not seem to use the convenient terminology *range of conics*, but says *pencil of line conics*.

The author's treatment is thorough and painstaking with emphasis on fundamentals. The reasoning is as sound as it can be in a non-postulational exposition. Particularly to be commended is the sparing use of the idea usually expressed by the phrase "in general." And as a matter of language, when one says *in general* one ought to mean *in all special cases*, but what is usually meant is *except in one or more special cases*. When we wish to convey the latter meaning why can we not say "ordinarily" instead of "in general" and thus preserve for the phrase its true meaning?

There is an opulent wealth of material in the book. The number of available exercises at the ends of sections and chapters is amazing—more than 1000 by actual count!

The print is clear; the typography excellent. The attractiveness and beauty of the cover characterize a very appropriately chosen dress for a dignified volume.

While not daring to assume too ostensibly the role of a prophet, the reviewer does venture the prognostication that, in its influence on the mathematical teaching of first-year graduates in the colleges and universities of the United States, and as a book of reference, Graustein's *Introduction to Higher Geometry* will eventually take rank along with Bôcher's *Introduction to Higher Algebra* and Eisenhart's *Differential Geometry*.

ERNEST P. LANE

Die Relativitätstheorie. By Professor Dr. Ludwig Hopf. Berlin, Julius Springer, 1931. 147 pages.

This German book on the relativity theory, intended to be a popular treatment, turns out to be very scholarly. This is not to be taken as a disparagement of the work. On the contrary, the reviewer considers it as one of the best books on the subject offered for educated laymen. To gain from it a full understanding of the theory, however, one must possess a fair previous acquaintance with the theory's basic ideas. The reader already informed in this way about the essentials of the theory will find in Professor Hopf's book an excellent means of broadening his knowledge.

The work consists of three parts; preliminaries, special relativity theory, and general relativity theory. The first part, which occupies about one fifth of the whole book, is the best of the three. It is quite intelligible even to the beginner in the study of the relativity theory. At full length and with great

lucidity it discusses the transformation of the mechanistic "world picture" into the electromagnetic one in the decades preceding the advent of the theory. This account of preliminary developments, not covered satisfactorily in other popular books on the subject, is a valuable distinguishing feature of the work; and prepares the reader admirably for the study of the main theme of the book, the relativity theory, presented in the other two parts.

For the novice these will prove rather disappointing. The fundamentals of the theory, which interest him first of all, are not given in a manner intelligible for him. He will also be confronted with other puzzles. The book, for instance, states (p. 62) that "*a body moving with light velocity cannot be accelerated any more; no force of the world would suffice for this. It follows that the mass of a body must grow with the velocity.*" This conclusion does not at all follow from the premises. The beginner will certainly not see why the mass must grow with the velocity for the reason that a velocity greater than that of light is impossible. He will not be helped over the difficulty by the whole discussion of the connection between the mass and velocity of a body: for this discussion is not clear. Some important paragraphs are worded so obscurely for the tyro that he will hardly be able to understand them. Such paragraphs are those treating the "dependence of mass upon velocity" (pp. 61–62), "light propagation in the accelerated laboratory" (99–101), "rotating bodies" (107–109), "clocks in the gravitational field" (111–113), "two methods of time measurement" (117–119), etc.

Professor Hopf's book has therefore not invalidated the reviewer's contention (this MONTHLY, January, 1932) that the relativity theory has not yet found the ideal expositor. Nevertheless, considering the needs of the advanced student as well as those of the beginner, Professor Hopf's book represents a most valuable contribution to the literature of the relativity theory.

MAX TALMEY

PROBLEMS AND SOLUTIONS

EDITED *by* B. F. FINKEL, OTTO DUNKEL, *and* H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3543. *Proposed by L. M. Berkeley, New York City.*

A, B, C, and D are points on a sphere. Given all the parts (sides and angles)

of the spherical triangle ACD, and given also the angles ABC and ABD, find AB.

3544. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

The orthogonal sphere of four variable spheres, with fixed centers, whose radii remain proportional, describes a coaxial pencil.

3545. *Proposed by George Y. Sosnow, Newark, N. J.*

The equation

$$\frac{a_1}{x+b_1}+\frac{a_2}{x+b_2}+\frac{a_3}{x+b_3}+\cdots+\frac{a_n}{x+b_n}=0,$$

will be an identical equation if

$$\sum_{i=1}^{n}(a_i)=0,\quad \sum_{i=1}^{n}(a_ib_i)=0,\quad \sum_{i=1}^{n}(a_ib_i^2)=0,\cdots,\quad \sum_{i=1}^{n}(a_ib_i^{n-1})=0.$$

3546. *Proposed by W. H. Rasche, Virginia Polytechnic Inst.*

Given that r is the radius of a circle, S its plane, O its center, P any point in the plane of the circle, distant $d(<r)$ from the center; l is the line through P perpendicular to line OP which makes with plane S the angle $\alpha=\cos^{-1}(d/r)$; show that if the circle be rotated about l as an axis it will generate a torus whose meridian section is two equal non-intersecting circles of radius d, whose centers are equi-distant from axis l.

Note: A solution suitable for college freshmen not familiar with solid analytical geometry is desired.

3547. *Proposed by Martin Rosenman, Brooklyn, New York.*

Consider n points in a plane. Join these in any order to form a closed polygon. Repeat the operation on the n midpoints of the sides of the polygon thus formed, etc. Prove that the successive polygons converge to a point.

3548. *Proposed by J. B. Reynolds, Lehigh University.*

A coal chute delivers coal at a point at distances a and b from two vertical walls at right angles to each other. Find the volume of a conical pile of coal under the chute if its radius is r, $r>(a^2+b^2)^{1/2}$, and the angle of repose of coal is A.

3549. *Proposed by Dewey C. Duncan, University of California.*

Show that for just one set of positive constants $a_1, a_2, \cdots, a_{k-1}$, the equation

$$x^{2k+1}-(x-1)^3\prod_{i=1}^{k-1}(x-a_i)^2=0$$

has $k-1$ double roots. These double roots are necessarily positive and the remaining pair are complex.

3550. *Proposed by H. L. Fisher, Annandale, New Jersey.*

How much will be bored from a cubical block, edge, e, by boring seven holes through the block; one each through the centers of three pairs of opposite faces and one each along the four diagonals as axes, the radius of the auger being r?

3551. *Proposed by Eugene M. Berry, Lynchburg College.*

Find the equation of a curve whose evolute is the same as one of its involutes but not the same as the original curve.

SOLUTIONS

3486[1931, 171]. *Proposed by E. C. Kennedy, College of Mines, El Paso, Texas.*

Find a function of x such that the integral of that function between the limits of 0 and x is equal to the reciprocal of the original function.

Solution by Byron D. Roberts, Parsons College.

Let y be the required function of x, such that $\int_0^x y dx = y^{-1}$. Differentiating this equation we get a simple differential equation of the first order in which the variables separate in the form $y^{-3}dy = -dx$. The general solution of this equation is $y = (2x+c)^{-1/2}$; and for the required function we must set $c=0$. The function is therefore $(2x)^{-1/2}$.

Also solved by H. T. R. Aude, A. G. Clark, Wei Liang Chow, J. D. Hill, V. F. Ivanoff, L. S. Johnston, J. H. Neelley, W. V. Parker, O. J. Ramler, H. D. Ruderman, F. Underwood, Kamcheung Woo, and the Proposer.

3487[1931, 227]. *Proposed by Vladimir F. Ivanoff, San Francisco, California.*

Prove the identities:

$$\frac{d^2y}{dx^2} + \frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3 \equiv 0, \tag{1}$$

$$\frac{d^3y}{dx^3}\left(\frac{dx}{dy}\right)^2 + \frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^2 + 3\frac{d^2y}{dx^2}\,\frac{d^2x}{dy^2} \equiv 0, \tag{2}$$

the derivatives d^ny/dx^n and d^nx/dy^n being based on the same functional relation $f(x, y) = 0$.

Solution by D. H. Richert, Bethel College

Since

$$\frac{dy}{dx} = 1\bigg/\frac{dx}{dy},$$

we have

$$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(1\bigg/\frac{dx}{dy}\right)\frac{dy}{dx} = -\frac{d^2x}{dy^2}\bigg/\left(\frac{dx}{dy}\right)^3$$

whence

$$\frac{d^2y}{dx^2}+\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3=0,$$

which is (1).

Again, we have

$$\begin{aligned}\frac{d^3y}{dx^3} &= -\frac{d}{dy}\left[\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3\right]\frac{dy}{dx}\\ &= -\left[\frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^3+\frac{d^2x}{dy^2}\,\frac{d}{dy}\left(\frac{dy}{dx}\right)^3\right]\frac{dy}{dx}\\ &= -\left[\frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^3+\frac{d^2x}{dy^2}\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)^3\frac{dx}{dy}\right]\frac{dy}{dx}\\ &= -\left[\frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^3+\frac{d^2x}{dy^2}\cdot 3\left(\frac{dy}{dx}\right)^2\cdot\frac{d^2y}{dx^2}\,\frac{dx}{dy}\right]\frac{dy}{dx}\\ &= -\left[\frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^4+3\frac{d^2y}{dx^2}\cdot\frac{d^2x}{dy^2}\cdot\left(\frac{dy}{dx}\right)^2\right].\end{aligned}$$

Whence, multiplying both sides of the equation by $(dx/dy)^2$ and transposing, we have

$$\frac{d^3y}{dx^3}\left(\frac{dx}{dy}\right)^2+\frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^2+3\frac{d^2y}{dx^2}\,\frac{d^2x}{dy^2}=0,$$

which is (2).

Also solved by A. G. Clark, W. R. Church, Louis Deutsch, J. D. Hill, J. F. G. Hicks, Theodore Lindquist, J. D. Leith, J. A. McLaughlin, Gertrude McCain, A. Pelletier, and E. P. Stark.

A Note by the Editors: These relations are derived on page 41 of vol. 1, Goursat-Hedrick, as we learn from a note by Norman Anning.

3488[1931, 228]. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, New York.*

Find the numbers m such that the square of each of the $\phi(m)$ numbers less than m and prime to it will be congruent to 1, modulo m.

Solution by J. D. Hill, University of California at Los Angeles

The numbers 1, 2, 3, and 4 are obvious solutions. In order to determine whether or not further solutions exist, we first note the following properties which such numbers necessarily possess:

(1) No odd number greater than 3 is a solution since 2^2 is not congruent to 1 for such a modulus.

(2) No square number greater than 4 is a solution. In view of the first property, assume m to be of the form $4x^2$ where x is greater than 1. But $4x^2$ cannot be a solution since $(2x-1)^2$ is not congruent to 1 for this modulus, although $2x-1$ and $4x^2$ are relatively prime.

(3) The number m is not a solution unless $m-1$ is a prime. For if $m-1$ is not a prime it has a prime factor p less than the square root of m, whereas p^2 is obviously not congruent to 1, modulo m.

(4) The number m to be a solution must be divisible by the product of all the primes less than its own square root. For if q is any prime less than the square root of m, which does not divide m, then it is evident that q^2 is not congruent to 1, modulo m.

For simplicity we first seek solutions among the integers 5 to 49 inclusive. To begin with, the odd numbers in the interval are discarded and after these the two squares, 16 and 36. Of the remaining even integers 10, 22, 26, 28, 34, 40, and 46 fail to satisfy (3), while 14, 20, 32, 38, 42, 44 and 48 do not meet condition (4). This leaves 6, 8, 12, 18, 24, and 30 as possible solutions, and of these numbers only 18 and 30 fail to fulfill the conditions of the problem.

That 1, 2, 3, 4, 6, 8, 12, and 24 are the only solutions of this problem may be proved as follows: Denote by P_n the product of all the primes equal to or less than n, and consider the numbers m defined by the inequality $n^2<m<(n+1)^2$, where $n\geqq 7$. By the property (4), m must be divisible by P_n in order to be a solution. But by the well known theorem that the sum of the natural logarithms of all primes $\leqq n$ equals n asymptotically,[1] it is easily shown that $P_n>(n+1)^2$, for $n\geqq 7$. Therefore no number greater than 49 could possibly be a solution, and we have the compact result that the conditions of the problem are satisfied by and only by the positive integral factors of 24.

Also solved by A. G. Clark, W. R. Church, Emma Lehmer, D. E. Rutherford, E. P. Stark, and F. L. Wilmer.

3489[1931, 288]. *Proposed by J. G. Deutsch, Columbia University.*

Given four points in a plane, no three of which are collinear, what is the necessary and sufficient condition that there exist a square containing the given points on its perimeter such that no two points are on the same side?

Solution by Roger A. Johnson, Brooklyn College

The problem is not well stated, for two reasons: first, there is no specification as to the order of the given points, and second, because the restriction that the points be not collinear is unnecessary. We may propose the problem in the following alternative form:

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[1] See Dickson's History, Vol. 1, page 439.

Given four points in a plane in definite order, to construct a square whose sides shall pass successively through these four points.

The result then is: if A, B, C, D are the points, so that A and C are to lie on opposite sides of the required square, B and D on the other pair of opposite sides, there always are two solutions except when AC and BD are perpendicular. In this exceptional case, if AC and BD are unequal there is no solution; if they are equal, there are infinitely many solutions.

Analysis. Let A, B, C, D be the given points; draw a rectangle $PQRS$ so that PQ, QR, RS, SP pass respectively through A, B, C, D. (It is sufficient to draw any parallels through A and C, and perpendiculars to them through B and D.) Let AH be perpendicular to RS, and DK to QR. In order that the rectangle may be a square, it is necessary and sufficient that $AH=DK$. Therefore, if the right triangle DBK were turned through 90° and D superposed on A, K would fall on H, and B, H, C would be collinear. Hence the construction.

Construction. Through A draw AE equal and perpendicular to DB. Then CE is the direction of the side of the required square which passes through C; and the square is at once constructed by drawing a line through A parallel to CE, and lines through B and D perpendicular to it.

Since AE can be drawn in either of two opposite directions, there are necessarily two solutions. The analysis and the construction work equally well when three or even four of the points are collinear. A failing case appears when AC and BD are perpendicular and unequal in length, for then AE falls on AC. If $AC=AE=BD$, any direction may be taken; i.e., any rectangle circumscribed about $ABCD$ is a square if AC and BD are perpendicular and equal. It can be proved at once that if A, B, C, D lie on the sides of a square and AC is perpendicular to BD, then $AC=BD$. Hence, *a priori* the case that AC and BD are perpendicular and unequal is impossible.

Also solved by E. B. Escott, H. E. Stelson, and F. Underwood.

3490 [1931, 288]. *Proposed by H. E. Stelson, Kent State College, Kent, Ohio.*

Sanford's *History of Mathematics*, page 264, tells of Pappus's (c. 300) method of trisecting an angle by the use of conics. He placed the angle in question with its vertex at the center of a circle. He then trisected the subtended chord AB and constructed a hyperbola having AA' two thirds of the chord as its transverse axis and $\sqrt{3}$ times the transverse axis for the conjugate one.

Prove analytically that points joined to the ends of the chord such that one of the angles thus formed is double the other lie on the above hyperbola.

Solution by Margaret M. Young, Brooklyn College of the City of New York

Let the axes be taken so that the coordinates of A, A', B are, respectively, $(a, 0)$, $(-a, 0)$, $(-2a, 0)$; and let $P(x, y)$ be a point chosen so that $\angle ABP = 2\angle BAP$. By the law of sines

$$\frac{BP}{AP} = \frac{\sin A}{\sin 2A} = \frac{1}{2\cos A}, \tag{1}$$

where

$$BP = [(x + 2a)^2 + y^2]^{1/2}, \quad AP = [(x - a)^2 + y^2]^{1/2}, \tag{2}$$
$$\cos A = (a - x)/AP.$$

From (1) and (2) we obtain after certain reductions the equation of the locus of P

$$4(x - a)^2(x + 2a)^2 = [(x - a)^2 - y^2]^2. \tag{3}$$

This equation is equivalent to two equations, one of which is

$$x^2 + 4ax + y^2 - 5a^2 = 0. \tag{4}$$

This equation (4) results from a triangle ABP where $180° - \angle ABP = 2\angle BAP$, and therefore $BA = BP$. This case is included in the analytical work, but is to be rejected. The desired equation is then the other part of (3)

$$3x^2 - y^2 - 3a^2 = 0. \tag{5}$$

It is an hyperbola of the kind described in the problem, and the branch through A' is the desired locus.

A Note by Otto Dunkel: Since it may be foreseen that the law of sines introduces an extraneous equation of a circle, it would be simpler to replace this law by the relation $\tan B = 2\tan A(1 - \tan^2 A)^{-1}$, where $\tan B = y/(x+2a)$, $\tan A = y/(a-x)$. After rejecting a factor y, we have the desired equation.

It should be observed that the construction given by Pappus requires the proof of the converse of the theorem at the end of the problem. This proof may be obtained by altering the form of equation (5) so as to introduce $\tan B$ and $\tan A$; in other words we simply reverse the order of the above proof.

If BA is prolonged to D so that $BD = 4a$, the branch through A' of the hyperbola of Pappus may be defined as the locus of P such that $DP = BP + 2a$. The proof may be now given by methods in use in the days of Pappus. Let M be the projection of P on BA, then $\overline{DM}^2 - \overline{MB}^2 = \overline{DP}^2 - \overline{BP}^2$. After factoring and using the above definition of P, this reduces to $2(DM - MB) = DP + BP = 2(BP + a)$. Hence $AM = DM - a = MB + BP$. Produce AB to Q so that $BQ = BP$, then $AM = MQ$, and the triangle AQP is isosceles. Hence $\angle ABP = 2\angle BQP = 2\angle BAP$.

Also solved by J. E. Burnam, A. Pelletier, O. J. Ramler, Wallace Smith, and F. Underwood.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

A professorship of mathematics has been established at Yale University in honor of Josiah Willard Gibbs. Professor E. W. Brown, previously Stirling professor of mathematics at the University, has been made the first incumbent of this new professorship.

Professor Oswald Veblen, of Princeton University, is on leave of absence during the second semester of the current academic year. On December 18, he lectured at Duke University on *Spinor analysis;* on January 8, 11, and 12, he lectured at the Rice Institute on the topics *The modern approach to elementary geometry, Analysis situs*, and *The new differential geometry;* and on January 25, he lectured at The University of California at Los Angeles on *Projective relativity*. In February he will go to Europe, and will lecture at the universities of Göttingen, Hamburg, and Berlin.

Professor H. N. Russell, of the department of astronomy of Princeton University, has been elected president of the American Philosophical Society.

The following have been appointed associate editors of the Transactions of the American Mathematical Society: E. P. Lane, Marston Morse, M. H. Stone.

The Scientific and Research Institute of Mathematics and Mechanics of the University of Moscow, has awarded to L. S. Pontrjagin, member of the Institute, a prize of 750 roubles for his paper "Über stätige algebraische Körper" which has just appeared in the Annals of Mathematics.

Assistant Professor J. A. Nyswander, of the University of Michigan, has been promoted to an associate professorship.

Mr. D. E. Whitford has been promoted to an assistant professorship at the Brooklyn Polytechnic Institute.

The following courses in mathematics are announced for the summer 1932:

University of California at Los Angeles, July 1 to August 10. By Professor Paul H. Daus: Foundations of arithmetic.

University of Maine, July 5 to August 12. In addition to the usual elementary work, the following advanced courses are offered. By Associate Professor Bryan: Teacher's course. By Associate Professor Jordan: Practical astronomy. By Professor Willard: Theory of functions of a complex variable.

Ohio State University, June 21 to September 2. In addition to the usual courses in trigonometry, analytic geometry, and the calculus, the following advanced courses will be offered. By Professor C. C. Morris: Theory of probability; Differential equations. By Professor J. H. Weaver: Projective geometry; Differential geometry. By Professor Lincoln LaPaz: Calculus of variations; Tensor analysis. By Dr. H. P. Theilman: Integral equations.

Stanford University, June 23 to September 3. In addition to the usual courses in calculus and differential equations, the following courses are offered. By Professor G. T. Whyburn of Johns Hopkins University: Theory of functions of a real variable; Foundations of geometry; Point-Set theory.

On February 28, there died at Providence, R. I., Dr. Arnold Buffum Chace, Chancellor (President of the Corporation) of Brown University. His edition of the Rhind Mathematical Papyrus is probably the most elaborate publication ever made of any ancient mathematical manuscript, and is certainly one of the most scholarly. For more than a half century he was a member of the Brown Corporation, and during most of this time he held either the office of treasurer or that of chancellor. He was educated at Brown and did advanced work in chemistry in Europe.

Miss Rose B. Wood, of Greenville Woman's College, died July 28, 1931. She was a charter member of the Association.

The announcement has been received of the death of Professor Doctor Heinrich Wieleitner, Oberstudiendirektor am neuen Realgymnasium und Honorarprofessor an der Universität, München, on December 27, 1931. Professor Wieleitner was well-known to all students in the history of mathematics. He was a serious scholar and a clear and accurate writer. His loss will be felt in this country as well as in Munich, where he worked for so many years.

It is known that Edouard Combescure (1824–1889), a docteur es-sciences of Paris (1858) and professor of Astronomy at Montpellier (1868) spent some time in the United States. It is desired to know if he was connected with any university here, and if he published any scientific papers while in this country. Any particulars relating to these matters will be welcome. Please send any information to Professor David Eugene Smith, 501 West 120th Street, New York.

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teachers' agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The prize will be awarded hereafter every three years. The last award was in December, 1929, to Professor T. H. Hildebrandt. The next award will be in December, 1932, for the period 1929-1931.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Sixteenth Summer Meeting of the Association, Los Angeles, Calif., Aug. 29-30, 1932.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1932 and reported to the Secretary.

ILLINOIS, Urbana, May 6-7.
INDIANA, Indianapolis, May 6-7.
IOWA, Cedar Falls, April 29-30.
KANSAS, Topeka, Feb. 13.
KENTUCKY.
LOUISIANA-MISSISSIPPI, Oxford, Miss., March 11-12.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7.
MICHIGAN, Ann Arbor, March 19.
MINNESOTA.
MISSOURI.
NEBRASKA.
OHIO, Columbus, Ohio, April 7.
PHILADELPHIA, Philadelphia, Pa., Nov. 26.
ROCKY MOUNTAIN, Laramie, Wyo., April 15-16.
SOUTHEASTERN, Gainesville, Fla., Mar. 18-19.
SOUTHERN CALIFORNIA, San Diego, March 26.
TEXAS, Austin, Jan. 30.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXIX, 1932

NUMBER 5, MAY

PUBLISHED BY THE ASSOCIATION

MENASHA, WIS., AND ITHACA, N.Y.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members

$5.00 a Year, Single Copies 60 cents, to Others

THE SEVENTH ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The seventh annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Natchitoches, La., on March 13 and 14, 1931, in connection with the annual meetings of the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics and the Louisiana Academy of Sciences. The session Friday afternoon was devoted largely to Section matters, Professor I. C. Nichols, chairman of the Section, presiding.

The attendance was well over one hundred, including the following fifteen members of the Association: Nola L. Anderson, T. A. Bickerstaff, Leora Blair, H. E. Buchanan, C. G. Jaeger, C. G. Killen, A. C. Maddox, B. E. Mitchell, I. C. Nichols, R. L. O'Quinn, M. C. Rhodes, S. T. Sanders, C. D. Smith, H. L. Smith, W. P. Webber; and two official representatives of institutional members: J. P. Cole and J. A. Hardin.

The following officers were chosen for the coming year: Chairman, A. C. Maddox, Louisiana State Normal College; Vice-chairmen, C. D. Smith, Mississippi A. and M. College and J. P. Cole, Louisiana Polytechnic Institute; Secretary-Treasurer, Nola L. Anderson, Sophie Newcomb College. The University of Mississippi was selected as the place for the 1932 annual meeting.

Three attractive features of the convention were the joint tea in the social room of the Normal College, served by the local branch of the American Association of University Women; the joint illustrated lecture by Doctor Frans Blom of Tulane University on "Middle American research," Professor George Williamson presiding; and the joint banquet in the College dining hall, Doctor Nichols presiding as toastmaster. About two hundred persons attended the banquet. Excellent music was furnished by the college band and glee clubs.

At the Friday session of the Section the following papers were presented:

1. "Ancient Egyptian problems" by Professor E. M. Shirley, Louisiana Polytechnic Institute, by invitation.
2. "The construction of the helium atom" by Professor H. E. Buchanan, Tulane University.
3. "The rise of a new geometry" by Professor C. D. Smith, Mississippi A. and M. College.
4. "Trigonometry in a space of n dimensions" by Professor Nola L. Anderson, Sophie Newcomb College.
5. "Certain partial derived functions" by F. A. Richey, Mandeville, La., by invitation.
6. "A certain theorem concerning two triangles" by Professor C. G. Jaeger, Tulane University.
7. "Sextantal analysis" by Professor B. E. Mitchell, Millsaps College.
8. "Maya numbers" by Professor Frans Blom, Tulane University, by invitation.

9. "Our national outlook" by Professor S. T. Sanders, Louisiana State University.

The program of the Council on Saturday morning was an integral part of the meetings. The following papers were presented:

1. "High school and college mathematics" by R. L. O'Quinn, Louisiana State University.
2. "How can mathematics be taught most effectively?" by Professor M. C. Rhodes, University of Mississippi.
3. "Developing the number sense" by Professor Leora Blair, Louisiana State Normal College.
4. "The place of algebra in the mathematical program" by Professor C. D. Smith, Mississippi A and M College.
5. "The investigative versus the traditional method of studying plane geometry" by L. S. Miller, Many, La.
6. "Seeing things" by J. T. Harwell, Byrd High School, Shreveport, La.
7. "The Rhind Mathematical Papyrus" by Professor J. P. Cole, Louisiana Polytechnic Institute.

I. C. NICHOLS

THE DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The Fall meeting of the Maryland-District of Columbia-Virginia Section was held at the U. S. Naval Academy on Saturday, December 5, 1931.

Thirty-nine members attended, bringing thirty-four guests, representing George Washington University, Catholic University, U. S. Coast and Geodetic Survey, U. S. Naval Observatory, Bureau of Ordnance, University of Maryland, Goucher College, Johns Hopkins University, St. John's College, U. S. Naval Academy and Naval Post Graduate School.

The members present were: G. F. Alrich, Ethel L. Anderton, Clara L. Bacon, G. A. Bingley, Archie Blake, C. C. Bramble, Paul Capron, C. N. Claire, J. S. Clayton, G. R. Clements, Abraham Cohen, Alexander Dillingham, J. A. Duerksen, Mary Ewin, Michael Goldberg, Harry Gwinner, W. M. Hamilton, F. E. Johnston, Solomon Kullback, W. D. Lambert, A. E. Landry, C. L. Leiper, Florence P. Lewis, G. A. Lyle, W. K. Morrill, F. D. Murnaghan, C. H. Rawlins, J. N. Rice, R. E. Root, J. B. Scarborough, Abraham Sinkov, J. H. Taylor, John Tyler, F. M. Weida, C. H. Wheeler, G. T. Whyburn, John Williamson, E. W. Woolard, Oscar Zariski.

The Chairman, Professor E. W. Woolard, conducted the meeting. In the absence of the Superintendent, Admiral Thos. C. Hart, Capt. Lewis B. McBride, head of the department of mathematics cordially welcomed the members and their guests. Capt. McBride expressed the hope that he might sometime hear from the Association concerning pedagogical problems.

At the morning session, four papers were presented:

1. "Analogues of De Moivre's theorem" by Professor John Tyler, U. S. Naval Academy.

2. "The theory of six numbers" by Professor Frank Morley, Johns Hopkins University, by invitation.

3. "On the conditions for the existence of a kinetic potential" by Professor J. H. Taylor, George Washington University.

4. "Probability in bombing as related to the angle of approach" by Professor C. C. Bramble, Postgraduate School, Naval Academy.

After lunch, all were guests of St. John's College; a remarkable exhibit had been arranged in the College Library by Professor G. A. Bingley, of books selected from the King William Collection of the College (the oldest extant of Colonial free public libraries: the Biblioteca Annapolitana of 400 volumes, presented in 1657), and from the private collections of President Gordon, Dr. R. T. H. Halsey, and Dr. Nathan Starr. The most noteworthy exhibits were:

Pacioli, Luca de Borgo: Divina Proportione; Venice 1509. (Plates attributed to Leonardo da Vinci.)

A second edition (1500) of Martiani Capella's famous encyclopaedia.

Numerous English works of the 18th century on building and architecture, one of them Wm. Pain's Practical House Carpenter, used by George Washington.

Maclaurin, Colin—Newton's Phil. Discourses 5th Ed. 1794.

Vol. 1 of Ozanam's Mathematical Course (5 vols.) London, 1712.

Tracts by Robert Boyle on Flame, Air and Explosives, London, 1672.

Freheri, D. Pauli—Theatri Vivorum Eruditione Clarorum, 2 vols. Nuremberg, 1688.

Two volumes of U. Aldrovanus, Bonn, 1667.

Andrew Wakely's Mariner's Compass and Tables, London, 1758.

The afternoon session completed the program with two papers:

5. "Magic squares" by H. M. Robert, U. S. Naval Academy, by invitation.

6. "Tritangent circles of the rational bi-cubic" by W. K. Morrill, Johns Hopkins University.

Abstracts of some of the papers follow:

2. Professor Morley spoke as follows: Ordinary algebra is concerned with unrestricted numbers, usually called complex. These numbers are represented by points of a plane, or by points of a sphere Ω, or by points of certain other rational surfaces. When we consider pairs of numbers, that is quadratics $q \equiv \alpha x^2+\beta x+\gamma$, the sphere is indicated. It is appropriate to call it the Celestial Sphere, and to regard it as the infinity of space around one. To two points x, y of Ω we attach the arc of a circle orthogonal to Ω, with the end points x, y. Thus an arc represents a quadratic q, and q names an arc.

Two quadratics q_1 and q_2 have the bilinear invariant $q_{12}=q_{21}$. When it vanishes, the arcs cut at right angles; let us say that they are normal. Two

quadratics have also the Jacobian $j_{12}=-j_{21}$. This is represented by the common normal of the two arcs. The form j_{12}/q_{12} is an absolute covariant of q_1 and q_2 (in this order). For any number of quadratics q_i, when ordered, the quadratic

$$j_{12}/q_{12}+j_{23}/q_{23}+\cdots+j_{n1}/q_{n1} \tag{1}$$

is of interest.

Taking six quadratics, $q_i=\alpha_i x^2+\beta_i x+\gamma_i$, subject to the relation

$$\left| \alpha_i^2 \beta_i^2 \gamma_i^2 \beta_i\gamma_i \gamma_i\alpha_i \alpha_i\beta_i \right| = 0, \tag{2}$$

or in other words taking six pairs of numbers which are in a biquadratic involution (symmetric two-to-two correspondence) we have in the unrestricted domain the results of the "hexagramma mysticum" in the unrestricted domain[1] where $\alpha_i:\beta_i:\gamma_i$ are real. Here (1), with $n=6$, takes the place of a Kirkman Point.

If we change the ordering by cyclic permutation of 2, 4, 6, we have the three Kirkman arcs

$$j_{12}/q_{12}+j_{23}/q_{23}+j_{34}/q_{34}+j_{45}/q_{45}+j_{56}/q_{56}+j_{61}/\varsigma_{61}$$

$$j_{14}/q_{14}+j_{43}/q_{43}+j_{36}/q_{36}+j_{65}/q_{65}+j_{52}/q_{52}+j_{21}/q_{21}$$

$$j_{16}/q_{16}+j_{63}/q_{63}+j_{32}/q_{32}+j_{25}/q_{25}+j_{54}/q_{54}+j_{41}/q_{41}$$

whose form is 0. The three such arcs have then a common normal, apart from the relation (2).

In general a theorem of planar projective geometry, when stated with reference to a conic, passes into a theorem of arcs by omitting the restriction of real numbers. And by expanding the universe so that the radius of Ω becomes ∞, we have a theorem on the lines of a Euclidean space. Thus the lines of a Euclidean space are a representation of the theory of quadratics (under homographics).

4. Professor Bramble said that under the assumptions of normal dispersion and no correlation of deviations in range and deflection, the probability of dropping a bomb on a rectangular target is written as a function of the angle of approach. This probability is a maximum for approach along the longer axis of the rectangle if the ratio of the longer axis to the angle of dispersion in range is not less than the ratio of the shorter axis to the average dispersion in deflection. In any case it is shown that the maximum probability occurs within a certain angle.

5. Professor Robert gave a brief account of the history of magic squares, their classification, methods of construction and illustrations of various types.

PAUL CAPRON, *Secretary*

[1] Salmon, Conic Sections.

THE INDIAN ORIGIN OF THE MODERN PLACE-VALUE ARITHMETICAL NOTATION

By SĀRADĀKĀNTA GĀṄGULI, Ravenshaw College, Cuttack, India

In the *Bakhshali Manuscript* (1927) edited by G. R. Kaye we read that "there is no sound evidence of the employment in India of a place-value system earlier than about the ninth century A.D." (page 82). This view has been so insistently given prominence in his articles published in various journals of India and Europe ever since July, 1907, that European and American scholars who have to depend for their knowledge of Indian mathematics on second-hand sources (i.e. on translations and criticisms of Indian mathematical and astronomical works in European languages) have had no other alternative than to accept the view. Accordingly we find the late Professor Cajori concluding his article[1] on the origin of the numerals as follows:

"As a by-product of the discussion of recent years we must admit that, on the evidence presented, the claim that our numerals and the zero were used in India as early as the fifth century must be abandoned, our earliest apparently reliable evidence belongs to the ninth century. . . .These corrections are due G. R. Kaye."

In this paper it is proposed to show that the above view of Kaye, accepted by Cajori and others, is not warranted by facts.

In the October (1927) issue of this MONTHLY I showed that an enunciation of the modern place-value arithmetical notation occurs for the first time in the *Āryabhaṭīyam* which was written by the elder Ārabhata in the year 499 A.D. Indian astrological, astronomical, and philosophical works[2] written after this date contain indisputable evidence in abundance to show that the modern notation has since that date been employed in India. As the use of the notation in India from the ninth century onwards is no longer disputed, I shall here discuss the evidence available in the following Indian works which are admitted by all authorities to have been written before the ninth century A.D.:

(1) Jīva Śarma's astrological work[3] (written before 550 A.D.).
(2) Varāhamihira's *Vhṛatsaṃhitā* (c. 550 A.D.).
(3) Varāhamihira's *Pañca-siddhāntikā* (6th century A.D.).
(4) Brahmagupta's *Brāhma-spuhṭasiddhānta* (628 A.D.).
(5) Brahmagupta's *Khaṇdakhādyaka* (665 A.D.).
(6) Lalla's *Siṣyadhīvṛddhida* (7th century A.D.).[4]

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[1] Scientific Monthly, Vol. 9, No. 5, November, 1919.

[2] I have excluded Subandhu's *Vāsavadattā*—a literary work of about 620 A.D. in which stars are compared to ciphers (*Śūnya-vindavah*); for, I hold that the cipher came into use long before the invention of the place-value arithmetical notation.

[3] Jīva Śarmā is anterior to Varāhamihira as he is mentioned as an authority by the latter (*Vrhajjātaka*, VII, 9).

[4] Sudhākar Dvivedī in his *Gaṇaka-taraṅginī* holds that Lalla preceded Varāhamihira. J. C. Ray in his *Āmāder Jyotisī O Jyotiṣ* takes Lalla to be a contemporary of Brahmagupta. A. B. Keith

(7) A commentary (*bhāṣya*) on Patañjali's *Yogasūtra* by Vyāsa (not later than the 7th century A.D.).[1]

(8) A commentary on *Brahma Sūtra* by Saṅkara (8th century)[2]

A quotation from Jīva Śarmā occurs in Bhattotpala's commentary on Verse 9 of Chapter VII of Varāhamihira's *Vṛhajjātaka* and contains two examples (viz., *dvi-yama* (22) and *vedāgni* [=*veda*+*agni*] (34)) of a method of stating, in terms of word-numerals,[3] numbers expressed in the modern place-value notation.

Two such number-expressions *nava*+*aṣṭa-pañca*+*aṣṭa*[4] (8589) and *śūnya-śara*+*aga-rāma* (3750) occur in the *Vṛhatsaṃhitā* (Chapter VIII, Verse 20). But similar number-expressions occur in great abundance in the works (3)—(6) mentioned above. Bühler holds that they "presuppose the existence of the decimal notation."[5] They also led Burnell to conclude, "It is thus perfectly clear that the Indians knew of numerals with a value according to position in the sixth century A.D."[6] As this opinion of these two authorities has been rejected by historians of mathematics under the influence of Kaye's writings, it is necessary to undertake a critical examination of these number-expressions in order to find the underlying notation.

Although it must be admitted that the figures found in any manuscript copy of a work can not always be taken faithfully to represent the actual forms of like figures of a distant earlier period, the same can not be said of any number stated not in figures but in words forming part of a Sanskrit verse. For example, let us take the expression *śūnya-śara*+*aga-rāma* quoted above. It stands for 3750. Either this expression must be copied as such or the component words must be so replaced by their equivalents that the metre may not be interfered with. The word *śūnya* which means 0 can not be changed into its equivalent *kha* or *ambara* as they have not the same number of syllables; but the word *pūrṇa* can be used for it as (i) *śūnya* and *pūrṇa* have the same number of syllables and (ii) the long syllable is followed by the short in both. It is difficult to replace the word *śara* (which means 5) without violating the metre. The word *aga* can be replaced by *adri*, *ṛṣi*, or *aśva*. The word *rāma* can

places him after Brahmagupta. For the sake of argument only I have here accepted the latest dates assigned to Lalla, Vyāsa, and Śaṅkara so that there may not arise any charge of antedating these authorities.

[1] S. N. Dās Gupta takes 400 A.D. to be the date of this commentary (*History of Indian Philosophy*, Vol. I, p. 212). J. H. Wood appears to assign it to the 6th century while A. B. Keith writes, that Vyāsa's "date is probably before Māgha" whom he places about 700 A.D. (*History of Sanskrit Literature*, pp. 490, 124).

[2] *Encyclopædia of Religion and Ethics*, Vol. II, p. 185.

[3] By a word-numeral I mean a word denoting a number not necessarily less than ten.

[4] In this and other number-expressions the word-numerals have been separated by the sign — or +, the latter sign indicating that the vowels separated by it have been joined in the text according to the rules for the conjunction of vowels.

[5] *Indian Antiquary*, Vol. XXXIII, 1904, Appendix, p. 861.

[6] *South Indian Palæography*, p. 62.

be changed into *kāla* or *loka*. Hence the expression *śūnya-śara+aga-rāma* can be altered into *pūrṇa-śara+adri-kāla* or *pūrṇa-śara+aśva-loka*. But in any case the notation remains the same. In dictating the number fifty expressed in the modern notation we may say "five zero" or "five naught." By replacing the word "zero" by the word "naught" we do not alter the notation. There is thus no reason to doubt that the notation behind the number-expressions which occur in the works of Jīva Śarmā and subsequent Indian authors referred to above existed as such at the times when these works were written.

What is this notation? In Varāhamihira's *Pañca-siddhāntikā* edited by Thibaut and Dvivedī we find four number-expressions, namely, *aṣṭa-yama-pakṣa* (228), *manu-śara* (514), *guṇa-kha-sapta* (703), and *aṣṭa-nava-rasa* (698) on the second page of the text. In the first, third and fourth of these expressions words have been used which denote numbers less than ten. In these three expressions we recognize the existence of the modern place-value notation with ten word-numerals including the zero. In such cases each word-numeral serves the purpose of a modern digit. But, when we come across number-expressions like *manu-śara* (514), *rada-veda*[1] (432), *rasa-tithi-yamala* (2156), and *danta+aṣṭa-yama* (2832) which consist of less word-numerals than the number of digits required to express the corresponding numbers in our modern notation, we are, at first sight, led to doubt the existence of any place-value notation. Nay, we may go farther and assert that there does not seem to be any sound principle behind these number-expressions or that the above two classes of number-expressions have originated from two different sources so that their occurrence in the same work is conclusive proof of its spuriousness.

The difficulty of reconciling the above-mentioned two classes of number-expressions vanishes when it is recognized that the expressions give the corresponding numbers expressed in our modern place-value notation by stating the digits (either one or two at a time to suit the metre) beginning with the units.[2] The practice of stating the digits of a number to save the time and labour of enumerating it fully is not now uncommon in the West, as is seen in reading telephone numbers. India also followed—and still follows—the same practice in metrical composition involving numbers, with this difference—that India generally[3] was more rational and scientific in this case than the West now is. For, when one begins with the digit in the units' place and enumerates the other digits in succession, the local value of each digit is apparent; whereas, when one begins with the digit in the highest place, the hearer is quite in the dark as

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[1] This and the following two expressions are taken from *Brāhma-sphuta-siddhānta* (I, 15; II, 3 and 4).

[2] This view has been taken also by Burgess who writes that "the method of writing numbers, large or small, is by naming the figures which compose them" (Introductory Note to his Translation of the *Sūrya Siddhānta*, p. 143).

[3] In his expressions for numbers in terms of letter-numerals Junior Āryabhata (10th century) states the digits beginning with the highest place (*Mahāsiddhānta*, pp. 4, 5, etc.). When he uses word-numerals, he begins with the units' place according to the existing convention (Ibid, Ch. XV).

to the local values of the digits until all the digits are exhausted.[1] As the year one thousand nine hundred and thirty-two is conveniently stated as "nineteen thirty-two," so the number 514 can be conveniently stated as "five fourteen" or "fifty-one four" according to Western practice or as "fourteen (*manu*) five (*śara*)," "four fifty-one" according to Indian practice. Hence the expression *manu-śara* for 514. Similarly we get the other expressions involving less word-numerals than the digits in the corresponding numbers. The expressions *śūnyāni sapta* (seven zeros) *rada-vedāḥ* (432) for the number 4320000000 and *khacatuṣṭaya-yama-śara-guṇa-śaśi-tri-veda*[2] (four zeros, two, five, three, one, three, four) for the number 4313520000 dispels all doubt as to the principle underlying all the number-expressions in terms of word-numerals which occur in the works mentioned above.

That the modern arithmetical notation is the basis of all the number-expressions which occur in the Indian works from the sixth century onwards will be clear from the following different ways[3] in which the number 123249 can be expressed with word-numerals in accordance with the demands of the metre used. The number can be stated thus:

(i) With six word-numerals

as *nanda-veda-netra-guṇa-pakṣa-śaśī* (1,2,3,2,4,9)
9 4 2 3 2 1

(ii) With five word-numerals

as (a) *tāna-netra-guṇa-pakṣa-śaśī* (1,2,3,2,49)
49 2 3 2 1

or as (b) *nanda-jina-guṇga-pakṣa-śaśī* (1,2,3,24,9)
9 24 3 2 1

or as (c) *nanda-veda-danta-pakṣa-śaśī* (1,2,32,4,9)
9 4 32 2 1

or as (d) *nanda-veda-netra-vikṛti-śaśī* (1,23,2,4,9)
9 4 2 23 1

or as (e) *nanda-veda-netra-guṇa-māsa* (12,3,2,4,9)
9 4 2 3 12

(iii) With four word-numerals

as (a) *tāna-danta-takṣa-śaśī* (1,2,32,49)
49 32 2 1

or as (b) *tāna-netra-vikṛti-śaśī* (1,23,2,49)
49 2 23 1

or as (c) *tāna-netra-guṇa-māsa* (12,3,2,49)
49 2 3 12

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[1] When a number is dictated to a copyist with a left-to-right script, the Western practice is more advantageous than the Indian.

[2] These two expressions are taken from *Brāhma-sphuta siddhānta*, Chapter I, Verses 15 and 52.

[3] The digits are stated from right to left, one or two at a time.

or as (d) *nanda-jina-vikṛti-śaśī* (1,23,24,9)
9 24 23 1
or as (e) *nanda-jina-guṇa-māsa* (12,3,24,9)
9 24 3 12
or as (f) *nanda-veda-danta-māsa* (12,32,4,9)
9 4 32 12

(iv) With three word-numerals
as *tāna-danta-māsa* (12,32,49)
49 32 12

As it is the previous knowledge of the modern notation which has made it possible in the West to state the number "one thousand nine hundred and thirty-two" as "one nine three two" or as "nineteen thirty-two," so it is the previous existence in India of the modern place-value arithmetical notation, which has made it possible for the Indian authors from the sixth century onwards to express the same number differently with suitable word-numerals as shown above.

In the February (1929) issue of the *Isis* I have shown that Vyāsa's commentary on Patañjali's *Yogasūtra* and Śaṅkara's commentary on *Brahma Sūtra* also point to the use of the modern place-value notation in India at least as early as the sixth century, A.D. For the convenience of my readers the following extracts are taken from the journal (page 135):

The following passage occurs in the *Vyāsa Bhāṣya* (*Bibhūtipāda*, 13th *sūtra*) of the *Yogasūtra* of Patañjali:

"Yathaikā rekhā śatasthāne śatam daśathāne daśa ekañcaikasthāne"[1] etc.

It may translated thus:

"Just as the same stroke (or figure) represents a hundred, a ten, or a unit according as it is in the hundred's place, the ten's place, or the unit's place," etc.

Professor J. H. Wood in this connection observes[2] that contrary to G. R. Kaye's opinion "the place system of decimals was known as early as the sixth century A.D."

Śaṅkara's (c. 750 commentary on *Brahmasūtra* contains the following passage:

"Yathā caikāpi satí rekhā sthānānyatvena niveśyamānaikadaśaśatasahasrādipratyayabhedamanubhavati" etc. (*Śārīrikabhāṣya*, 2.2.17).

It may be translated thus:

"Just as the same stroke (or figure) conveys different ideas such as a unit, a ten, a hundred, a thousand, etc., according to the place in which it is set down," etc.

"The use of such passages for the purpose of elucidation of abstruse philosophical doctrines shows that the place-value decimal notation was extremely popular at the time when the above commentaries were written."

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[1] My attention was kindly drawn to this passage and the next one by B. B. Datta.

[2] *English Translation of the Yoga System* of Patañjali (HOS), p. 216, foot-note.

Taking into consideration the absence of facilities of communication in those early times, it would not be extravagant to suppose that the place-value decimal system of notation was at least two centuries old at the time of Vyāsa.[1]

It will thus be seen that there is no reason to be surprised at Severus Sebokht's (7th century) highly appreciative reference to the Indian method of computation by means of nine signs (meaning thereby the employment of the place-value decimal notation),[2] the presence of the Indian figures in Syria in 662 A.D.[3] and the fact that the Arab writers from the early ninth century on acknowledged the Indian origin of the modern place-value notation.[4]

COMPARISON OF CERTAIN PROBABILITIES

By WILLIAM DOWELL BATEN, University of Michigan

Probabilities of a continuous variable may be considered as areas under a probability curve between certain limits. The entire area under the curve for the interval of definition is unity, which represents the sum of all possible probabilities of the variable under consideration. The probability that the variable x lies on the interval (a, b) is $\int_a^b f(x)dx$, where $f(x)dx$ is the probability that x lies on the interval $(x, x+dx)$.

When the probability function or probability law for the variable x is known, the probability laws for functions of x can be determined.[5] In some cases it is necessary to compare probabilities of various functions of x. These comparisons are obtained by considering two areas under the respective probability laws on an interval of comparison. Generally this interval of comparison is a small interval which includes the abscissa of the greatest value of the law. In error laws the interval which is of most importance is that which includes the probable value, or the mean.

In the following theorems several intervals of comparison will be considered. Frequently the theorems are only true in the interval mentioned.

THEOREM 1. *A necessary and sufficient condition that the probability of bx is greater than the probability of x, is that the positive constant b is less than 1; provided the interval of comparison is $(-c, d)$ and that x extends over the interval $(-k, h)$, where $k>c$ and $h>d$.*

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[1] If the date (400 A.D.) assigned to Vyāsa by S. N. Dās Gupta be correct, the date of invention of the modern notation in India would be pushed back several centuries earlier. But in that case it would be difficult to explain the presence of the enunciation of the notation in the *Āryabhaṭīyam*.

[2] D. E. Smith, *History of Mathematics*, Vol. II (1925), pp. 64 and 65.

[3] Ibid, p. 72. Also Kaye, *Indian Mathematics*, p. 31.

[4] Smith and Karpinski, *The Hindu-Arabic Numerals*, pp. 4 and 10. Also Cajori, *A History of Mathematics*, (1922), pp. 102 and 103.

[5] See Dodd: *The frequency law of a function of one variable*, Bulletin of the American Mathematical Society, Vol. 31 (1925), pp. 27–31.

PROOF: Let $f(x)dx$ be the probability that the variable x lies on the interval $(x, x+dx)$, then the probability that, $y=bx$, lies on the interval $(y, y+dy)$ is[1]

$$F(y)dy = (1/b)f(y/b)dy.$$

If the probability that y lies on the interval $(-c, d)$ is greater than the probability that x lies on this interval, then

$$(1/b)\int_{-c}^{d} f(y/b)dy > \int_{-c}^{d} f(x)dx,$$

or

$$\int_{-c/b}^{d/b} f(z)dz > \int_{-c}^{d} f(x)dx.$$

If this is true the interval $(-c/b, d/b)$ is greater than the interval $(-c, d)$, or

$$(1/b)(d + c) > (d + c),$$

or

$$b < 1.$$

The sufficiency of the condition can be proved by reversing the steps of the foregoing proof.

It is readily seen that the quantity c must be non-negative.

The quantity k may be zero, in which case c is zero. If the maximum of the law is not at the origin, the origin can be changed to the maximum or greatest value of the probability function.

If each variable x_i $(i=1, 2, 3, \cdots, n)$ is subject to the same probability law defined on the interval $(-k, h)$, then the law for the sum of n variables can be found.[2] Let $y=(1/n)\sum_{i=1}^{n} x_i$, then according to the above theorem the probability of the mean for the interval $(-c, d)$ is greater than the probability for the sum, since here $b=1/n$, and is less than unity when n is greater than 1. The probability law for the mean may be zero in part of $(-c, d)$.

THEOREM 2. *Let the probability law for the variable x be $f(x)$, which is defined on the interval $(-c, a)$, where c and a are positive quantities. If the positive quantity b is less than* 1, *there exists an interval of comparison* $(0, d)$ *in which the probability of bx^k is greater than the probability of x^s, provided $k>s>1$.*

PROOF: If the law for x is $f(x)$, then the law for $y=bx^k$ is[3]

$$F(y) = \frac{p \cdot y^{(1-k)/k}}{kb^{1/k}} \cdot f\{(y/b)^{1/k}\},$$

and the law for $z=x^s$ is

$$\Psi(z) = \frac{q \cdot z^{(1-s)/s}}{s} \cdot f\{(z)^{1/s}\},$$

[1] Dodd, loc. cit., Bulletin.

[2] Dodd, *The frequency law of a function of variables with given frequency laws*, Annals of Mathematics, Ser. 2, vol. 27 (1925) pp. 12–20.

[3] Dodd, loc. cit., Bulletin.

where p and q are 2 if k and s are even and 1 if k and s are odd. The probability that bx^k lies on the interval $(0, d)$ is greater than the probability that x^s lies on this interval if

$$\int_0^d F(y)dy > \int_0^d \Psi(z)dz, \text{ or if}$$

$$\frac{p}{kb^{1/k}}\int_0^d y^{(1-k)/k}f\{(y/b)^{1/k}\}dy > \int_0^d (q/s)z^{(1-s)/s}f\{(z)^{1/s}\}dz,$$

or if

$$\int_0^{(d/b)^{1/k}} f(t)dt > \int_0^{d^{1/s}} f(v)dv.$$

Let $s=rk$, where $r<1$. The last inequality is true if

$$(d/b)^{1/k} > (d)^{1/rk},$$

or if

$$\log d < \{r/(r-1)\} \log b.$$

The quantity $r/(r-1)$ is negative and the logarithm of b is negative; hence the logarithm of d is less than some positive number, which determines a limit for the interval of comparison $(0, d)$.

The theorem is true if $s=k$ as can be seen from

$$(d/b)^{1/k} > (d)^{1/k}.$$

The theorem is also true if $s=1$.

THEOREM 3. *If $\sum_{i=1}^n b_i<\sum_{i=1}^m k_i$, then the probability of $\sum_{i=1}^n b_ix_i$ is greater than the probability of $\sum_{i=1}^m k_ix_i$, provided the probability law for each variable is*

$$f(x_i) = \frac{h}{\pi(h^2 + x_i^2)},$$

and the interval of comparison is $(-c, d)$.

PROOF: The probability function for $y=b_ix_i$, is

$$F(y) = \frac{b_ih}{\pi}\cdot\frac{1}{b_i^2h^2 + y^2},$$

the probability function for $\sum_{i=1}^m k_ix_i=u$, is

$$\Psi(u) = \frac{\sum_{i=1}^m k_ih}{\pi}\cdot\frac{1}{h^2\left(\sum_{i=1}^m k_i\right)^2 + u^2},$$

the probability function for $w=\sum_{i=1}^{n} b_i x_i$, is

$$\theta(w) = \frac{\sum_{i=1}^{n} b_i x_i}{\pi\left(h^2\left(\sum_{i=1}^{n} b_i\right)^2 + w^2\right)}.$$

The probability that the sum $\sum_{i=1}^{n} b_i x_i$ lies in the interval $(-c,d)$ is greater than the probability that the sum $\sum_{i=1}^{m} k_i x_i$ lies in this interval if

$$\frac{h\sum_{i=1}^{n} b_i}{\pi}\int_{-c}^{d}\frac{du}{h^2\left(\sum_{i=1}^{n} b_i\right)^2 + u^2} > \frac{h\sum_{i=1}^{m} k_i}{\pi}\int_{-c}^{d}\frac{dw}{h^2\left(\sum_{i=1}^{m} k_i\right)^2 + w^2},$$

and this is true since $\sum_{i=1}^{n} b_i < \sum_{i=1}^{m} k_i$.

Nothing was stated about each b_i and each k_i being less than 1. In fact the theorem is true if some are greater than 1 and some less than 1, as long as the sum of the b's is less than the sum of the k's. If $k_i=1$ then the probability of $\sum_{i=1}^{n} b_i x_i$ is greater, for the interval $(-c, d)$, than the probability of $\sum_{i=1}^{n} x_i$, if $\sum_{i=1}^{n} b_i < n$.

THEOREM 4. *If the probability function for the variable* x_i, $(i=1, 2, \cdots, n)$, *is the Gaussian function, then the probability that*

$$-c \leqq \sum_{i=1}^{n} b_i x_i \leqq d,$$

is greater than the probability that

$$-c \leqq \sum_{i=1}^{m} k_i x_i \leqq d,$$

provided

$$\sum_{i=1}^{n} b_i^2 < \sum_{i=1}^{m} k_i^2.$$

PROOF: If the law for x_i is $h/\sqrt{\pi}\cdot e^{-h^2 x_i^2}$, then according to a method given by Dodd[1] the laws for the sums $\sum_{i=1}^{n} b_i x_i = u$ and $\sum_{i=1}^{m} k_i x_i = w$ are respectively

$$F(u) = \frac{h}{(\pi)^{1/2}\left(\sum_{i=1}^{n} b_i^2\right)^{1/2}} e^{-h^2u^2/\sum_{i=1}^{n} b_i^2},$$

$$\Psi(w) = \frac{h}{(\pi)^{1/2}\left(\sum_{i=1}^{m} k_i^2\right)^{1/2}} e^{-h^2w^2/\sum_{i=1}^{m} k_i^2}.$$

[1] Dodd, loc. cit., Annals.

The theorem is true if

$$\int_{-c}^{d} F(u)du > \int_{-c}^{d} \Psi(w)dw,$$

which is true if

$$\sum_{i=1}^{n} b_i^2 < \sum_{i=1}^{m} k_i^2 .$$

This shows that the probability of the sum $\sum_{i=1}^{m} x_i$ is greater than the probability of the sum $\sum_{i=1}^{n} x_i$, on the interval $(-c, d)$, if $m < n$.

In a weighted mean the sum $\sum b_i = 1$, hence the arithmetic mean has the greater probability for the interval mentioned. If however the $\sum_{i=1}^{n} b_i^2$ is less than $1/n$, then this kind of weighted mean has greater probability than the probability of the mean.

If the law for the individual variable x_i is an even function the probability for the sum $\sum_{i=1}^{n} b_i x_i$ can be compared with the probability for the sum $\sum_{i=1}^{m} k_i x_i$ by examining two integrals of the types[1]

$$p_{nb}(u) = (1/\pi) \int_0^{\infty} f(b_1 t) f(b_2 t) \cdots f(b_n t) dt,$$

$$p_{mk}(u) = (1/\pi) \int_0^{\infty} f(k_1 t) f(k_2 t) \cdots f(k_m t) dt.$$

The nature of the functions under the integral signs will have to be examined in each particular case before it can be stated that the probability of the first is greater than the probability of the second for some interval of comparison.

THEOREM 5. *Under the Gaussian Law of error the probable value of any given power of the absolute error of one weighted mean with weights* $p_1, p_2, \cdots, p_n$, *is less than that of a second weighted mean with weights* $q_1, q_2, \cdots, q_m$, *if* $\sum_{i=1}^{n} p_i^2 < \sum_{j=1}^{m} q_j^2$.

PROOF: If the law for the individual variables is $he^{-h^2x^2}/\sqrt{\pi}$, then the law for the first weighted mean is

$$P_n(u) = \frac{P}{(\pi)^{1/2}} e^{-P^2u^2}, \text{ where } P^2 = h^2 / \sum_{i=1}^{n} p_i^2 ;$$

and the law for the second weighted mean is

$$Q_m(w) = \frac{Q}{(\pi)^{1/2}} e^{-Q^2w^2}, \text{ where } Q^2 = h^2 / \sum_{i=1}^{m} q_i^2 .$$

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[1] Dodd, loc. cit., Annals.

The probable value of the absolute error of the first raised to the power k is less than the absolute error of the second raised to this same power if

$$\int_{-\infty}^{\infty} |u|^k P_n(u)du < \int_{-\infty}^{\infty} |w|^k Q_m(w)dw,$$

or if

$$1/P^2 < 1/Q^2, \text{ or if } \sum_{i=1}^{n} p_i^2 < \sum_{j=1}^{m} q_j^2 .$$

This shows that under this error law the probable value of any given positive power of the absolute error of the arithmetic mean is less than that of any other weighted mean. In this case $m=n$ and the weights are each equal to $1/n$.

If $p_i=1/m$ and $q_j=1/n$, where $i=1, 2, \cdots, m$ and $j=1, 2, \cdots, n$, then the probable value of any given positive power of the absolute error of the first weighted mean, (here the real mean), is less than that of the second mean if $m<n$. This shows that the mean with a greater number of variates is the better. This is as it should be, or what would be expected. But the p's and q's can be such that even if $m>n$ the probable value of any given positive power of the absolute error of the first weighted mean is greater than that of the second weighted mean. The weighted mean with the most variates is not as good as that of the second with a smaller number of variates.

FOURIER SERIES IN THREE DIMENSIONS

By W. O. PENNELL, Bell Telephone Company, St. Louis

Introduction. The classical Fourier Series represents a function in a given interval and then repeats the same values in the next and subsequent intervals. In other words if $S(x)$ is the classical Fourier Series representing $f(x)$ in the interval $0<x<a$ then $S(x)=f(x-na)$ where n takes on the values $n=0, \pm 1, \pm 2, \cdots$ corresponding to the various intervals $na<x<(n+1)a$.

The author has[1] shown how a generalized Fourier Series $S_1(x)$ may be obtained representing a function in the intervals $na<x<(n+1)a$ as follows:

$$S_1(x) = b^n f(x - na)$$

where n takes on the values $n=0, \pm 1, \pm 2, \cdots$, and b is any real constant.

In this paper will be described a still more general Fourier Series $S_{11}(x)$ representing a function in the intervals $na<x<(n+1)a$ as follows:

$$S_{11}(x) = [b^n f(x - na)]_{n\psi}$$

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[1] A Generalized Fourier Series Representation of a Function, by W. O. Pennell—American Mathematical Monthly, Nov. 1930.

where corresponding to the above intervals n takes on the values $n=0, \pm 1, \pm 2, \cdots$, b is any real constant, and the subponent notation $n\psi$ denotes the rotation of the plane of the curve about the X axis through an angle $n\psi$ with the XY plane.

In other words $S_{11}(x)$ represents a curve in planes passing through the X axis as follows:

Interval	Value of $S(x)$	Angle with XY Plane of Plane of Curve
$0<x<a$	$f(x)$	0
$a<x<2a$	$bf(x-a)$	ψ
$2a<x<3a$	$b^2f(x-2a)$	2ψ
$\cdots$	$\cdots$	$\cdots$
$-a<x<0$	$b^{-1}f(x+a)$	$-\psi$
$-2a<x<-a$	$b^{-2}f(x+2a)$	-2ψ
$-3a<x<-2a$	$b^{-3}f(x+3a)$	-3ψ
$\cdots$	$\cdots$	$\cdots$

Deduction of Formula. Let $f(x)$ be a function which satisfies the condition necessary for representation by a classical Fourier Series in the interval $0<x<a$. Then in general $f(x)b^{-x/a}e^{-i\psi x/a}$, where b, a, and ψ are real constants, can be represented by a Fourier Series. Let such a series by denoted by $S(x)$. Then

$$S(x) = f(x - na)b^{-(x-na)/a}e^{-i\psi(x-na)/a} \tag{1}$$

when n takes on the values $n=0, \pm 1, \pm 2, \cdots$, corresponding to the intervals $na<x<(n+1)a$. From (1)

$$S(x)b^{x/a}e^{i\psi x/a} = f(x - na)b^n e^{i\psi n}. \tag{2}$$

Now plot $y=f(x-na)b^n e^{i\psi n}$ as follows:

Since x is limited to real values, it is plotted as usual along the X axis; but y is a vector or complex variable. The Y plane will be taken perpendicular to the X axis and passing through the Y axis. Values of y are plotted in this plane in the usual way, real values along the Y axis and imaginary values at right angles to it. Having chosen and located a value of x, the point corresponding to y is obtained by drawing from the point x, a vector parallel and equal to the y vector as located in the Y plane.

With these conventions in mind, it is readily seen that $e^{in\psi}$ is a rotor turning the plane of the curve about the X axis through an angle $n\psi$.

The rule for obtaining the space Fourier Series representing a function $b^n f(x-na)$ in planes passing through the X axis and making angles $n\psi$ with the XY plane where n has the values $0, \pm 1, \pm 2, \cdots$, corresponding to the intervals $na<x<(n+1)a$ can then be briefly stated as follows:

"Find the classical Fourier Series $S(x)$ for $f(x)b^{-x/a}e^{-i\psi x/a}$. Multiply the series so found by $b^{x/a}e^{i\psi x/a}$. The result $S(x)b^{x/a}e^{i\psi x/a}$ will be the space Fourier Series desired."

Some Characteristics of the Series. Since the space series is a Fourier Series multiplied by $b^{x/a}e^{i\psi x/a}$ it follows that it has the same kind of convergence as the Fourier Series and is subject to the same rules regarding term by term differentiation and integration as the classical series.

In the classical Fourier Series for $f(x)$, the value of $f(x)$ at a point of finite discontinuity, for example, $x=a$ is given by $\{f(a+0)+f(a-0)\}/2$. From its derivation, it is obvious that the space series follows the same rule except in this case $f(a+0)$ and $f(a-0)$ are vectors and the sum must be interpreted as the vector sum.

If the angle ψ is commensurable with 2π the curve, after a certain number of intervals periodically returns to a plane in which it was formerly located. If ψ and 2π are incommensurable the curve will never occupy the same plane a second time.

An Illustration. Find a Fourier Series representing a function in planes passing through the X axis as follows:

Interval	Value of Function	Angle of Plane of Curve With XY Plane
$0<x<a$	1	0
$a<x<2a$	b	ψ
$2a<x<3a$	b^2	2ψ
. . . .	. . . .	. . . .
$-a<x<0$	b^{-1}	$-\psi$
$-2a<x<-a$	b^{-2}	-2ψ
$-3a<x<-2a$	b^{-3}	-3ψ
. . . .	. . . .	. . . .

Going through the process described for the deduction of the series the result is:

$$(3)\ y = \frac{b^{x/a-1}e^{i\psi(x/a-1)}(be^{i\psi}-1)}{\log_e b + i\psi}$$
$$+\frac{2(be^{i\psi}-1)}{be^{i\psi}}\sum_{n=1}^{\infty}\frac{b^{x/a}e^{i\psi x/a}[\log_e b+i\psi]\cos\dfrac{2n\pi x}{a}+2n\pi b^{x/a}e^{i\psi x/a}\sin\dfrac{2n\pi x}{a}}{4n^2\pi^2+(\log_e b+i\psi)^2}.$$

If $b=1$ and $\psi=0$, (3) reduces to unity, which is the value of the function under these conditions.

At the point of discontinuity $x=a$, the value of $f(a+0)=be^{i\psi}$ and $f(a-0)=1$. So $\{f(a+0)+f(a-0)\}/2$ is $(be^{i\psi}+1)/2$. By substituting in (3) $x=a$ this is the value obtained.

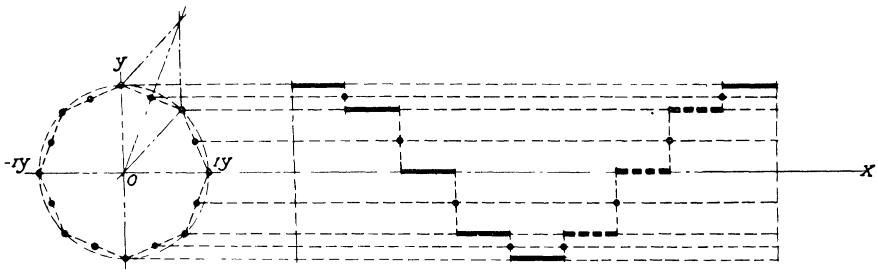


Fig. 1
PROJECTION

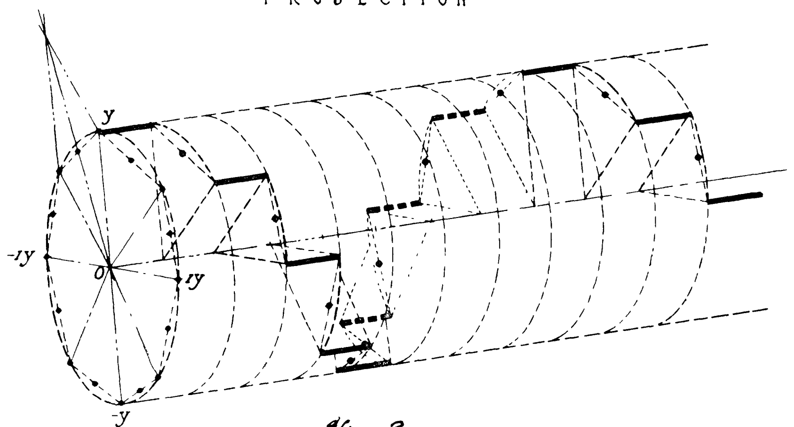


Fig. 2
PERSPECTIVE

In Fig. 1 is shown a projection upon the XY and upon the Y planes of the curve represented by (3) for $b=1$ and $\psi=\pi/4$. In Fig. 2 is shown an isometric view of the same.

The values at the points of discontinuity are shown by dots and it will be seen that they equal one half the vector sum of the values just before and just after the discontinuity.

Other examples can be obtained from the Author's paper[1] on "A Generalized Fourier Series" by substituting in the examples there solved, for b the vector $be^{i\psi}$.

Remarks on the Approximation Curves. It is interesting to note that in general the approximation curves, even for a single interval, are not confined to one plane, but are curves in space twisting about the ultimate curve they are to represent and coming nearer and nearer to it as the number of the approximation curves increases.

[1] American Mathematical Monthly, November, 1930.

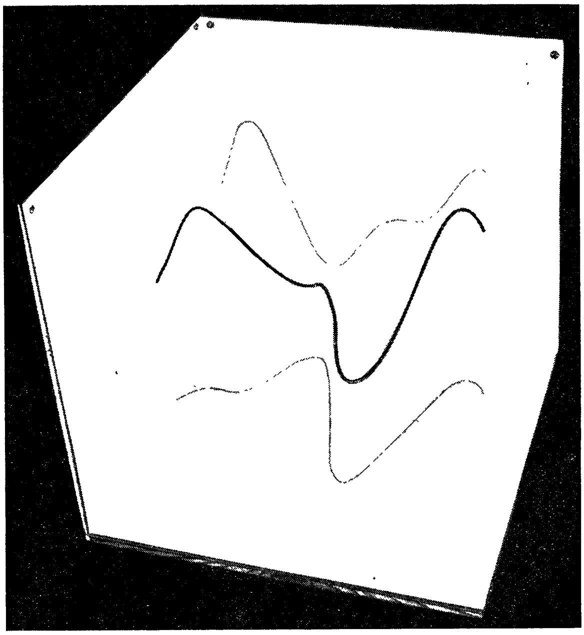


FIG. 3

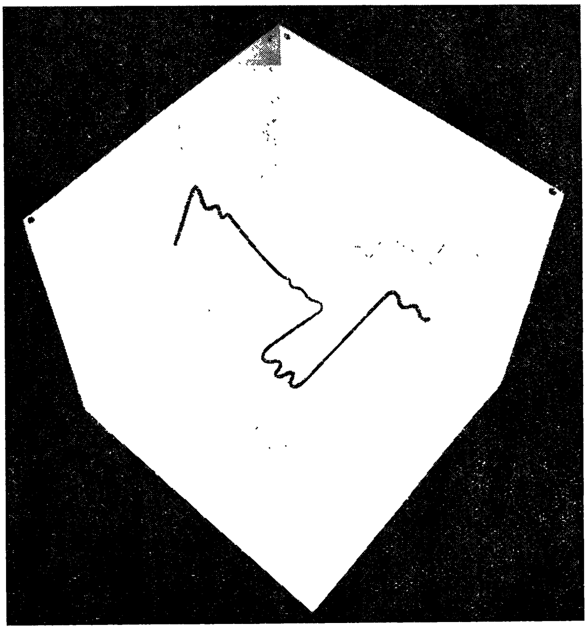


FIG. 4

Figures 3 and 4 respectively show the first and third approximation curve for the example illustrated in equation 3 when $b=1$ and the angle $\psi=5\pi/4$.

Various values of x and y were calculated from the formula and the corresponding projections on the XY and the $X\ i\ Y$ planes plotted. From these projections an actual model was constructed, the curve being represented by a wire bent into the proper shape. The model was then photographed together with the projections on the two planes mentioned above.

Acknowledgment. The author wishes to acknowledge his indebtness to Miss Elizabeth Harris who has calculated and plotted the approximation curves for a number of examples.

The author is also indebted to Mr. John Casey and Mr. W. F. Uphouse who constructed the models of the approximation curves and supervised the taking of the photographs.

VANISHING AGGREGATES OF DETERMINANTS AND THEIR RELATIONSHIPS

By R. A. BEAVER, New York State College for Teachers

In the years since Bézout's important work of 1779 a large number of papers have been published on the subject of vanishing aggregates of determinants. Many of the results given seem at first sight quite unrelated, but on closer observation prove to be intimately connected. The principal object of this paper is to point out which aggregates are fundamental and to show how the others are related to them.

§1

Deruyts (1882) and Muir (1888) proved the following theorem:[1]

THEOREM A: *If any two determinants A and B of the nth order be taken, and from these two sets of determinants be formed, namely, first, a set of* $_nC_r$ *determinants each of which is in r rows identical with A and in the remaining rows with B; and, second, a set of the same number of determinants each of which is in r columns identical with A and in the remaining columns with B, then the sum of the first set is equal to the sum of the second set.*

Denote this theorem by

$$R\begin{pmatrix} A & B \\ a & b \end{pmatrix} = C\begin{pmatrix} A & B \\ a & b \end{pmatrix}$$

where $a+b=n$.

In 1917 Metzler generalized it, giving the theorem[2]

[1] Muir and Metzler, Theory of Determinants, (1929), §319. This work will hereafter be referred to as Text.

[2] Text, §320. Vanishing Aggregates, *Proceed. R. Soc. Edinburgh*, xxxvii, pp. 324–326.

By the use of *Theorems I, II, III* we may deduce not only known theorems but also new ones worthy of note. Using the notation of the last paragraph of §1 with respect to the array

$$\begin{matrix} a_1 & b_1 & c_1 & r_1 & s_1 & t_1 & x_1 & y_1 & z_1 \\ a_2 & b_2 & c_2 & r_2 & s_2 & t_2 & x_2 & y_2 & z_2 \\ a_3 & b_3 & c_3 & r_3 & s_3 & t_3 & x_3 & y_3 & z_3 \end{matrix}$$

we find that upon applying *Theorem I* to the three determinants

$$\begin{vmatrix} (12) & [4] & (789) \\ [2] & [4] & [3] \\ [2] & [4] & [3] \end{vmatrix}, \quad \begin{vmatrix} [2] & [4] & [3] \\ [2] & (3456) & [3] \\ [2] & [4] & [3] \end{vmatrix}, \quad \begin{vmatrix} [3] & [6] \\ [3] & [6] \\ [3] & (456789) \end{vmatrix},$$

there results the relation

$$\begin{aligned} |a_1b_2c_3| \cdot |r_1s_2t_3| \cdot |x_1y_2z_3| &= |a_1b_2x_3| \cdot |r_1s_2c_3| \cdot |t_1y_2z_3| \\ + |a_1b_2x_3| \cdot |r_1c_2t_3| \cdot |s_1y_2z_3| &+ |a_1b_2x_3| \cdot |c_1s_2t_3| \cdot |r_1y_2z_3| \\ + |a_1b_2y_3| \cdot |r_1s_2c_3| \cdot |x_1t_2z_3| &+ |a_1b_2y_3| \cdot |r_1c_2t_3| \cdot |x_1s_2z_3| \\ + |a_1b_2y_3| \cdot |c_1s_2t_3| \cdot |x_1r_2z_3| &+ |a_1b_2z_3| \cdot |r_1s_2c_3| \cdot |x_1y_2t_3| \\ + |a_1b_2z_3| \cdot |r_1c_2t_3| \cdot |x_1y_2s_3| &+ |a_1b_2z_3| \cdot |c_1s_2t_3| \cdot |x_1y_2r_3|, \end{aligned}$$

which expresses the product of three determinants of order three in terms of like products.

To obtain a similar expression for the product of four determinants of order three *Theorem I* would be applied to four determinants of order twelve, nine of the rows of each of which would contain only zero elements.

The general theorem to which we are led is a generalization of Sylvester's theorem of 1851 referred to above and may be enunciated thus:

The product of p determinants of order n is equal to the sum of like products obtained from the original as follows: Replace a definite set of k columns of the first determinant by any set of k columns of the pth. Replace any set of k columns of the second by those columns displaced from the first, and so on, finally replacing the columns taken from the pth determinant by those displaced from the $(p-1)$th. The replacement of k columns by k columns is effected by replacing the first column of the one set with the first column of the other, the second of the one with the second of the other, and so on.

FLOW OF HEAT IN A DISC HEATED BY A GAS STREAM

By R. L. PEEK, JR., Bell Telephone Laboratories

A problem in the flow of heat, for which the solution does not appear to be recorded elsewhere, arises when a gas passes through a cylindrical vessel which is broken up into a number of chambers by flat floors or partitions perpendicular

to the axis of the vessel. This is the case in various processes in which a solid material passes from hearth to hearth in a vertical cylindrical vessel of this type, while a stream of gas flows in the opposite direction. The heat problem in question relates to the amount of heat which is taken up from the gas and flows outward through the circular partition or floor, assuming that the periphery of the latter is integral with the outside of the cylindrical vessel and is cooled so as to be at a constant temperature.

The problem is not difficult, and the solution is readily obtained by the methods discussed in courses dealing with the partial differential equations of mathematical physics. It is believed, however, that it may be of interest to teachers of such courses since it involves the expansion of a function in an orthogonal trigonometric series whose terms are not harmonic. Examples of this sort having genuine physical significance and yet simple enough to be used as problem material are rather rare.

It is assumed that the heat transfer from the gas to the plate is proportional to the temperature difference between them, i.e. that:

$$K\frac{\partial\theta}{\partial x} = \pm\, k(\theta_1 - \theta) \text{ when } (x = \pm\, a), \tag{1}$$

where a is the half thickness of the plate, K is the conductivity of the plate, θ its (variable) temperature, θ_1 the (constant) temperature of the gas, and k the coefficient of heat transfer. Either side of equation (1) represents the heat entering a unit area of either surface of the plate. The problem is then to solve the heat equation for a steady state:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{dr^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} = 0 \tag{2}$$

subject to the boundary conditions given by equation (1) and the condition that the edge of the plate be at the constant temperature θ_0, i.e.:

$$\theta = \theta_0 \text{ when } r = R. \tag{3}$$

A solution to equation (2) may be written in the form:

$$\theta = \theta_1 + \sum A_m J_0\left(\frac{im\pi}{a}\, r\right)\cos\left(\frac{m\pi}{a}\, x\right). \tag{4}$$

By substitution of this expression in equation (1) it is seen that this condition will be satisfied if the values of m appearing in the different terms of the series are given by $m\pi/a = \lambda_m$ where λ_m is a root of the equation:

$$\tan(a\lambda) = p/\lambda, \tag{5}$$

where $p = k/K$. As there are an infinite number of such roots, there may be an infinite number of terms in the series. The remaining condition, equation (3),

may be satisfied by choosing the value of A_m for each term so that the series is (for $r=R$) an expansion of $\theta_0-\theta_1$ from $x=0$ to $x=a$, i.e.:

$$\theta_0 - \theta_1 = \sum A_m J_0(i\lambda_m R) \cos(\lambda_m x). \tag{6}$$

An expansion of this sort, where the values of λ_m are roots of equation (5), may be readily accomplished, as the series is orthogonal.[1] When the function to be expanded is a constant, as in this case, the coefficients of the cosine terms are readily evaluated and yield:

$$A_m J_0(i\lambda_m R) = \frac{2(p^2+\lambda_m^2)}{a\lambda_m^2 + ap^2 + p}\,\frac{\sin a\lambda_m}{\lambda_m}(\theta_0 - \theta_1). \tag{7}$$

The total quantity of heat flowing out through the edge of the disc per unit time (Q/t) is given by the quantity $-K(\partial\theta/\partial r)\partial x dr$ integrated over this surface, or

$$\begin{aligned} Q/t &= 4\pi RK(\theta_0-\theta_1)\int_0^a \sum A_m iJ_1(i\lambda_m R)\cos(\lambda_m x)dx \\ &= -8\pi RK(\theta_1-\theta_0)\sum \frac{p^2}{\lambda_m(a\lambda_m^2+ap^2+p)}\,\frac{iJ_1(i\lambda_m R)}{J_0(i\lambda_m R)} \\ &= 8\pi RKS(\theta_1-\theta_0) \end{aligned} \tag{8}$$

where S is written for the absolute value of the summation appearing in the expression, which has always a negative value.

To evaluate the series S, it is first necessary to determine the value of λ_m for each term included in the computation: in general two or three terms suffice to evaluate S to four significant figures or better. Values of k and K are given in various tables. The former is an empirical constant dependent on the gas velocity and the state of the gas, for which values are given in various engineering texts and in the International Critical Tables. (This work, among others, gives values of K.) With k, K, and a known, values of λ_m are obtained by solution of equation (5).

For engineering purposes, equation (5) would probably be solved graphically, as by determining the intersections of $y=ap/x$ with $y=\tan x$; which would give values of $a\lambda_m$. For greater accuracy the roots could be obtained from the series expansion:

$$a\lambda_{n+1} = n\pi\left[1+\frac{ap}{(n\pi)^2}-\frac{(ap)^2}{(n\pi)^4}-\left\{\frac{1}{3(n\pi)^4}-\frac{2}{(n\pi)^6}\right\}(ap)^3+\cdots\right], \tag{9}$$

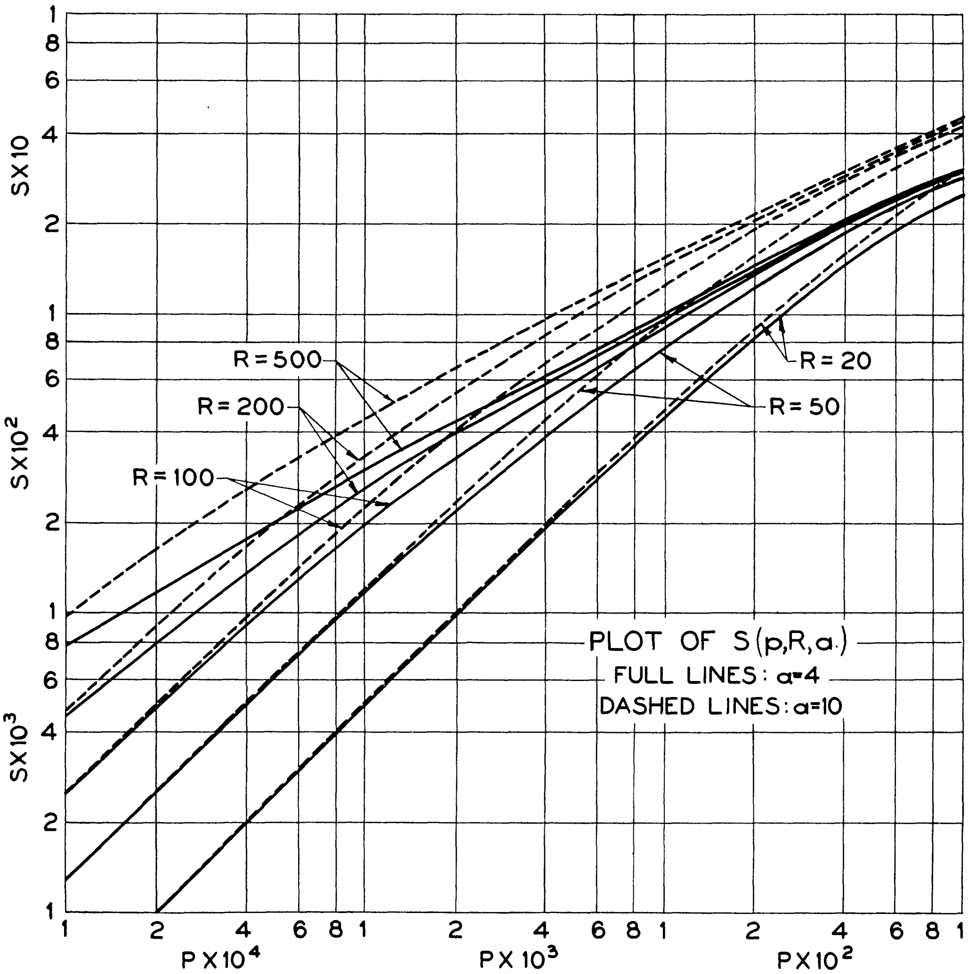
which applies to all roots except the first $(n=0)$.

Equation (9) is obtained by writing equation (5) as $ap=(n\pi+q_n)\tan q_n$, expanding the right-hand side in a power series, and subjecting the series to inversion. The same process can be used to obtain a series for the first root $(a\lambda_1)$, except that in this case the inversion of the series will lead to fractional powers of ap.[2]

[1] Carslaw, Mathematical Theory of Heat Conduction, 2nd Ed., 1921, p. 132.

[2] The writer is indebted to Dr. T. C. Fry of these Laboratories for the development of equation (9).

For engineering purposes a solution to a problem of this sort should, when possible, be reduced to a form less cumbrous than that of equation (8). In this case, curves were prepared showing the quantity S of equation (8) as a function of p, R and a. For selected pairs of values of R and a, curves of S vs. p were plotted on logarithmic paper. The curves were grouped in families for which a



had a constant value but in which R varied from curve to curve. Two such families are shown in Fig. 1, that for which $a = 4$ cms. being shown in full lines, while that for which $a = 10$ cms. is shown in dashed lines. It was found that with two additional groups ($a = 1.0$ cm. and $a = 2.0$ cms.) values of S for all values of a and R in the range indicated could be obtained by interpolation with sufficient accuracy.

In the actual problem each plate had a central orifice for the passage of gas

from one chamber to the other. As the temperature gradient near the center will be low the error introduced by ignoring the effect of such an orifice is small. In practice also the gas velocity and hence the value of k will not be uniform over the surface of the plate. Even if such an effect could be considered in developing the solution, the data required for its evaluation would not be likely to be available, and it is worthy of consideration only in estimating the accuracy of the result.

THE PRODUCT OF A CIRCULANT MATRIX AND A SPECIAL DIAGONAL MATRIX

By J. WILLIAMSON, Johns Hopkins University

Introduction. In the following paper we discuss an n-rowed square matrix

$$B = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots a_{n-1} \\ \omega a_{n-1} & \omega a_0 & \omega a_1 & \cdots \omega a_{n-2} \\ \omega^2 a_{n-2} & \omega^2 a_{n-1} & \omega^2 a_0 & \cdots \omega^2 a_{n-3} \\ \cdot & \cdot & \cdot & \cdot \\ \omega^{n-1} a_1 & \omega^{n-1} a_2 & \omega^{n-1} a_3 & \cdots \omega^{n-1} a_0 \end{pmatrix},$$

where ω is a primitive nth root of unity.[1] This matrix B is the product ΩA of the diagonal matrix

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & \cdots 0 \\ 0 & \omega & 0 & \cdots 0 \\ 0 & 0 & \omega^2 & \cdots 0 \\ \cdot & \cdot & \cdot & \cdots \\ 0 & 0 & 0 & \cdots \omega^{n-1} \end{pmatrix}$$

and the circulant matrix[2]

$$A = \begin{pmatrix} a_0 & a_1 \cdots a_{n-1} \\ a_{n-1} & a_0 \cdots a_{n-2} \\ \cdot & \cdot \\ a_1 & a_2 \cdots a_0 \end{pmatrix}$$

---

[1] That is a root of the equation $x^n - 1 = 0$ but is not a root of any equation $x^m - 1 = 0$, where $m < n$.

[2] The determinant of a circulant matrix is known. See T. Muir, *The Theory of Determinants*, page 189.

We prove that the nth power of the matrix B is the product of the determinant of A by the unit matrix. The proof is interesting and is peculiar, in as much as it is easier, when n the order of the matrix is a prime. For this reason the proof is given first, when n is a prime, and is later modified to suit the case when n is composite.

§1. *The order n a prime.* The elements in any one row or column of the matrix A are permutations of the n elements $a_0, a_1, \cdots, a_{n-1}$, and, if for brevity we write $a_i = i$, then it is apparent that the element in the ith row and jth column of A is $j-i$ or $n+j-i$ according as $i \leqq j$ or $i > j$, and may therefore be denoted by $j-i$, where $j-i = a_k$ if $j-i \equiv k \bmod n$, $0 \leqq k < n$. We now prove the following lemmas.

LEMMA 1. *The matrices of the m-rowed principal minors of A formed from the i_1th, i_2th, $\cdots$, i_mth rows and columns and formed from the j_1th, j_2th, $\cdots$, j_mth rows and columns are equal, if and only if*

$$i_r - i_s \equiv j_r - j_s \bmod n, \quad r, s = 1, 2, \cdots, m. \tag{1}$$

For the elements in the sth row and the rth column of the two matrices are respectively $i_r - i_s$ and $j_r - j_s$. But (1) is true for all values of r and s, if

$$i_r - i_{r-1} \equiv j_r - j_{r-1} \bmod n, \quad r = 2, 3, \cdots, m. \tag{2}$$

Let $j+k_0$, $j+k_1$, $\cdots$, $j+k_{m-1}$ be m integers, $m<n$, where $k_1 \not\equiv k_r \bmod n$ if $i \neq r$; then the residues of these m integers modulo n are distinct and form a combination of the integers $0, 1, 2, \cdots, n-1$, which we shall denote by $M(k,j)$. By giving j the values $0, 1, 2, \cdots, n-1$ in turn we have n such combinations $M(k,0)$, $M(k,1)$, $\cdots$, $M(k,n-1)$, which form a set denoted by M_k.

LEMMA 2. *If m is relatively prime to n, the combinations of the set M_k are distinct.*

Let the members of the combination $M(k,r)$ coincide in some order with the members of the combination $M(k,s)$. Then certainly

$$mr + k_0 + k_1 + \cdots + k_{m-1} \equiv ms + k_0 + k_1 + \cdots + k_{m-1} \bmod n.$$

Hence

$$mr \equiv ms \bmod n,$$

or, since m and n are relatively prime,

$$r \equiv s \bmod n,$$

and, since both r and s are less then n,

$$r = s.$$

Accordingly this proves the lemma.

LEMMA 3. *If two sets M_k and M_t have one combination in common, then the combinations of M_k are the same as those of M_t except for a re-arrangement in order.*

Let $M(k,j)$ coincide with $M(t,r)$. Then for every value of $i=0, 1, 2, \cdots, m-1$, there must be an integer p $(0 \leqq p < m)$, such that $j+k_1 \equiv r+t_p$ mod n. But the combinations of the set M_k are $M(k,j+u)$, $u=0, 1, \cdots, n-1$, where $M(k,j+u)$ is identical with $M(k,j+u-n)$ if $j+u \geqq n$. Hence every integer in the combination $M(k,j+u)$ is congruent to an integer of the form $j+k_i+u$ and is therefore congruent to $r+t_p+u$ modulo n. Accordingly the combination $M(k,j+u)$ coincides with the combination $M(t,r+u)$ and so the sets M_k and M_t coincide.

If now we consider two sets to be the same, if the combinations of the one are identical with the combinations of the other, and notice that every combination of m integers chosen from the integers $0, 1, \cdots, n-1$, must occur in at least one set, we have the following result.

LEMMA 4. *If m is an integer less than n and if n and m are relatively prime, the combinations of the integers $0, 1, \cdots, n-1$, taken m at a time, can be arranged into distinct sets, where each set contains n distinct combinations.*

Corollary. Since the number of combinations of n things m at a time is $\binom{n}{m}$, the number of distinct sets is $\binom{n}{m}/n$.

LEMMA 5. *Let $t_0, t_1, \cdots, t_{n-1}$, be the residues modulo n of the sums of the integers in the combinations $M(k, 0)$, $M(k, 1), \cdots, M(k, n-1)$ respectively. Then $t_0, t_1, \cdots, t_{n-1}$ form a complete residue system modulo n, if m is relatively prime to n.*

For,

$$t_1 \equiv i + k_0 + i + k_1 + \cdots + i + k_{m-1} \bmod n,$$
$$\equiv mi + g, \text{ where } g = k_0 + k_1 + \cdots + k_{m-1}.$$

But if m is relatively prime to n, the integers $mi+g$, $i=0, 1, \cdots, n-1$, form a complete residue system modulo n.[1]

We are now in a position to prove the following theorem for the case in which n is a prime number.

THEOREM 1. *The nth power of the matrix B is equal to the product of the determinant of A by the unit matrix*; or

$$B^n = (-1)^{n-1} |B| E = |A| E,$$

where $|A|$ denotes the determinant of A and E is the unit matrix.

The characteristic equation of B is $|B-\lambda E|=0$, or $\lambda^n - b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + (-1)^n b_n = 0$, where b_m is the sum of the m-rowed principal minors of B. If we let the m-rowed principal minor formed from the $(j+k_i)$-th rows and

[1] L. E. Dickson, *Introduction to the Theory of Numbers*, Theorem 9, p. 8.

columns of B, $(i=0, 1, \cdots, m-1)$ correspond to the combination $M(k, j)$, by lemma 4 the $\binom{n}{m}$ m-rowed principal minors of B may be divided into $\binom{n}{m}/n$ sets, each set consisting of n minors. But by lemma 1 the minors corresponding to the set M_k differ from each other only by a power of ω. In fact, if Δ_k denote the value of the m-rowed principal minor of A corresponding to any combination of the set M_k, the value of the m-rowed principal minor of B corresponding to the combination $M(k, j)$ is, with the notation of lemma 5, $\omega^{t_j}\Delta_k$. Therefore the sum of the n principal minors determined by the set of combinations M_k is

$$\begin{aligned}&\Delta_k(\omega^{t_0} + \omega^{t_1} + \cdots + \omega^{t_{n-1}})\\ &= \Delta_k(1 + \omega + \omega^2 + \cdots + \omega^{n-1}) \text{ by lemma 5,}\\ &= 0.\end{aligned}$$

Hence the coefficients b_m of the characteristic equation of B vanish for all values of $m=1, 2, \cdots, n-1$. Accordingly the characteristic equation of B is

$$\lambda^n + (-1)^n|B| = 0.$$

But every square matrix satisfies its own characteristic equation[1] and therefore

$$B^n = (-1)^{n-1}|B|E, \text{ where } E \text{ is the unit matrix.}$$

But, as we shall see later, $|B|=(-1)^{n-1}|A|$, and so the theorem is proved, when n is a prime.

§2. *The order n composite.* If n is not a prime number, for some value of m less than n, m and n will not be relatively prime and accordingly lemma 2 will no longer be true and the above proof must be modified.

If m and n are not relatively prime, let p be the greatest common divisor of m and n so that

$$m = pq,\ n = pt,\ q \text{ and } t \text{ are relatively prime.} \tag{3}$$

Since the truth of lemma 3 does not depend on the fact that m and n are relatively prime, we can still divide the $\binom{n}{m}$ combinations of n integers m at a time into sets, where each combination occurs in one and only one set, but, since lemma 2 is no longer true, the same combination may be repeated in its set. We can however prove the subsidiary lemma.

LEMMA 6. *In any set M_k the t combinations*

$$M(k, rt),\ M(k, rt+1),\ \cdots,\ M(k, rt+t-1) \textit{ are distinct.}$$

For, if the combination $M(k, rt+f)$ coincides with the combination $M(k, rt+s)$, then as in lemma 2, $mf \equiv ms \bmod n$ and by (3) $qf \equiv qs \bmod t$. But since q and t are relatively prime and f and s are both less than t, this last result is only true if $f=s$.

Thus the set M_k can be divided into p subsets consisting of t combinations each, the combinations in each subset being distinct. Further, if one combina-

[1] L. E. Dickson, *Modern Algebraic Theories*, page 48.

tion appear in two subsets, these subsets coincide. The proof is similar to that of lemma 3. Let the combination $M(k, rt+f)$ coincide with the combination $M(k, st+d)$. Then certainly

$$mrt = mf \equiv mst + md \bmod n,$$
$$mf \equiv md \bmod n \text{ by } (3),$$
$$qf \equiv qd \bmod t,$$

and therefore $f=d$. Hence the combination $M(k, rt+f)$ coincides with the combination $M(k, st+f)$ and as a result the combination $M(k, rt+i)$ coincides with the combination $M(k, st+i)$, for all values of i, $0 \leqq i < t$. Accordingly it is possible to divide the $\binom{n}{m}$ combinations of n integers m at a time into $\binom{n}{m}/t$ distinct sets of t combinations each, where each set consists of combinations of the type, $M(k, rt+i)$, $i=0, 1, \cdots, t-1$.

As before the sum of the principal minors of B corresponding to this subset is

$$\Delta_k \sum_{i=0}^{t-1} \omega^{k_0+k_1+\cdots+k_{m-1}+mrt+mi},$$

$$= \omega^g \Delta_k \sum_{i=0}^{t-1} \omega^{mi}, \; g = k_0 + k_1 + \cdots + k_{m-1} + mrt.$$

We now wish to show that $\sum_{i=0}^{t-1} \omega^{mi} = 0$, where ω is a primitive nth root of unity.

By (3) the set of integers $0, q, 2q, \cdots, (t-1)q$ forms a complete residue system modulo t and is therefore congruent modulo t to the set $0, 1, 2, \cdots, t-1$ in some order. Accordingly the set $0, m, 2m, \cdots, (t-1)m$ is congruent modulo n to the set $0, p, 2p, \cdots, (t-1)p$ in some order. Therefore,

$$\sum_{i=0}^{t-1} \omega^{mi} = \sum_{i=0}^{t-1} \omega^{pi} = \sum_{i=0}^{t-1} (\omega^p)^i = 0,$$

for, if ω is a primitive nth root of unity ω^p satisfies the equation $x^{t-1}+x^{t-2} + \cdots + x + 1 = 0$. Thus the coefficients b_m of λ^m in the characteristic equation of B vanish for $m=1, 2, \cdots, n-1$ also when m is composite, and for all possible values of n

$$B^n = (-1)^{n-1} | B | E.$$

But, if n is odd,

$$| B | = \omega^{n(n-1)/2} | A | = (\omega^n)^{(n-1)/2} | A | = | A | .$$

On the other hand, if n is even,

$$| B | = \omega^{n(n-1)/2} | A | = (\omega^{n/2})^{n-1} | A | = (-1)^{n-1} | A | = - | A | .$$

Accordingly, for all values of n, $B^n = |A| E$ and theorem 1 is proved.

If

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

and $\omega_1, \omega_2, \cdots, \omega_n$ are the nth roots of unity[1]

$$|A| = \prod_{i=1}^{n} f(\omega_i) = k^n.$$

Then $(B/k)^n = E$, and the latent roots[2] of B/k are $\omega_1, \cdots \omega_n$ i.e. are $1, \omega, \cdots, \omega^{n-1}$. But $B = \Omega A$, and so there exists a non-singular matrix X such that

$$X\Omega A/kX^{-1} = \Omega, \text{ or } A/k = \Omega^{-1}X^{-1}\Omega X.$$

This last result is simply an alternative statement of theorem 1.

Moreover the matrices $\omega^i B$, $i=0, 1, \cdots, n-1$, are permutable with each other and the elementary symmetric functions, of degrees $1, 2, \cdots, n-1$ in these n matrices, are all zero and consequently are equal to the corresponding elementary symmetric functions of the latent roots of B. Accordingly we have proved;

THEOREM 2. *The matrices $\omega^i B$ form a complete set of conjugate matrices*[3] *all of which satisfy the characteristic equation of B.*

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

DEFINITIONS OF PROBABILITY[4]

By B. H. CAMP, Wesleyan University

A good definition of probability should have at least the following characteristics: it should be an idealization of one or more of the notions commonly associated with the term probability; it should admit proof of the sum and product theorems; and, taken together with these theorems, it must not lead to an inconsistency. Let us consider first the case in which there are but a finite number of possibilities or elements in the universe sampled. It is well understood that a satisfactory definition can be devised, provided one can define the words "equally probable." Some authors choose to escape this difficulty by asserting that this phrase is not definable, and that what is meant by it is understood intuitively. This attitude is logically defensible, but it is a defeatist attitude,

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[1] T. Muir, loc. cit.

[2] Bôcher, *Introduction to Higher Algebra*, page 283.

[3] Taber, American Journal of Mathematics, Vol. 13, p. 159.

[4] Part of a paper presented at a meeting of the American Statistical Association, December, 1931, under the title, "*Some fundamental concepts in statistics.*" This part has special reference to some remarks on probability in this *Monthly* by W. A. Wilson, vol. 38 (1931), pp. 578, 579.

for one cannot avoid the suspicion that the notion is declared to be intuitive because the definition is difficult.

There are three general problems to which the term probability is sometimes applied, and it is useful to distinguish between them before attempting a definition. So-called a priori probability has to do with the problem of piercing the veil which separates the future from the present, a posteriori probability with the problem of piercing the veil which separates the unknown past from the present, and there is finally the problem of passing from the known at any instant to the unknown at the same instant, as in the query: Are there mammals on Mars? With these last two problems we shall not be concerned here. They do not always have a meaning, and when they do, it is expressible in terms of a priori probability. The fundamental question with which we shall have to do resolves itself, therefore, into this: *What is meant by the statement that heads and tails are equally probable when a coin is tossed?*

There appear to be but two popular notions of probability which it is useful to idealize: (a) the long run idea, and (b) the idea of balanced causation. These two ideas rest on two mutually inconsistent philosophies. The first philosophy involves the supposition that the same set of antecedent events would not always produce the same result. The physical world is supposed to be so constituted that, if the trial of coin tossing were repeated indefinitely, *in exactly the same way each time*, heads and tails would happen equally often in the long run. Idealizing this notion, we have the familiar limit definition,

$$\lim_{n\to\infty} \frac{x}{n} = \frac{1}{2},$$

where x is the number of heads and n the number of trials. The second philosophy makes the requirement that any prescribed set of antecedent events is the determining cause of the result, and that the same result would always be produced from the same causes. We are, of course, not now concerned with the tenableness of either of these philosophies in man's world; we are merely assuming worlds in which the one or the other is true while constructing our definitions.

(*a*) The limit definition admits of the sum and product theorems, but unless carefully stated it leads to such inconsistencies as have been noticed recently by T. C. Fry[1] and W. A. Wilson.[2] First it is necessary to remark that the limit in question is not to be supposed uniform for all possible sequences. The major difficulty, however, is not a matter of uniformity; it has to do with the fact that the sequence we are considering does not exist *in the sense that* its various terms are not postulated or predetermined. It is not like the ordinary sequences of mathematics in which the law of formation of the terms is known, or at least contained implicitly in a set of known formulae. The inconsistencies mentioned

[1] *Probability and its Engineering Uses*, D. Van Nostrand Company (1928), pp. 89, 90.

[2] *Loc. cit.*, p. 578.

by Fry and Wilson are connected with this fact; also they are essentially the same though differing in form. The seeming paradox to which they allude may be put in a third form as follows: For any sequence of the sort demanded by our definition there would exist a number N (unknown) and a dependent number M such that, if the first M trials should be all heads, then the remaining $(N-M)$ trials would all be tails. It would appear, therefore, that, to an observer who might watch the first M trials, the probability of heads on the next trial would not be one-half, since that trial would be certain to result in tails. Such, however, is not the case, if one holds rigidly to the definition, for the sequence to which that next trial would belong, and of which it would constitute the first term, would also be one which would converge to one-half. There is nothing contradictory about tails being certain (in the popular sense) on that next trial and heads and tails being equally likely (in the sense of the definition).

These ideas may be expressed more succintly if one uses the language of point sets. There are an infinity of conceivable infinite sequences, of the sort demanded by our definition. Let a represent one such sequence, A the totality of the a's. Clearly A is a definite aggregate. Now choose $\epsilon>0$, and for each a denote by N the smallest n for which

$$\left|\frac{x}{n}-\frac{1}{2}\right|<\epsilon, \quad n \geqq N. \tag{1}$$

Next determine a subset A' of A such that, for each sequence of A', the following conditions hold: (*i*) There exists an M such that the first M trials are all heads and the next $(N-M)$ trials are all tails. Obviously M equals $N/2$, approximately. (*ii*) M is also such that, if the next $(N-M)$ trials were not all tails the inequality (1) would not hold. Clearly, A' is a definite subset. Now our postulate of convergence says that the sequence that interests us belongs to A. To obtain our so-called paradox one must postulate further that it also belongs to A'. This was done originally by the use of the phrase beginning, "*if* the first M trials should all be heads." This amounts to a partial particularization of the sequence, and it should not cause surprise that by such particularization one should also partially predetermine it. If one is to find a paradox in such partial particularization, one might as well go the limit and completely particularize it. A complete particularization would amount to a predetermination of every term. Let us do this, and let us again contemplate the immortal that Wilson describes, playing out the successive terms of the particular sequence we select. As Wilson states, N is unknown but fixed, now that the sequence is. In the same sense every term of this sequence is fixed, for we are now assuming that this immortal is actually playing out some particular a of A. This does not lead to a contradiction because, after any finite number of trials, he still stands at the threshold of a sequence of A. The probability of heads on his next throw is then one-half in the same sense as it was when he began.

Although this limit definition does seem to me to be free from real inconsistency, there are nevertheless valid objections to it. In fact, the idea it sets

out to idealize is not wholly satisfactory. One of the strongest objections is that it ignores what may happen in the short run. Suppose it were true that the coin tossing experiment would always lead to a peculiar subset B of A in which every sequence b had certain peculiar characteristics, such as long runs of heads or tails, then the condition of the definition would still be satisfied, but we might be quite unwilling to agree that heads and tails were under such conditions to be considered as equally probable. A part of this difficulty, but not all of it, is removed if one adds the requirement, which is in fact usually supposed implicit in the definition itself, that the conditions must remain true after any finite number of trials.

(*b*) Using the idea of balanced causation, and the deterministic type of philosophy, one may define heads and tails as being equally probable on the ensuing throw *if the causes which tend to produce the one are exactly equally balanced with those tending to produce the other in so far as we can measure them, provided that in truth they are not exactly balanced.* The last clause is needed in order to eliminate the cases where the coin will stay on edge. The major difficulty with this definition is that it is subjective, but this cannot be avoided; it is intrinsic in the idea of balanced causation. For, if we may imagine a god, all wise so far as past and present are concerned, and able to determine immediately the effect of all causes, to him probability could not exist in this sense. If the causes were not truly balanced he would know which way the coin would fall, and if they were exactly balanced it would stay on edge. Hence this idea can exist only for imperfect human intelligences; in other words it is of necessity subjective.

In the opinion of the author either of these definitions is logically tenable, but *not both at once.* They rest on two mutually exclusive philosophies, and much of the confusion that exists in discussions of probability is due to the fact that they are not kept distinct.

Turning now to the case where the number of elements is infinite, it would seem to be unnecessary for the statistician to consider it. Sampling from an infinite population is a convenient phrase commonly employed in statistics in the sense of sampling from a finite population with replacements. Further, there seems to be no reason why a frequency curve may not always be considered merely as a convenient approximation to a histogram, which in turn approximates a finite group. In fact, since in mathematics the word "infinite" applies to a process rather than to a fixed number, the physical concept of a bag containing an infinite number of balls does not exist. It is to be expected, therefore, that if one applies to the infinite realm phrases which have been given a meaning only for finite groups all sorts of paradoxes may appear, including the striking one suggested by Wilson.[1]

[1] *Loc. cit.*, p. 579.

A Simplification in the Numerical Computation of Multiple Integrals

By W. M. GOODHUE, Instructor in Electrical Engineering, Harvard University

In physics and in engineering there are many problems in which double integrals are to be evaluated, and in which the rapid determination of reasonably good approximate values is of prime importance. Suppose for example, that it is desired to compute the temperature due to a uniformly distributed source of heat, in a body whose shape and boundary condition permits the assumption that the problem is two-dimensional. This leads to the integral,

$$T_1 - T_0 = \frac{-\rho \operatorname{Div} J}{2\pi} \iint \left(\log \frac{r_1}{r_0}\right) dS, \tag{1}$$

where $T_1 - T_0$ is the temperature difference between two fixed points P_1 and P_0, r_1, r_0 are distances of dS from P_1 and P_0, J is the heat flow density vector, and ρ is the mean thermal resistivity of the material.

If this is to be computed by means of one of the usual approximation formulas, employing say n points for a single integration, then something like n^2 determinations of the integrand, depending on the shape of the region, and their sum, are required for the double integration.

The present paper suggests a transformation to make the integrand of the inner integral a function independent of the outer variable, so that the evaluation of this inner integral amounts to finding the area under a fixed curve, but between variable limits (the limits varying with the outer variable). Step by step integration natural'y yields an *integral curve*, which is independent of the variable limits involved.

Expanding equation (1), and setting $dS = dx\,dy$,

$$T_1 - T_0 = \frac{-\rho \operatorname{Div} J}{2\pi} \iint \left(\log \frac{r_1}{y_1} - \log \frac{r_0}{y_0} + \log \frac{y_1}{y_0}\right) dxdy. \tag{2}$$

$$\text{Set } \frac{r_1}{y_1} = \sqrt{1 + \left(\frac{x_1}{y_1}\right)^2} = (1 + t_1^2)^{1/2} \tag{3}$$

where x_1 and y_1 are measured from point P_1 instead of the origin.

Equation (2) may now be further expanded:

$$T_1 - T_0 = \frac{-\rho \operatorname{Div} J}{2\pi} \left\{ \int y_1 dy \int_{t_1'}^{t_1''} \log (1 + t_1^2)^{1/2} dt_1 \right.$$
$$\left. - \int y_0 dy \int_{t_0'}^{t_0''} \log (1 + t_0^2)^{1/2} dt_0 + \int (x'' - x') \log \frac{y_1}{y_0} dy \right\}.$$

Note that the inner integrals represent areas under the *same* curve, the dif-

ference being solely in the limits. The third term is merely a single integration. Allowing n points to each integration, the total number of points required is $3n$, assuming the inner integral curve available from previous problems, as it may be preserved and used indefinitely (otherwise n extra points).

A Note by O. D. Kellogg, Harvard University: Mr. Goodhue is here calling attention, by means of a specific example, to the reduction in labor of approximate computation which follows when it is possible so to change the variables in an iterated integral that the integrand becomes a product of functions, each of a single variable,—or even if it merely becomes the sum of a small number of such products.

Conditions for the possibility of such transformations are not difficult to set up, and, in the case of two variables, amount to a single differential equation for the two functions giving the transformation. As this is an underdetermined system, it looks as if the transformation were always possible. However, the equation is somewhat involved, and it would be of interest to determine broad classes of cases in which the reduction is possible by simple assigned methods. Such an investigation would also be of interest in the case of triple integrals, where the problem is no longer underdetermined.

On Equilateral Triangles

By J. R. MUSSELMAN, Western Reserve University

§1. In a recent number[1] of this Monthly, the following problem was proposed by Professor W. H. Echols: "*At the corners of any equilateral triangle $A_1B_1C_1$ let there be hinged three equilateral triangles $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ of any sizes or positions. Then will the mid-points of each of the sets of three segments $(A_2C_3,\ B_3C_2,\ A_3B_2)$, $(B_1C_3,\ C_1B_2,\ B_3C_2)$, $(A_2B_1,\ B_3A_1,\ A_3B_2)$, $(A_1C_2,\ C_1A_3,\ A_2C_3)$ be the corners of an equilateral triangle.*"

As there are a number of equilateral triangles connected simply with two given equilateral triangles and more connected with three given equilateral triangles, it seems worth while to call attention to them; and secondly, to point out in connection with Professor Echols's figure the existence of fifty-five equilateral triangles of which he mentions four. In conclusion I wish to state a theorem which is rather interesting and I believe entirely novel.

The most elegant method of proof follows from the use of a complex number as the coordinates of a point in the plane. Thus if the vertices of a triangle $P_1P_2P_3$ be given the complex numbers p_1, p_2, p_3 respectively, Professor Frank Morley[2] has shown that the condition that $P_1P_2P_3$ be a positively equilateral triangle i.e., that the verticesro tate into each other in a counter-clockwise direction, is

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[1] American Mathematical Monthly, vol. 39 (1932), p. 46. The notation in the problem has been changed to the above.

[2] Harkness and Morley, *Treatise on The Theory of Functions* (1893), p. 26.

$$\begin{vmatrix} p_1 & p_2 & p_3 \\ 1 & \omega & \omega^2 \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

or $p_1+\omega p_2+\omega^2 p_3=0$ where ω and ω^2 are the two complex cube roots of unity.

§2. Let $A_1A_2A_3$ and $B_1B_2B_3$ be the vertices of two positively equilateral triangles of any size or position whose coordinates in the plane are the complex numbers a_i, $b_i (i=1, 2, 3)$ respectively. These are subject to the conditions

$$a_1 + \omega a_2 + \omega^2 a_3 = 0 \tag{1}$$

$$b_1 + \omega b_2 + \omega^2 b_3 = 0. \tag{2}$$

If we divide the sum of (1) and (2) by the quantity 2 we obtain

$$(a_1 + b_1)/2 + \omega(a_2 + b_2)/2 + \omega^2(a_3 + b_3)/2 = 0, \tag{3}$$

whence we have the theorem *that the midpoints of* A_1B_1, A_2B_2, A_3B_3 *are the vertices of an equilateral triangle.* As both equations (1) and (2) are true when multiplied through by ω or ω^2, it is possible to combine them in two more ways, obtaining the result *that the midpoints* of A_1B_2, A_2B_3, A_3B_1; *also of* A_1B_3, A_2B_1, A_3B_2 *form the vertices of an equilateral triangle.* A simple way of obtaining them is in (3) to keep the a_i fixed and permute the b_i cyclically. Hence connected with two positively equilateral triangles of any size or position we find three other equilateral triangles.

§3. Let $C_1C_2C_3$ be a third positively equilateral triangle of any size or position whose vertices have been given the complex numbers c_1, c_2, c_3 respectively as coordinates. Then

$$c_1 + \omega c_2 + \omega^2 c_3 = 0. \tag{4}$$

If we combine (1), (2) and (4) by pairs in all possible ways as indicated in paragraph 2 we can obtain in all 9 equilateral triangles. Included among these are the three last triangles mentioned in the problem of Professor Echols.

If we add (1), (2) and (4) we obtain

$$(a_1 + b_1 + c_1) + \omega(a_2 + b_2 + c_2) + \omega^2(a_3 + b_3 + c_3) = 0. \tag{5}$$

Upon dividing this by the quantity 3 we have the theorem that *given three positively equilateral triangles* $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ *then the centroids of the three triangles* $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ *themselves form a positively equilateral triangle.* Inasmuch as (1), (2) and (4) are true if the a_i, b_i, c_i $(i=1, 2, 3)$ are permuted cyclically, we see that (5) is only one of nine ways in which the points A_i, B_i, C_i can be arranged so that the centroids of the three triangles formed are themselves an equilateral triangle. To obtain all nine keep the a_i in (5) fixed and permute the b_i and c_i cyclically. Moreover we note that *the centroid of each of the nine equilateral triangles thus formed is the same point.*

§4. If the three positively equilateral triangles be now so placed in the plane that $A_1B_1C_1$ forms a positively equilateral triangle, we have the further requirement that

(6) $$a_1 + \omega b_1 + \omega^2 c_1 = 0.$$

Multiply (2) by ω, (4) by ω^2 and add to their sum (1). Making use of (6) we obtain

(7) $$(b_3 + c_2) + \omega(a_2 + c_3) + \omega^2(a_3 + b_2) = 0.$$

Dividing (7) by the quantity 2 we have thus proven the first statement in Professor Echols's problem that *the midpoints of B_3C_2, A_2C_3, A_3B_2 form an equilateral triangle.*

If we add (2), (4) and (6) we obtain

(8) $$(b_1 + c_1 + a_1) + \omega(b_2 + c_2 + b_1) + \omega^2(b_3 + c_3 + c_1) = 0$$

Upon dividing (8) by the quantity 3 we have the next theorem that *the centroids of the three triangles $B_1C_1A_1$, $B_2C_2B_1$, $B_3C_3C_1$ form a positively equilateral triangle.* To write down the nine equilateral triangles that can thus be formed keep the last letter in each term of (8), namely $a_1b_1c_1$, fixed and permute the other b_i and c_i cyclically. Thus for example, we obtain the theorem that the centroids of $B_3C_1A_1$, $B_1C_2B_1$,[1] $B_2C_3C_1$ form an equilateral triangle.

The existence of fifty-five equilateral triangles connected with the figure of three equilateral triangles, so placed in a plane that one vertex of each also forms an equilateral triangle, is now evident. If we take the equations (1), (2), (4) and (6) by pairs, from each pair by the method of paragraph 2 we obtain 3 equilateral triangles or 18 in all. If we take the equations by threes from each triad we obtain 9 equilateral triangles or 36 in all. Finally by combining all four equations we obtain one more, thus making a total of 55 equilateral triangles which can easily be constructed.

§5. If we place three positively equilateral triangles $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ so that $A_1B_1C_1$ is likewise a positively equilateral triangle we saw that the four conditions (1), (2), (4) and (6) could be combined to produce

(7′) $$(a_2 + c_3) + \omega(b_2 + a_3) + \omega^2(c_2 + b_3) = 0.$$

Hence if in addition, we require $A_2B_2C_2$ to be a positively equilateral triangle, namely

$$a_2 + \omega b_2 + \omega^2 c_2 = 0,$$

then (7′) becomes $c_3 + \omega a_3 + \omega^2 b_3 = 0$, whence $A_3B_3C_3$ is likewise an equilateral triangle. Or in other words we have proven the following theorem: *given two positively equilateral triangles $A_1B_1C_1$, $A_2B_2C_2$ of any size or position in the plane, if we construct the positively equilateral triangles $A_1A_2A_3$, $B_1B_2B_3$ and $C_1C_2C_3$ then $A_3B_3C_3$ itself is a positively equilateral triangle.*

In exactly similar way we can prove the theorem: *given two positively equilateral triangles $A_1B_1C_1$, $A_2B_2C_2$ of any size or position in the plane, if we construct the negatively equilateral triangles $A_1A_2A_3$, $B_1B_2B_3$,* and *$C_1C_2C_3$ then $A_3B_3C_3$ is a positively equilateral triangle.*

[1] The centroid of $B_1C_2B_1$ is the point on the segment B_1C_2 one third of the distance B_1C_2 from B_1.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

A Treatise on Algebraic Plane Curves. By J. L. Coolidge. Oxford University Press, 1931. xxiv+513 pages. $10.00.

In the words of the author "the object of the volume before us" is "to give an account of the present status of the theory of algebraic plane curves, and their relations to various fields of geometry and analysis." A mere inspection of the table of contents would be almost enough to convince one that the author has done all that is possible with five hundred odd pages to accomplish his purpose.

The material is presented in four Books which deal respectively with the Elementary Theory (194 pp.), the Singular Points (53 pp.), Systems of Points on a Curve (127 pp.), and Systems of Curves (116 pp.). There is also appended an index of authors and a subject index.

The range of topics covered in these four books is extraordinarily wide. In the first book there are ten chapters devoted respectively to fundamental properties of polynomials, elementary properties of curves, real curves, real circuits of curves, elementary invariant theory, projective theory of singular points, Plücker's equations and Klein's equation, the genus, covariant curves, and metrical properties of curves. The point of view in this book is that of projective or Euclidean geometry.

The second book contains chapters on the reduction of singularities, developments in series, clustering singularities, and adjoint curves and Plücker's equations. Here Cremona transformations are used to reduce higher singularities to the ordinary type, and birational transformations to remove ordinary multiple points in favor of nodes.

The third book is mainly concerned with the algebraic geometry on a curve with excursions into the allied transcendental theory and a return at the close to the projective geometry of the rational curve. Its nine chapters discuss the general theory of linear series, abelian integrals, singular points of correspondences, moduli and limiting values, curves of special type, non-linear series of groups of points on a curve, higher theory of correspondences, parametric representation of the general curve (a sketch), and rational curves.

The eight chapters of the final book treat the postulation of linear systems by points, the transformation of linear systems, ternary apolarity, special curves in linear systems, non-linear systems of curves, the general Cremona transformations, and groups of Cremona transformations.

The encyclopedic character of the treatise, indicated by the chapter headings above, is strongly confirmed by an inspection of the topics considered in the

126 sections of the book. Many of these sections, after preliminary explanations, contain lists of theorems whose proofs, if reasonably direct, are merely indicated, or left entirely to the reader as exercises. In this way the scope of the work is much increased. In our opinion the volume not only will meet a long felt need on the part of students for a comprehensive survey in English of modern plane algebraic geometry, but also will be of real service to more advanced workers.

It would seem that on the whole the author has made a reasonably happy choice from the mass of material which was available for exposition. It is true that the excursions into the transcendental field are so sketchy that only expert readers can follow them, but doubtless little more could be attempted in one volume.

The arrangement of the content is not as logical as might be expected. One would naturally look for topics which belong essentially to projective geometry and the theory of invariants in the first book. But the study of the rational curve is postponed to Chap. IX of Book III, and that of ternary apolarity to Chap. III of Book IV. The references to the literature seem adequate for further orientation particularly in view of the existence of current surveys.

Some notations such as $\partial f/\partial 1$ may offend the conventional mind. At times the writer seems to prefer, for no obvious reason, the use of plain English to that of an appropriate technical term. Thus the classic separation of algebraic integrals into those of the first, second, and third "kinds" is always referred to as a separation into "sorts." The phrase—"a correspondence without value" (Wertigkeit)—will have an odd ring to one accustomed to the technical term "valence."

So far as may be inferred from the preface this volume was prepared for the press without the aid of readers. As a result there are numerous slips of various kinds—misprints, mistakes in cross references, and faulty statements. For example, the theorems 13, 16, 21 (pp. 413–15) are either incorrect or badly stated. In the haste to reach results some errors in logic appear. Thus, on p. 206, in order to prove that the process of resolving a higher singularity eventually comes to an end, the theorem (p. 119) that the genus of an irreducible curve is not negative is used, even though in the intervening pages the theorem itself, and a method for calculating the genus, have not been derived for a curve with a higher singularity. Another example occurs on p. 487 where the following theorem is stated:

"A necessary and sufficient condition that a set of positive integers $n, r_1, r_2, \cdots, r_r$ which satisfy the relations,

$$3n - r_1 - r_2 - \cdots - r_r = 3, \quad n^2 - r_1^2 - r_2^2 - \cdots - r_r^2 = 1,$$

should correspond to an actual Cremona transformation is that there should exist positive integers s_j and such positive or zero integers α_{ij} that" the customary equations which condition these numbers should be satisfied.

The proof is merely a reiteration that the order n may be continually reduced by multiplication by a quadratic transformation until it becomes unity.

But if $n=5$, $r_1=r_2=3$, $r_3=r_4=\cdots=r_8=1$, a quadratic transformation at most reduces the case to $n=3$, $r_1=r_2=1$, $r_3=-1$, $r_4=r_5=\cdots=r_8=1$, and further reduction is impossible. It is easy in this particular case to verify that integers s_j, α_{ij} do not exist; but it would not be easy to make this verification in higher cases, and one must therefore assume provisionally that such integers do exist. There is of course little doubt that the theorem is true. It is likely that any valid proof will rest upon the existence of inequalities satisfied by the integers $n, r_1, \cdots, r_r$. The theorem is ascribed, in part at least, to S. Kantor but the reference given is incorrect.

Nevertheless these lapses do not seriously impair the value of the work as a whole. A very desirable feature is an exposition of the symbolic notation of Aronhold and Clebsch. The use of this notation should be as much a matter of course in projective and algebraic geometry as the use of vectors in metric geometry.

ARTHUR B. COBLE

The Macmillan Table Slide Rule. By J. P. Ballantine. New York, The Macmillan Company, 1931. $0.50.

This "Table Slide Rule" is a table of numbers printed on cardboard, in such a fashion that with the aid of cardboard slides it can be used somewhat as one does an ordinary slide rule, to read off directly the results, correct to three significant figures, of the common arithmetic operations of multiplication, division, root extraction, and so forth. There is the additional advantage that five figure accuracy is easily available from the table, though in this case not instantly. The idea is an ingenious one and deserves some attention. Since the table slide costs only fifty cents and seems to do the work, it is conceivable that it may go a considerable way toward displacing the slide rule in common use.

In comparison with the ordinary slide rule it is apparently at a disadvantage in at least two respects, namely portability and durability. The ordinary slide rule can be carried in the pocket and used in the two hands, but the table rule is not a pocket instrument and for comfortable operation requires two or three square feet of clear surface on which it can be laid out flat. Moreover the ordinary rule will outlast the present type of table rule many times over, but this difference is possibly compensated by the difference in initial cost.

An apparent advantage for the table rule is in the matter of setting. The ordinary slide rule must be set accurately to get good results. With the table rule there is a very comfortable margin of error in setting that produces no change at all in reading. This may well be a valuable contribution to the lessening of fatigue in extensive computation.

In an effort to get an impartial comparison, the present reviewer selected an intelligent member of his freshman class who had never seen or even heard of any kind of slide rule. This student was given both an ordinary slide rule and the table slide rule, with printed directions for the use of each. He was told to

learn how to multiply and divide and report forty-eight hours later for a proficiency test. It is an interesting fact that in the test, this student used the table slide not only more accurately but also more rapidly.

R. E. GILMAN

Bericht über Neuere Untersuchungen und Probleme aus der Theorie der Algebraischen Zahlkörper. By Helmut Hasse. Leipzig and Berlin, G. B. Teubner, 1930. Teil I: *Klassenkörpertheorie*, 134 pages. Teil II: *Reziprozitätsgesetz*, iv +204 pages.

During the last decade certain phases of the Theory of Algebraic Numbers have been materially advanced by the great work of Takagi and of Artin. Hasse has undertaken the task of presenting an exposition of the work of these mathematicians and of their followers.

A considerable portion of the first part of Hasse's report centers around Takagi's generalization and extensions of Hilbert's results on the class-field (Klassenkörper) of an Algebraic field. Takagi inculcated a new spirit in the treatment of relative abelian fields, by means of which that theory was greatly simplified and some previously unsolved problems settled with finality.

The *motif* of the second part of the report is Artin's reciprocity law, conjectured by Artin in 1923, and proved by him in 1926. Artin's reciprocity law may be heralded as the final reciprocity law which so many devotees of the Theory of Numbers have sought for a century. Its importance can hardly be overestimated and can be realized only by reading Hasse's report, from which it appears, among other matters, that the classical reciprocity laws of the Theory of numbers are consequences of Artin's reciprocity law.

A detailed bibliography and a long list of unsolved problems included in the report will be of invaluable assistance and serve as a stimulus to specialists in this fascinating but difficult branch of mathematics.

LOUIS WEISNER

The Queen of the Sciences. By E. T. Bell. Baltimore, The Williams and Wilkins Company, 1931. 138 pages. $1.00.

This little book belongs to "A Century of Progress Series" which, according to the announcement of the publishers, comprises "volumes by well-known scholars presenting the essential features of those fundamental sciences which are the foundation stones of modern industry." It is surprising how comprehensively Dr. Bell has presented the fundamental concepts of mathematics in a short non-technical treatise.

After an enlightening discussion of the object of mathematics in Chapter I, in which the fact is emphasized that creative mathematicians of all time have been inspired by "the art of mathematics rather than by any prospect of ultimate usefulness," the author continues in the next chapter with a description of mathematics and presents the famous definitions of Russell, Peirce, Klein and

Hilbert. The discussion leads naturally to the postulational method, algebra being treated by this method in Chapter III. The chapter presents also the parallel postulate and the characteristic features of the geometries of Lobachevsky and Riemann. Chapters following show that common algebra can be "realized" with the classes of rational, real, and complex numbers and present the notions of transformation, group, invariant, and matrix. To the "unruly domain" of arithmetic a chapter is devoted which contains, by way of interest, several simple classic, proved and unproved, theorems of the theory of numbers. Later chapters dwell on the notion of the infinite and on the power and present state of analysis. The concluding chapter is concerned with mathematical existence and with the modern critical attack on mathematical reasoning.

Dr. Bell, incidentally a successful novelist, has written in simple style, mathematically unconventional perhaps, but not flippant. The misspelling of the great name "Peirce" (page 18) and the statement of "88" for "68" (page 80) are obviously misprints. The reviewer, frankly enthusiastic, would like to recommend the book to everybody. It would be an unimaginative person who would not, from even a cursory reading, catch some of the glorious spirit of mathematics. The teacher who has not enjoyed the advantage of extensive modern graduate study will be instructed and oriented. The apprentice in the guild of mathematicians will be inspired. The university lecturer will put the book down feeling that he has reaffirmed his faith.

PAUL H. LINEHAN

Mathematics. By B. B. Low. London, Longmans, Green and Co., 1931. vii+443 pages.

The author states that he has avoided the title *Practical Mathematics* because of the difference of opinion as to what is practical, but his use of *Mathematics* and the purpose and scope of this volume are made clear by the following statement on page 1: "The scientific worker must understand certain parts of pure mathematics and must be able to apply his knowledge to obtain numerical results when these are required. He uses what has been called practical mathematics or what the author prefers to call *Mathematics*." This book should prove valuable to this scientific worker, either if he is confronted with some problem for whose solution his mathematical equipment is inadequate, or if he wishes to add to his knowledge of mathematics, within the author's meaning.

The subjects treated are elementary algebra, trigonometry, plane and solid analytic geometry, differential and integral calculus, derivation of empirical laws, calculus of finite differences, harmonic analysis, and differential equations. Very little previous mathematical training is assumed and there are numerous worked examples to illustrate each principle. Where it is impossible to make each subject complete in itself, the author attempts to make the way easier for the reader who seeks knowledge of some special topic by a fairly complete system of references to preceding chapters. The mathematically curious, however, will find need of other texts because, as is natural in this type of book,

the mathematical theory is developed no further than is necessary for the immediate end in view. A thoroughly sound list of such texts is found in Chapter I. There are numerous sets of not very difficult exercises, including many of scientific or technical interest.

L. T. MOORE

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide. All club activity manuscripts for the academic year 1931–1932 should be submitted for publication not later than June 1, 1932.

CLUB TOPICS

1932 AS A CENTENNIAL YEAR IN THE HISTORY OF MATHEMATICS

BY WALTER CROSBY EELLS, Stanford University, California

In continuation of previously published lists[1] of centennial dates in the history of mathematics, the following group of important 1932 centennial dates is presented.

B.C. 569.[2] Approximate date (according to Ball) of Pythagoras founder of the Pythagorean school of such great importance in the development of Greek mathematics.

A.D. 1532. Birth of Xylander, who translated the first six books of Euclid's *Elements* and works of other Greek mathematicians into German.

A.D. 1632. Birth of Christopher Wren, celebrated architect, who rectified the cycloid and determined its center of gravity.

A.D. 1632. Publication of Oughtred's *Circles of Proportion and the Horizontal Instrument* containing a description of his invention of the circular slide-rule.

A.D. 1632. Publication of Galileo's *Dialogo dei due massimi sistemi del mondo*, expounding the Copernican theory.

A.D. 1632. Death of Burgi, Swiss mathematical genius, who invented logarithms independently of Napier.

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[1] This Monthly, vol. 38 (1931) pp. 100–101 for a list of 1931 centennial events, and for references to previous volumes for corresponding lists from 1925 to 1930.

[2] This Monthly, vol. 38 (1931) p. 100, footnote, for justification of 569 B.C. as a centennial date for 1932.

A.D. 1632. Death of Girard, Dutch mathematician, author of treatise on trigonometry which, according to Ball, contains the first use of the abbreviations *sin*, *tan*, and *sec*.

A.D. 1732. Birth of Karsten, inventor of an interesting graphic representation of imaginary logarithms.

A.D. 1732. Birth of Rittenhouse, noted colonial mathematician and astronomer. "Without literary friends or society, and with but two or three books, he became, before he reached his four and twentieth year, the rival of two of the greatest mathematicians of Europe." (Cajori, *Mathematics in the United States*, Washington, 1890, pp. 38, 41, 42.)

A.D. 1832. Birth of Fiedler, who brought out German editions of Salmon's works on higher algebra and geometry.

A.D. 1832. Birth of Sylow, Norwegian mathematician, whose lectures materially extended the field of substitution groups. "Sylow's Theorem."

A.D. 1832. Publication of Jerrard's *Mathematical Researches* containing the reduction of the quintic equation to trinomial form.

A.D. 1832. Publication of Steiner's *Systematische Entwicklung der Abhängigkeit geometrischer Gestalten von einander*, one of the principal foundations upon which modern synthetic geometry rests.

A.D. 1832. Publication of Wolfgang Bolyai's *Tentamen juventutem in elementa mathaseos purae . . . introducendi*, the second volume of which, published the following year, included in an appendix his son's fundamental contribution on non-Euclidean Geometry.

A.D. 1832. Dirichlet gave a proof for Fermat's Theorem on the impossibility of $X^n + Y^n = Z^n$ in integers for $n = 14$.

A.D. 1832. Death in a duel at the age of 20 of the youthful genius, Galois, to whom credit is due for important advances in group theory and the theory of algebraic equations of higher degree.

A.D. 1832. Death of Sadi Carnot whose essay *Réflexions sur la puissance motrice du feu*, according to Ball, "may be taken as initiating the modern theory of thermodynamics."

A.D. 1832. Suspension of publication of *The Mathematical Diary*, early American mathematical periodical.

A.D. 1832. Birth of Reverend C. L. Dodgson (Lewis Carroll) whose mathematical work was mainly in connection with Euclid and Symbolic Logic. (See "The Life and Letters of Lewis Carroll" by his nephew, S. Dodgson Collingwood, The Century Company, 1899.)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3552. *Proposed by J. M. Feld, Brooklyn College of the City of New York.*

Prove that

$$\begin{vmatrix} e_{11} & 0 & 0 & \cdots 0 & A_1 \\ e_{21} & e_{22} & 0 & \cdots 0 & A_2 \\ e_{31} & e_{32} & e_{33} & \cdots 0 & A_3 \\ \cdot & \cdot & \cdot & \cdots \cdot & \cdot \\ e_{n1} & e_{n2} & e_{n3} & \cdots e_{n,n-1}, & A_n \end{vmatrix}$$

$$= A_n - \sum A_{n/p_i} + \sum A_{n/p_ip_j} - \cdots + (-1)^s A_{n/p_1p_2\cdots p_s},$$

where (1) $e_{ij}=1$ if j is a divisor of i and $e_{ij}=0$ if j is not a divisor of i; (2) $p_i, p_j \cdots$ are the distinct prime factors of n.

3553. *Proposed by A. A. Bennett, Brown University.*

Let c_r be defined for $r=1, 2, \cdots$, by

$$c_r = 0, \text{ if } r \text{ is not of the form } m(3m\pm 1)/2,$$
$$c_r = (-1)^m, \text{ if } r \text{ is of the form } m(3m\pm 1)/2.$$

Thus

r	1	2	3	4	5	6	7	8	9	10	11	$12\cdots$
c_r	-1	-1	0	0	$+1$	0	$+1$	0	0	0	0	$-1\cdots$

Show that the sum of the kth powers of the roots of

$$x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_{n-1}x + c_n = 0,$$

(for $k=1, 2, \cdots, n$), is $s_k=$the sum of the divisors of k, (independent of n). Thus,

k	1	2	3	4	5	6	7	8	9	10	11	$12\cdots$
s_k	1	3	4	7	6	12	8	15	13	18	12	$28\cdots$

3554. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Show by the processes of descriptive geometry how to determine the H- and V-projections of an elliptical cylinder (that is, a cylinder whose right section is an ellipse) whose H-base is a given ellipse and whose V-base is a circle.

3555. *Proposed by E. C. Kennedy, Texas College of Mines.*

If F_1 and F_2 are factors of the number 32,228,143 show that the inequality $3.2F_1 > F_2 > 3F_1$ cannot hold. Show that $2.2F_1 > F_2 > 2F_1$ and find the value of the two factors. F_1 and F_2 may be prime or composite but their product is equal to the original number. The problem is to be solved without making a single division or using any factor table, or table of primes.

3556. *Proposed by the late Artemas Martin, Washington, D. C.*

A hollow sphere, external and internal radii R and r, rolls down an inclined plane in time t; after the cavity is half filled with water it rolls down the same plane in time t'. Determine the specific gravity of the sphere.

See the *Annals of Mathematics*, vol. 8 (1894), p. 104.

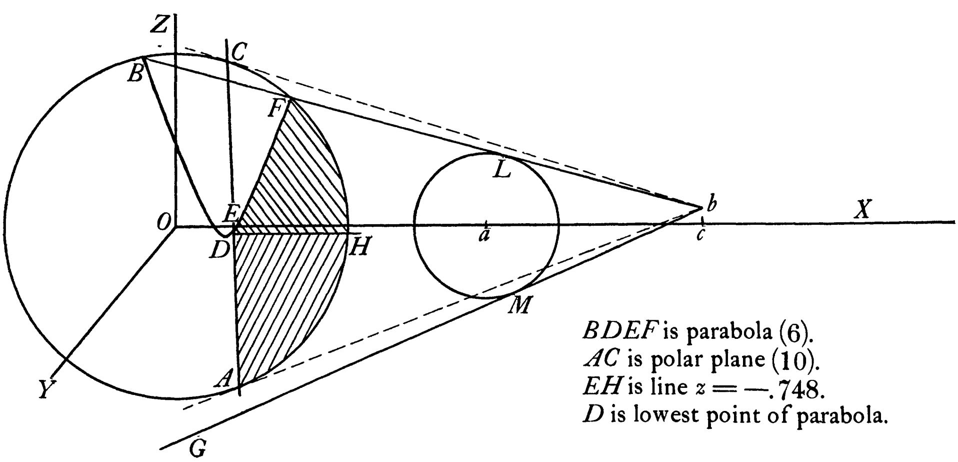
SOLUTIONS

353 (1914, 55; 1919, 312; 1931, 171). *Proposed by Richard P. Lochner.*

The center of a sphere, radius $R=5$ inches is $a=10$ inches above the surface of a sphere, radius $r=12\frac{1}{2}$ inches. There is a point of light at $b=1$ inch horizontally from a point $c=10$ inches vertically above the surface of the first sphere. What is the area of the shadow which the upper sphere casts on the lower one?

Solution by Wallace Smith, New River State College, Montgomery, West Virginia

Let us choose the centers of the spheres and point c on the X-axis and the position of the light above the X-axis in the direction of the Z-axis. Take the center of the sphere with radius $r=12\frac{1}{2}$ as the origin.



The above figure shows the shadow projected on the XZ-plane. The equation of the sphere at point a is

$$S_a \equiv (x - 45/2)^2 + y^2 + z^2 - 25 = 0. \tag{1}$$

The equation of the polar plane of point b with respect to S_a is

(2) $$P_b \equiv 15(x - 45/2) + z - 25 = 0.$$

Hence a surface tangent to S_a at its intersection with P_b is given by

(3) $$P_b^2 + kS_a = 0.$$

Now choose k so that the surface (3) passes through point b, then $k = -201$. Therefore,

(4) $$P_b^2 - 201S_a = 0$$

is a conical surface with vertex at b and tangent to S_a. The equation of the sphere at the origin is

(5) $$S_0 \equiv x^2 + y^2 + z^2 - 625/4 = 0.$$

If we eliminate y between (4) and (5), we get the projection on the XZ-plane of the intersection of (5) with (4), which is

(6) $$225x^2 + 30x(z - 61) + z^2 - 725z + 13075/4 = 0,$$

or

(6′) $$15x = 61 - z \pm 3(67)^{1/2}(z + \tfrac{3}{4})^{1/2}.$$

This is the equation of a parabola with real points for $z \geqq -\frac{3}{4}$ and with a tangent parallel to the x-axis at $x = 247/60 = 4.1167$, $z = -\frac{3}{4}$. We now find the intersection of (6) with the great circle of (5) in the XZ-plane. Set $y = 0$ in (4) and we get

(7) $$12x^2 + 15x(z - 61) - 25[4z^2 + (29/2)z - 1387/2] = 0,$$

or

(7′) $$24x = 15(61 - z) \pm 5(201)^{1/2}(z - 1),$$

which is a pair of straight lines in the XZ-plane. If we set $y = 0$ in (5), we get

(8) $$x^2 + z^2 - 625/4 = 0,$$

the trace of S_0 on the XZ-plane By eliminating x between (7′) and (8), the positive radical furnishes the real values $z = 9.077$ and $z = 12.217$, while the negative radical gives only imaginary values. These are the values of z for the intersections of (6) with (8) also.

The polar plane of point b with respect to S_0 is

(10) $$(75/2)x + z - 625/4 = 0.$$

Hence the illuminated part of S_0 when S_a is removed is the zone of S_0 toward b to right of (10). Therefore, the shaded part of S_0 is bounded by (4) and (10). We now find the intersections of (6) with (10), and (8) with (10) to be $x = 4.186$, $z = -.748$ and $x = 4.479$, $z = -11.670$ respectively. Now to determine the area we must consider the shaded portion as projected on the XZ-plane in two sec-

tions: first, take that portion between the parabola (6′), the great circle (8), the point at $z=9.077$, and the line $z=-.748$; second, take that portion between the polar plane (10), the great circle (8), the line $z=-.748$, and the point at $z=-11.670$. Hence the area of the shaded portion of S_0 is given by the sum of the two integrals:

$$A = 25\int_{-.748}^{9.077}\int_{f_1(z)}^{f_3(z)} y^{-1}dxdz + 25\int_{-11.670}^{-.748}\int_{f_2(z)}^{f_3(z)} y^{-1}dxdz,$$

where

$$y = (625/4 - x^2 - z^2)^{1/2}, \quad f_1(z) = [61 - z + 3(67)^{1/2}(z+\tfrac{3}{4})^{1/2}]/15$$

$$f_2(z) = 25/6 - 2z/75, \qquad f_3(z) = (625/4 - z^2)^{1/2}.$$

Note: Professor Dunkel suggested the simple method of writing the equations (4) and (10) which lead to the finding of the desired limits.

3496. [1931, 340] *Proposed by H. E. Stelson, Kent State College.*

In the triangle ABC, a normal is drawn to the circumcircle cutting the lines BC and AC at X and Y respectively. Find the locus of intersection of AX and BY.

Solution by A. Pelletier, Montreal, Canada

Let O, the center of the circumcircle, be taken as the origin so that the coordinates of A, B, C are $(-a, b)$, (a, b), (c, d), respectively, and $y=mx$ is the equation of YOX. The coordinates of X are then $(b-ka)\,(m-k)^{-1}$, $m(b-ka)(m-k)^{-1}$, where $k=(d-b)\,(c-a)^{-1}$. The equation of AX is then

$$(1)\qquad (y-b)(b+ma-2ka) = k(b-ma)(x+a).$$

The equation of BY is obtained by replacing a by $-a$ and k by $k'=(d-b)(c+a)^{-1}$.

Eliminating m from these two equations, we get for the equation of the locus of the intersection P of AX and BY

$$(2)\qquad (y-b)^2(ka-k'a-b) - 2kk'a^2(y-b) + kk'b(x^2-a^2) = 0.$$

An examination of (2) shows that, if O lies within ABC, the locus is an hyperbola; if O lies outside ABC, it is an ellipse; if O lies on AB, it is a straight line through C parallel to AB.

A Note by Otto Dunkel: If O is any fixed point in the plane of the given triangle ABC, the lines OXY cut BC and AC in two projective ranges of points X and Y. Hence P, the intersection of AX and BY, describes a conic through A and B with tangents OA and OB. The vertex C is also on the conic, since at C X, Y, and P coincide. If CO cuts AB in N, and if N' is taken on AB so that

N', A, N, B are harmonic, then $N'C$ is the tangent at C. The three tangents at A, B, C intersect in two other points O_a and O_b, and any one of the points O, O_a, O_b may be used to generate the conic with a suitable pair of vertices of ABC.

If O lies anywhere within angle B, or its vertical angle, i.e. within the infinite angular region containing the area ABC, it is easily seen that OXY may be drawn so that P lies within the triangle ABC. Hence in such cases the conic must be an hyperbola. A similar remark applies to the angular region at A. There are other regions for O which determine an hyperbola. If AX and BY are parallel, P is at infinity. Consider then the equal pencils AX, BY, where corresponding rays are parallel. Then the straight line XY envelopes a conic tangent to AC and BC. When X is at C, Y is at infinity on AC; and hence this conic is tangent to AC at infinity. The conic is therefore an hyperbola with AC and BC as asymptotes. If $A'B'$ is the straight line symmetric to BA with respect to C, it is clear that $A'B'$ is tangent to the hyperbola. By symmetry it follows that AB is also a tangent. Hence if O lies within this hyperbola, the locus of P is an ellipse; if O lies outside, the locus of P is an hyperbola; if O lies on the hyperbola, the locus of P is a parabola. If O lies on any side of ABC, the locus of P degenerates.

In the special case in which O is the center of the circumcircle, the only position of O that needs additional consideration is one for which O and C are separated by AB. In this case it is obvious that in the triangle OO_aO_b the points A, B, C lie on and within the segments of its sides, and we surely get an ellipse as the locus of P. This shows that in this case the center of the circumcircle lies within the hyperbola.

Also solved by D. C. Duncan, F. Underwood, and F. L. Wilmer.

3497. [1931, 340] *Proposed by J. Rosenbaum, Milford, Conn.*

Denoting the coefficients of the expansion of $(x+y)^n$, n being a positive integer, by $a_0, a_1, \cdots, a_n$; find the sum of $a_0, a_3, a_6, \cdots, a_{3k}$, where $(n-2) \leqq 3k \leqq n$.

Solution by Morgan Ward, California Institute

Suppose that

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n \tag{1}$$

is any polynomial in z of degree n, and that it is desired to evaluate the sum

$$S = a_s + a_{s+r} + a_{s+2r} + \cdots + a_{s+kr}$$

where

$$s + kr \leqq n < s + (k+1)r, \quad s < r.$$

Let ω be a primitive rth root of unity, so that

$$\begin{aligned}(2)\qquad \omega^t + \omega^{2t} + \omega^{3t} + \cdots + \omega^{rt} &= 0,\ t \not\equiv 0 \bmod r\\ &= r,\ t \equiv 0 \bmod r.\end{aligned}$$

Multiply (1) by z^{r-s}, set $z=\omega^t$, and then sum from $t=1$ to $t=r$. On simplifying the resulting expression by the use of (2), we obtain the general formula

$$(3)\qquad S = \frac{1}{r}\{f(\omega)/\omega^s + f(\omega^2)/\omega^{2s} + \cdots + f(\omega^r)/\omega^{rs}\}.$$

Now let us assume that $s=0$, and that $f(z)=(1+z)^n$. Then on taking $\omega=e^{2\pi i/r}$, we have for any positive integer t,

$$1+\omega^t = 2e^{\pi it/r}\cos(\pi t/r),\ \ 1+\omega^{r-t} = 2e^{-\pi it/r}\cos(\pi t/r)$$

so that

$$(4)\qquad f(\omega^t) + f(\omega^{r-t}) = 2^{n+1}\cos\frac{n\pi t}{r}\left(\cos\frac{\pi t}{r}\right)^n.$$

On simplifying the right side of (3) by the use of (4), we find after a few reductions that

$$S = \frac{2^{n+1}}{r}\left\{\tfrac{1}{2} + \cos\frac{n\pi}{r}\left(\cos\frac{\pi}{r}\right)^n + \cos\frac{2n\pi}{r}\left(\cos\frac{2\pi}{r}\right)^n + \cdots + \cos\frac{kn\pi}{r}\left(\cos\frac{k\pi}{r}\right)^n\right\}.$$

k here is the greatest integer in $(r-1)/2$.

In the case at hand, $r=3$, so that $k=1$ and

$$\begin{aligned}S &= \frac{2^{n+1}}{3}\left\{\tfrac{1}{2} + \cos\frac{n\pi}{3}\left(\cos\frac{\pi}{3}\right)^n\right\}\\ &= 2^n/3 + (-1)^n 2/3,\ \ n \equiv 0 \bmod 3\\ &= 2^n/3 + (-1)^{n+1}/3,\ \ n \not\equiv 0 \bmod 3.\end{aligned}$$

Also solved by R. P. Agnew, Frank Ayres, H. D. Grossman, H. L. Olson, W. V. Parker, A. Pelletier, and F. Underwood.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

THE SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The Sixteenth Summer Meeting of the Mathematical Association of America will be held at the University of California at Los Angeles on Monday and Tuesday, August 29–30, 1932, in conjunction with the summer meeting and colloquium of the American Mathematical Society. The program of the Association sessions will be sent to members the middle of July. The colloquium lectures will be given by Professor J. F. Ritt of Columbia University on "Differential equations from the algebraic standpoint." By invitation Professor D. N. Lehmer will address the Society on "The continued fraction representing cubic and higher irrationalities," and Professor Tibor Radó on "Recent work in the problem of Plateau." The joint dinner will occur Wednesday evening. On Thursday afternoon and evening an excursion will be held to visit the California Institute of Technology and the Mount Wilson Observatory at Pasadena. Opportunity will also be given to visit the University of Southern California, Occidental College, Pomona College, and other institutions of this region. Arrangements will be made for tennis and golf.

Accommodations for the mathematicians will be furnished in Mira Hershey Hall, a new women's dormitory of the University of California at Los Angeles. The charge for a single room will be $1.50 per person per day, and $1.00 per person per day when the room is occupied by two, provided the room is engaged for a minimum of three days. Suitable arrangements can be made for families with children. Meals will be served at Mira Hershey Hall at the rate of 50c for breakfast, 75c for luncheon, and 75c for dinner.

The railroads are planning the usual summer rates to California and it will be possible to obtain a round-trip rate whether one uses a northern route or not, thus making it possible to go or come from the east by a southern or central route and to make the other trip through the Northern Rockies or Canada, including a visit to the Pacific Northwest, without incurring any extra railroad charges. This makes it possible for those planning to attend the meeting at Los Angeles to arrange a vacation in any part of the Rockies or along any portion of the Pacific Coast.

The Olympic Games are to be held at Los Angeles during July and August, most of the main track events being in August. Those attending these games should make hotel reservations well in advance.

The French government has conferred the title "Officer de l'Academie" on Professor E. R. Hedrick, of the University of California at Los Angeles, "for services rendered to the cause of culture and of science."

Dr. Irving Langmuir, of the General Electric Company, has been awarded the prize of the Popular Science Monthly for notable scientific achievement.

Sir James Hopwood Jeans has been elected an honorary member of the Washington Academy of Sciences.

Professors Wilhelm Blaschke, of the University of Hamburg, in 1931 Visiting Lecturer of the American Mathematical Society, lectured in February, 1932, at the Universities of Annamalai and Calcutta, and will lecture in April at the Universities of Tokyo and Sendai, in May at Stanford University, and during the summer session at the University of Chicago. His subjects will be differential geometry, topology, and continuous groups.

Professor Albert Einstein has been appointed Rouse Ball lecturer at the University of Cambridge for the year 1931–32. Professor Einstein has been in residence at the California Institute of Technology and the Mount Wilson Observatory during January and February, 1932.

Dr. Willem de Sitter, director of the Astronomical Observatory of the University of Leiden, delivered a series of lectures in January, 1932, at the University of California, on *The astronomical aspects of the theory of relativity*, and *The system of astronomical constants*. He has also lectured recently at several other American universities.

Dr. W. A. Shewhart, of the Bell Telephone Laboratories, will lecture at the University of London in the spring of 1932 on the role of statistical method in industrial standardization, and will hold a series of conferences on theoretical and applied statistics in Great Britain and on the Continent. Dr. Shewhart has recently been elected a fellow of the Royal Statistical Society of England.

The summer meeting of the American Association for the Advancement of Science will be held at Syracuse N. Y., June 20 to 25. Section A of the Association will meet on Tuesday, June 21. The program of this section will be as follows: Professor H. M. Gehman, University of Buffalo, will speak on "Homeomorphic Geometry of the Projective Plane" at 10:00 A.M. Professor W. A. Hurwitz, Cornell University, on "Logical Foundations for Groups and Fields" at 11:15 A.M. Professor J. Shohat, University of Pennsylvania, on "Interpolation" at 2:00 P.M. The mathematicians attending this meeting will have lunch together, and after the afternoon session there will be a picnic at a nearby lake.

The announcement has been received of the death of Professor Heinrich Wieleitner, Oberstudiendirektor am neuen Realgymnasium und Honorarprofessor an der Universität, München, on December 27, 1932. Professor Wieleitner was well known to all students in the history of mathematics. He was a serious scholar and a clear and accurate writer. His loss will be felt in this country as well as in Munich, where he worked for so many years.

The thirteenth annual meeting of the National Council of Teachers of Mathematics was held on February 19–20, 1932, at Washington, D. C. The following papers were presented:

1. "Curriculum adjustments" by W. C. Myers, Washington, D. C.
2. "A comparative study of the teaching of mathematics in the United States and Germany" by Professor W. D. Reeve, New York, N. Y.
3. "Improving America's mathematics" by H. C. Barber, Exeter, N. H.
4. "What do we owe to the brighter pupil?" by Beulah I. Shoesmith, Chicago, Ill.
5. "Regents' and College Entrance Board examinations in mathematics" by Professor W. S. Schlauch, New York, N. Y.
6. "What mathematics means to the world" by Professor E. R. Hedrick, Los Angeles, Calif.

All of these papers will appear in forthcoming issues of The Mathematics Teacher.

Three new standing committees were established, one on individual differences, one on geometry, and a third on policy, which is to formulate a body of first principles that should govern the teaching of secondary mathematics. The committee on cooperation with official examining boards was continued for another year, as was the committee on handbook in mathematics.

The annual election resulted as follows: President, William Betz, Specialist in mathematics, Rochester, N. Y.; Second vice-President, Mary A. Potter, Supervisor of mathematics, Racine, Wis.; Directors, Mrs. Elsie P. Johnson, High School, Oak Park, Ill.; J. P. Everett, State Teachers College, Kalamazoo, Mich.; and Raleigh Schorling, University of Michigan, Ann Arbor, Mich. E. W. Schreiber, State Teachers College, Macomb, Ill., continues as Secretary-Treasurer.

The National Council of Teachers of Mathematics, which had a very modest beginning in Cleveland, Ohio, February 1920, when approximately seventy-five charter members created the organization, has now grown to a membership of nearly six thousand, representing every state in the union and many foreign countries.

The following courses in mathematics are announced for the summer 1932:

University of Southern California, first term, June 17 to July 29. In addition to the usual elementary work, the following advanced courses are offered: By Professor L. D. Ames: Selected topics—Algebra; Theory of functions of a complex variable. By Professor L. E. Gurney: Mathematical astronomy. By Associate Professor D. V. Steed: Vector analysis; Riemannian geometry. Second term, July 29 to September 2. By Professor Ames: Infinite processes; Modern higher algebra. By Professor Gurney: Differential equations; Mathematical astronomy.

The Mathematical Association of America

Invites all persons interested in the promotion of mathematics in America, especially in the collegiate field, to become members of this Association, and it urges you who are now members to extend this invitation personally and most cordially to all non-members within your circle of convergence.

The Association is living well within its budget, but some of its revenues are affected by the prevailing depression. Hence this added reason for a constantly increasing membership. Also note the statement of the Secretary on page 134 of the March, 1932 "Monthly."

It was a deep conviction of the late lamented Professor Young that the Association should extend its influence far more widely not only among college students but also among intelligent people in general, to the end that the rôle which mathematics plays in the world might be more widely understood and appreciated. This conviction was strongly set forth in his retiring presidential address printed in the January, 1932 "Monthly," pages 1-15. This address will bear reading and rereading.

The Editorial Board has some plans on foot which may be a beginning toward a partial realization of some of Professor Young's ideals, at least so far as these may be realized in the present Departments of the "Monthly."

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Sixteenth Summer Meeting of the Association, Los Angeles, Calif., Aug. 29-30, 1932.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1932 and reported to the Secretary.

ILLINOIS, Urbana, May 6-7.
INDIANA, Indianapolis, May 6-7.
IOWA, Cedar Falls, April 29-30.
KANSAS, Topeka, Feb. 13.
KENTUCKY, Lexington, May.
LOUISIANA-MISSISSIPPI, Oxford, Miss., March 11-12.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7.
MICHIGAN, Ann Arbor, March 19.
MINNESOTA, River Falls, Wis., May 7.
MISSOURI.
NEBRASKA, Omaha, May, 6-7.
OHIO, Columbus, Ohio, April 7.
PHILADELPHIA, Philadelphia, Pa., Nov. 26.
ROCKY MOUNTAIN, Laramie, Wyo., April 15-16.
SOUTHEASTERN, Gainesville, Fla., Mar. 18-19.
SOUTHERN CALIFORNIA, San Diego, March 26.
TEXAS, Austin, Jan. 30.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXIX, 1932

NUMBER 6, JUNE-JULY

PUBLISHED BY THE ASSOCIATION

MENASHA, WIS., AND ITHACA, N.Y.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members

$5.00 a Year, Single Copies 60 cents, to Others

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JOHN WESLEY YOUNG

Appreciation by President Hopkins of Dartmouth College

Perhaps the informal and companionable attributes of Professor Young's character can best be illustrated by the remark of a mutual friend the day of Professor Young's death. This man, bespeaking his sense of loss, said: "I never knew until this morning that Cy's name was John Wesley." This was actually true, I believe, of many another. Certainly over nearly two decades of acquaintanceship and friendship I never knew any official or unofficial occasion when formality had to be summoned to substantiate the position of either one of us in our mutual relationships.

The quality of his scholarship was beyond question, of course, but he was a scholar who liked to teach. His interest, moreover, was as keen in individual students as in the subject. He loved and respected both, I think. As an official associate the breadth of his interest in affairs not only of the curriculum but of general academic policy made him invaluable in discussion and deliberation concerning College affairs. He was wise in counsel, loyal in friendship, highly qualified for work of distinction in his chosen field, and indefatigable in his eagerness to know about and understand the world in which he lived. We who knew him well value greatly the privileges which have been ours in association with him, and in intimate thought we cherish his memory.

Appreciation by a Student Friend

(From The Dartmouth of February 20, 1932)

VALE: CY YOUNG

A week ago: your hand, in my hand resting—
The shy, queer smile that never went away.
A week ago: your jest, to crown my jesting—
Sunlight at nightfall on a gloomy day.

Long I shall cherish that last careless meeting,
That picture, stamped unfading on the brain:
Symbol of you, who offered life a greeting
Of banter, for its surly gift of pain.

Great heart, grown quiet, merely to have seen you
Wear, as an honored garment, such great ruth,
Made us your debtors. Death cannot demean you
Who taught us manhood in our stumbling youth.

A. L.

John Wesley Young, 1879-1932

Appreciation on behalf of the Association by H. E. Slaught

John Wesley Young rendered great and lasting service to the Mathematical Association of America in various significant ways. First, as a charter member he allied himself with that large group of forward looking men and women who from the outset sensed the importance of the principles for which the Association was to stand. He was among the first elected members of the Board of Trustees and served two terms (five years) in that capacity. Later he was elected vice-president and finally president for a two-year term, 1929–1930. Thus he served the Association in official capacity during a total of nine years and always with sane and constructive counsel.

Another kind of official service, quite different in character from the above and perhaps the most important of all that he rendered, was his significant work as chairman of the National Committee on Mathematical Requirements which was organized under the auspices of the Mathematical Association of America. The work of this committee, whose membership included outstanding representatives of both collegiate and secondary instruction, doubtless constituted the most exhaustive study of the teaching of secondary mathematics ever undertaken in this country. This study extended over a period of eight years, involving numerous face-to-face discussions by the entire committee (thirteen members), besides eight special investigations carried on by individual members and other invited contributors. More than one hundred organizations of teachers of mathematics contributed suggestions and criticisms for the consideration of the committee. The report of the committee, as finally published by the Association in 1923, a volume of 652 pages entitled "The Reorganization of Mathematics in Secondary Education," had a circulation of several thousand volumes and undoubtedly has produced, and will continue to produce, a profound influence wherever the teaching of secondary mathematics is under discussion.

Only the members of this committee and others who were in close touch with Professor Young can realize how masterful he was in directing the many lines of investigation and discussion, how effective he was in securing the active and helpful cooperation of so large a number of individuals and organizations, and how convincing he was in presenting the need and importance of this work to the General Education Board thereby securing the necessary financial support for conducting the investigations and publishing the report.

Professor Young rendered important service in connection with the College Entrance Examination Board, as a member of the American Committee of the International Commission on the Teaching of Mathematics, and finally, up to the very end of his active career, as chairman of the joint Committee on Funds representing both the Society and the Association. In this latter capacity he exhibited in high degree his fitness for leadership, his judicial and fair-minded consideration of all sides of a question, and his intense and ardent desire to serve

the best interests of the two mathematical organizations in their need of financial support.

Finally Professor Young rendered important service to the Association as a member of the Carus Monograph Committee and in particular as the author of monograph number four on *Projective Geometry* which has received favorable comment from many sources. It was in connection with the preparation and publication of this monograph that the writer came in closest contact with him and learned more completely to admire his many fine qualities.

Professor Young was a staunch supporter of the Mathematical Association of America and his great desire was to see its influence still further widened. To this thesis he directed his retiring presidential address in which he set forth a plan calculated to bring the Association into closer touch with undergraduates in our colleges and universities. That plan, Professor Young realized, is not feasible at present on account of the necessary financial support which it would involve. However, it is believed that some improvements along this line can be effected, at least partially, through modifications of certain departments of the Monthly. The Board of Editors have these matters under consideration and definite announcement may be expected in the near future. Meanwhile a re-reading of Professor Young's presidential address may serve to inspire us all with his spirit and to spur us on to more effective service in the cause which was so dear to him and for which, in a sense, he gave his life.

His Life and Scientific Activities by L. L. Silverman

John Wesley Young died at Hanover on February 17, 1932 at the early age of fifty-two. His death brought a feeling of profound sorrow throughout the country; for he was not only a devoted friend to his colleagues in Hanover, but a personal friend to many mathematicians throughout the land.

He was born November 17, 1879, at Columbus, Ohio. His father, William Henry Young, was a college professor and later an American consul in Baden-Baden, Germany. His mother, Marie Louise Widenhorn Young, was born in Paris, of French and German ancestry. In 1907 Professor Young was married to Mary Louise Aston of Columbus, Ohio. Their daughter, Mary Elizabeth, was graduated from Smith College in 1930 and was married in May of this year.

Professor Young received most of his early education in Germany. During the years of his father's consular service in Baden-Baden, 1889–1895, he attended the German Gymnasium and prepared himself to enter the university. He attended the institution in his native city and was graduated from Ohio State University in 1899. He then went for his graduate work to Cornell, where he received the doctor's degree in 1904. He taught at Northwestern, Princeton, University of Illinois, University of Kansas, University of Chicago and Dartmouth College. He came to Dartmouth in 1911 as head of the department of mathematics.

He was active in research; but he was also interested in the problems of teaching both elementary and advanced mathematics. He was appointed to many positions to which he devoted his energies and ability; and he was elected to positions of honor as a result of the recognition of the importance of his achievements. He was editor of the Bulletin, 1907–1925; on the editorial board of the Mathematics Teacher, 1921–1928; chief examiner in geometry, College Entrance Examination Board, 1913–1915. When in 1916 the Mathematical Association of America organized the National Committee on Mathematical Requirements for the purpose of investigating possible reforms in the teaching of secondary school mathematics, Professor Young was appointed chairman of this committee. To this work he devoted eight years, accomplishing results which will long affect American mathematics. He was also editor of a series of mathematical texts published by Houghton, Mifflin and Company. Professor Young was president of the Connecticut Valley Section of the Association of Teachers of Mathematics in New England, 1926–1927; vice-president of the American Mathematical Society, 1928–1930; and president of the Mathematical Association of America, 1929–1930. He was a member of Phi Beta Kappa and Sigma Xi, and of many learned societies both here and abroad.

He began his mathematical research[1] in the subject of groups, in which he always retained a keen interest. But his interest continually widened. He soon began to work in projective geometry and in logical foundations; and in 1910 he wrote, in collaboration with Professor Veblen, the well-known book on projective geometry, building the geometry upon a set of independent axioms, which were quite different from any that had hitherto been used. It was Professor Young's opinion that the axiomatic method was the best way to teach projective geometry in the university. Nevertheless he later wrote for the non-specialist an excellent book on projective geometry, the Carus Mathematical Monograph, No. 4, where the axiomatic method was entirely abandoned.

One paper written by Professor Young about twenty-five years ago has been widely quoted in the literature and is still highly spoken of by specialists in the field. This paper is entitled *General theory of approximations by functions involving a given number of arbitrary parameters*, and appeared in the Transactions of the American Mathematical Society in 1907. Tchebychev was the first to consider the problem of determining the polynomial $p(x)$ of given degree n which gives the closest approximation to the given function $\phi(x)$ on the interval (a, b). Tchebychev defines the approximation to be the "closest" when the maximum of $f(x)=|p(x)-\phi(x)|$ in the interval (a, b) is as small as possible. Professor Young considers the following more general problem. Let the class C contain $g(x; a_0, a_1, a_2, \cdots, a_n)$, a function of x and the parameters $a_0, a_1, \cdots, a_n$; and let f be identified with $U(g)$, the result of operating on g with some functional

[1] A complete bibliography of the books and articles by Professor Young is being prepared for publication.

operation U. If there exists a set of parameters a_i which renders the maximum of $U(g)$, as x varies over a given finite interval, as small as possible, the resulting function g is called a function of approximation in the class C with reference to U. It is then shown that under very general conditions, such a function of approximation exists, and a uniqueness theorem is obtained.

Professor Young felt that there should be in our colleges a course on the history of science, which would give the non-specialist the fundamental ideas of science as they developed through the ages. He urged the introduction of such a course at Dartmouth and himself undertook to teach the course for the first few years. It required a wide knowledge of science and much labor in preparation. The course which resulted was most valuable and stimulating to the students who were in it.

Professor Young's mathematical outlook may be described by four characteristics:

He was interested in the *unity* of mathematics. His was not a narrow interest in some special field. He listened with sympathy and understanding to ideas in every field; and often through his natural keenness he was able to make suggestions to specialists in fields far removed from his own. But while having an interest in and an understanding of a wide field of mathematical subjects, he always felt that there was an inner unity in mathematics. And this feeling applied also to the elementary field, where in his book, *Elementary Mathematical Analysis* (written in collaboration with F. M. Morgan), he tried to relate all the ideas of algebra, trigonometry and analytic geometry to the central notion of function.

He was interested in the process of *generalization*. Moore's General Analysis appealed to him greatly. The article on the general theory of approximation, referred to above, is an example of his desire for generalization. Another illustration is his article, *A new formulation for general algebra*, which appeared in the Annals in 1927. Nevertheless he did not approve of mere generalization as such; the generalization must be significant. In his retiring presidential address to this Association he refers to the generalization of the idea of continuity. He goes on to say: "The means employed was the concept of the 'development' of the range of the general variable. On the basis of this concept he (Professor Pitcher) arrived very naturally at the definition of 'continuity with respect to a given development.' He then proved the surprising theorem that *every function is continuous with respect to some development of the range.* The result may be of interest—I am inclined to think it is—but as a generalization of continuity it is obviously futile."

What appealed to Professor Young most, perhaps, was the *beauty* of mathematics. He often spoke of mathematics as a fine art; and like Professor Bôcher, he saw in the combination of the creative and imaginative activities of the mathematician something which resembled the activities of the artist. On a number of occasions he made popular addresses on this subject, taking his illus-

trations largely from the introduction of ideal elements in projective geometry.

The *popularization* of mathematics was an idea close to Professor Young's heart. Here is a subject full of beautiful ideas, relations and generalizations. Would it not be splendid to eliminate the technical side so far that these beautiful things might be appreciated also by those students who do not specialize in mathematics? In answer to this question, his book of *Lectures on the fundamental concepts of algebra and geometry* is a masterpiece. He was particularly interested in the Synoptic Course at Dartmouth, a survey course endeavoring to give to the non-specialist, with the minimum of technique, some understanding and appreciation of a number of topics in advanced mathematics.

Professor Young lived the life of a philosopher. He was fearless in his convictions, sympathetic to his friends, fairminded to his opponents. These features were such an outstanding part of his character that they exerted a marked influence on all who came in contact with him. His life was regrettably short. A consolation to his friends is that it was a very happy one.

EDITORIAL POLICY

The American Mathematical Monthly is "devoted to the interests of collegiate mathematics"—if one accepts the statement on the cover. But if one looks further than the cover page, he may find reason to doubt the whole-heartedness of this devotion. There has been a rapidly increasing demand for space in those American journals whose function is the publication of the results of mathematical research; and the devotion of the Monthly to collegiate mathematics has not been so complete as to leave it entirely indifferent to the attractions in this somewhat different direction. It has been, and is being, tempted.

In his retiring presidential address which appeared in the January issue of the Monthly, the late Professor Young called attention to the "marked shift in the appeal" of the Monthly in recent years; and he suggested the establishment of a "new magazine devoted to the mathematical interests of undergraduates." But the necessary funds and facilities are not available for such an undertaking, and for the present the Monthly must see to it that this field is not entirely neglected.

While no sharp change of policy is contemplated, it is the desire of the editors that every number of the Monthly shall contain a certain amount of material that will be of interest to the more capable undergraduate. Good expository papers will be especially acceptable—papers which begin with ideas easily understood by the better college senior and then lead him on into unfamiliar fields. Some research papers will be accepted as they have been in the past, but subject to certain restrictions as to length and style of presentation. It is clear that

a style suitable for a paper in the Transactions, for instance, assuming on the part of the reader a general knowledge of, and easy access to, the literature of the subject, familiarity with technical language and symbols, and a high degree of mathematical maturity and experience, would not be a style suitable for the presentation of the same material to the readers of the Monthly.

But it is in the departments of *Problems and Solutions*, *Questions andDiscussions*, and *Mathematics Clubs* that we have our best opportunity to offer mathematical stimulant to our hypothetically thirsty undergraduate. Beginning this fall, probably with the October number, we expect to subdivide the department of *Problems and Solutions* into *Elementary Problems* and *More Difficult Problems*. Professor W. F. Cheney of the Connecticut Agricultural College will take charge of the division of elementary problems. He is hoping for the cooperation of all our readers in securing a steady supply of suitable material. The problems should be relatively easy without being too trivial, and should have, as far as possible the sort of novelty that will arouse interest. Much material will doubtless be sent in which, for one reason or another, can not be used; but it is hoped that this will not offend or discourage contributors.

The department of *Questions and Discussions* is to be expanded into *Questions, Discussions, and Comments* in order to allow somewhat more latitude in the type of papers included; and the department will use some papers of quite elementary character when they appear to be appropriate and sufficiently interesting.

In the department of *Mathematics Clubs* we welcome not only the reports of club activities, but material suitable for club programs.

There is need for the cooperation of our readers in another direction. Very few college undergraduates subscribe to the Monthly. It will reach them only as college teachers call their attention to it, and make the issues regularly available in a library or otherwise. Institutional membership in the Association provides two copies of the Monthly, one for the library and one for the individual use of mathematics club members. If teachers will, from time to time call attention to particular problems or discussions that might possibly interest their better students, we can hope to function to some extent as a stimulus to undergraduate mathematical activity.

THE MATHEMATICAL ASSOCIATION OF AMERICA

The following twenty-nine persons and one institution have been elected to membership in the Association on applications duly certified:

To Individual Membership

G. A. BAKER, Ph.D. (Illinois) Head of Dept., Shurtleff Coll., Alton, Ill.

C. J. BLACKALL, M. S. (Notre Dame) Instr., Math. and Physics, St. Thomas Coll., St. Paul, Minn.

J. L. BOTSFORD, A.B. (Washington) Teaching fellow, California Inst. of Tech., Pasadena, Calif.

A. L. BUCKMAN, A.M. (California) Teaching fellow, Univ. of California, Berkeley, Calif.

Iris Callaway, A.M. (Peabody) Asso. Prof., State Teachers Coll., Athens, Ga.

A. S. Corbin, B.S. (Worcester Poly. Inst.) Instr., Coll. of William and Mary, Williamsburg, Va.

D. Cromwell, A.B. (Baylor) Asst. Prof., John Tarleton Jr. Coll., Stephenville, Tex.

U. P. Davis, A.M. (Florida) Instr., Univ. of Florida, Gainesville, Fla.

Ola Eskelson, M.S. (Illinois) Teacher, High School, Borger, Tex.

J. D. Fitzpatrick, A.B. (Marquette) Instr., Creighton Univ., Omaha, Neb.

Eunice Hutto, A.M. (Alabama) Head of Dept., Bob Jones Jr. Coll., College Point, Fla.

Louise Johnson, A.B. (Colorado) Grad. Asst., Univ. of Colorado, Boulder, Colo.

Harris Jones, B.S. (Mass. Inst. of Tech.) Lt. Col., U. S. Army, Prof., U. S. Milit. Acad., West Point, N. Y.

E. R. Keller, A.M. (Tennessee) Asst., Univ. of Kentucky, Lexington, Ky.

Kate K. Knight, A.B. (East Central S.T.C.) Prof., East Central State Teachers Coll., Ada, Okla.

Ruth G. Mason, Ph.D. (Chicago) Private research, Berkeley, Calif.

Polly P. Nelson, A.M. (Chicago) Instr., Hunter Coll., New York, N. Y.

W. V. Parker, Ph.D. (Brown) Prof., Mississippi Woman's Coll., Hattiesburg, Miss.

H. H. Pixley, Ph.D. (Chicago) Instr., Coll. of the City of Detroit, Detroit, Mich.

E. S. Quade, B.S. (Florida) Instr., Univ. of Florida, Gainesville, Fla.

J. H. Rose, Ph.B. (Wisconsin) Asst., Univ. of Wisconsin, Madison, Wis.

R. G. Sanger, Ph.D. (Chicago) Instr., Univ. of Chicago, Chicago, Ill.

H. L. Schug, B.S. (Lafayette) Research Engineer, The Hoover Co., N. Canton, Ohio

V. B. Teach, M.S. (Ohio State) Asso. Prof., Armour Inst. of Tech., Chicago, Ill.

V. Thébault, Inspecteur d'assurances, Le Mans, France

A. M. Wallace, M.S. (Oklahoma) Prof., East Central State Teachers Coll., Ada, Okla.

Irene L. Wente, M.S. (Chicago) Instr., State Coll., Brookings, S. D.

Ada H. West, A.M. (Kansas) Asst. Prof., State Teachers Coll., Peru, Neb.

L. S. Winton, B.S. (Grove City) Grad. student, Oberlin Coll., Oberlin, Ohio

To Institutional Membership

Georgia State College for Men, Tifton, Ga.

W. D. Cairns, *Secretary*

THE NINTH ANNUAL MEETING OF THE MICHIGAN SECTION

The ninth annual meeting of the Michigan Section of the Mathematical Association of America was held in conjunction with the Michigan Academy of Science at the University of Michigan on Saturday, March 19, 1932. Professor J. P. Everett, chairman of the Section, presided.

There were over eighty persons in attendance, including the following forty-four members of the Association: H. M. Ackley, N. H. Anning, W. L. Ayres, J. W. Baldwin, W. D. Baten, W. M. Borgman, Jr., J. W. Bradshaw, J. B. Brandeberry, W. D. Cairns, R. V. Churchill, R. W. Clack, C. C. Craig, S. E. Crowe, Wayne Dancer, Albertus Darnell, L. C. Emmons, C. M. Erikson, J. P. Everett, Peter Field, George Fisanick, K. W. Folley, W. B. Ford, B. C. Getchell, V. G. Grove, T. H. Hildebrandt, L. A. Hopkins, E. E. Ingalls, C. E. Love, A. L. Nelson, H. L. Olson, L. C. Plant, J. E. Powell, G. Y. Rainich, C. C. Richtmeyer, L. J. Rouse, T. R. Running, R. H. Schoonover, E. R. Sleight, C. E. Stout, A. G. Swanson, J. F. Thomson, T. O. Walton, R. L. Wilder, J. B. Winslow.

At the business meeting the following officers were elected for the ensuing year: Chairman, R. W. Clack, Alma College; Secretary-Treasurer, W. L. Ayres, University of Michigan; Member of the Executive Committee, S. E. Crowe, Michigan State College. A motion was made and carried recommending to the new Executive Committee that they consider the advisability of holding two meetings of the Section each year, one, the annual meeting at the University in conjunction with the Michigan Academy of Science, and the other in October at some other institution. A luncheon, attended by fifty-five persons, was held at noon at the Michigan League.

The following papers were presented at the morning meeting:

1. "Evaluation of a certain summation" by Professor W. D. Baten, University of Michigan.
2. "Heat conduction and convection from tall, hot vertical cylinders and high walls at uniform temperature" by Professor W. S. Kimball, Michigan State College, by invitation.
3. "Systems of linear difference equations with constant coefficients" by Professor C. M. Erikson, Michigan State Normal College.
4. "On the Ostogradsky-Hermite method of integration of certain types of functions" by D. K. Kazarinoff, University of Michigan, by invitation.
5. "Edge conditions for double integrals in the calculus of variations" by J. E. Powell, Michigan State College.
6. "Character through mathematics" by Professor R. W. Clack, Alma College.
7. "A mathematical analysis of the design of spark plugs" by Professor R. V. Churchill, University of Michigan.
8. "Some applications of mathematics in physical chemistry" by L. O. Case, University of Michigan, by invitation.
9. "The imaginary in analytic geometry" by H. M. Ackley, Western State Teachers College.
10. "Functions whose ranges are simply ordered sets" by Professor K. W. Folley, College of the City of Detroit.
11. "A method for approximating real roots of equations by the principle of areas" by Professor T. R. Running, University of Michigan.

At the afternoon session of the Section the following program was given:

12. "An undergraduate course leading to the study of wave mechanics" by Professor W. D. Cairns, Oberlin College (Guest speaker).
13. "A special slide rule for a woolen mill problem" by Professor E. E. Ingalls, Albion College.
14. "Focal and intersector hypersurfaces from a projective standpoint" by W. M. Borgman, Jr., College of the City of Detroit.

LOUIS A. HOPKINS, *Secretary*

THE 1932 MEETING OF THE TEXAS SECTION

The 1932 meeting of the Texas Section of the Mathematical Association of America was held in Waggener Hall of the University of Texas, Austin, Texas, on Saturday, January 30, 1932. Local arrangements were made by a committee consisting of Professors H. J. Ettlinger, chairman, E. L. Dodd and R. G. Lubben.

The attendance was forty-four, including the following twenty-seven members of the Association: W. N. Barnes, P. M. Batchelder, L. W. Blau, A. A. Blumberg, L. M. Blumenthal, H. E. Bray, J. E. Burnam, E. L. Dodd, Nat Edmonson, Jr., H. J. Ettlinger, G. C. Evans, G. W. Evans, A. E. Finlay, L. R. Ford, C. A. Gilley, H. H. Halperin, J. W. Harrell, Goldie P. Horton, R. G. Lubben, R. L. Moore, C. A. Murray, Brother Norbert, P. K. Rees, W. A. Rees, B. P. Reinsch, C. R. Sherer, F. W. Sparks.

At the business meeting Professor C. R. Sherer of Texas Christian University was elected chairman of the Section for the coming year and W. A. Rees of the Houston Junior College was chosen vice-chairman for the same period.

The entertainment features of the meeting consisted of a sight-seeing trip about Austin conducted by members of the University of Texas mathematics department immediately following the afternoon session; and a dinner, complimentary to the members of the Texas Section and their guests, given by the University of Texas. The principal speaker at the dinner was Dr. H. Y. Benedict, President of the University.

The following program was presented:

1. "The isometric circle" by Professor L. R. Ford, The Rice Institute.
2. "Relations between projective geometry and trigonometry" by Professor M. B. Porter, University of Texas, by invitation.
3. "The postulate of continuity in dynamics" by Dr. C. H. Dix, The Rice Institute, by invitation.
4. "The cooling problem for spherical regions" by Dr. W. M. Rust, The Rice Institute, by invitation.
5. "Some recent trends and applications of statistics" by Professor E. L. Dodd, University of Texas.
6. "Simplifications for high school mathematics" by G. W. Evans.

Abstracts of these papers follow:

1. This paper gave a history of the isometric circle with some of the major results achieved by its use, and indicated some open problems together with possible avenues to their solution.

2. This paper discussed the place of projective geometry in undergraduate teaching together with some application to trigonometry. The straight line equation was interpreted as a projective transformation in one dimension, and the fundamental formulae of analytic trigonometry were derived.

3. In all classical theories dynamical quantities have been assumed to vary

over a continuous range of values. It is plain that a mathematical continuum corresponds to no direct observation and so enters the theory for reasons other than observational. In studying this problem the author has been led to deny the postulate and investigate any consequences of this denial. In place of Hamilton's principle consider

$$\delta \sum_{i=1}^{n} L(x(t_i),\ [x(t_i)]_t,\ t_i)\Delta_+ t_i = 0,$$

where $[x(t_i)]_t$ is the velocity $\{x(t_{i+1})-x(t_i)\}/\{t_{i+1}-t_i\}$ and $\Delta_+ t_i = t_{i+1}-t_i$. L is the Lagrangian function and δ indicates the first variation of the sum. We here postulate a finite sequence of instants $t_1, t_2, \cdots, t_{n+1}$. The result is the analog of the Lagrangian equation

$$\frac{\partial L}{\partial x} - \left[\frac{\partial L}{\partial [x]_t}\right] = 0,$$

where

$$[f(t_i)]_{\bar{t}} = \{f(t_i) - f(t_{i-1})\}/\{t_{i+1} - t_i\}.$$

The transformation given by

$$p - \frac{\partial L}{\partial [x]_t} = 0$$

$$H(x, p, t_i) = [x]_t p - L(x, [x]_t, t_i)$$

leads to the Hamiltonian equations

$$[p]_{\bar{t}} = -\frac{\partial H}{\partial x}, \qquad [x]_t = \frac{\partial H}{\partial p}.$$

It will be noticed that partial derivatives occur in spite of the discontinuity in the time, but these enter when we select from a continuum of possible $x(t_i)$'s the one, for each value of t_i, that makes the first variation of the sum zero. At this point it should be mentioned that in quantum mechanics the possible values of p and q are a continuum; but it cannot be said that the transitions are continuous. In conclusion it may be stated that the postulate of an exactly given sequence of instants can no more be placed on an observational basis than the postulate of a continuum. The discontinuous time furnishes a new parameter but at the same time introduces difficulties when we use coordinates other than rectangular ones since the square of the velocity as it appears in L is no longer in the simple form $[x]_t^2 + [y]_t^2$.

4. This paper treats the solution of the cooling problem for spherical regions by a slight modification of the methods used for linear regions in another paper.[1]

[1] *Integral equations and the cooling problem for several media,* American Journal of Mathematics, vol. 54 (1932), p. 190.

A general uniqueness theorem is established and a solution satisfying this theorem is constructed by use of the fundamental solution

$$\frac{1}{r(t-t')^{1/2}} e^{-a^2(r-r')^2/4(t-t')}$$

of the differential equation for the flow of heat in spherical regions. This solution is in terms of the solutions of a set of integral equations. It is shown that these integral equations can be solved by constructing a set of Volterra integral equations with bounded kernels, whose solutions are shown to be absolutely continuous and to have derivatives which satisfy the original set of integral equations.

5. As the title indicates, this paper consisted of a discussion of some late developments in the field of statistics. Particular attention was paid to applications in the biological sciences.

6. This paper discussed simple and straightforward methods already in use in some schools for arithmetic and algebra and proposed methods in geometry. These methods do not represent innovations, but a return to the ancient ways, with no omission of essential topics and with emphasis on the connection between "great ideas and ordinary thoughts," on generality and on self-criticism.

NAT EDMONSON, JR., *Secretary*

THE EIGHTEENTH ANNUAL MEETING OF THE KANSAS SECTION

The eighteenth annual meeting of the Kansas Section of the Mathematical Association of America was held in the High School Building, Topeka, Saturday, February 13, 1932. The morning session was a joint meeting with the Kansas Association of Mathematics Teachers. The two organizations held separate meetings in the afternoon. Professor J. J. Wheeler, chairman of the Section, presided at both the morning and afternoon sessions.

Sixty-four persons were in attendance, including the following twenty-six members of the Association: (Note: This was a fifty per cent attendance of the Kansas membership.) C. H. Ashton, R. W. Babcock, Wealthy Babcock, Florence L. Black, R. D. Daugherty, Lucy T. Dougherty, W. H. Garrett, W. A. Harshbarger, W. H. Hill, A. S. Householder, Emma Hyde, W. C. Janes, H. E. Jordan, C. F. Lewis, Anna Marm, Thirza A. Mossman, Arthur Ollivier, O. J. Peterson, A. W. Philips, P. S. Pretz, B. L. Remick, J. A. G. Shirk, G. W. Smith, W. T. Stratton, A. E. White, Ferna E. Wrestler.

At the business session, the following officers were elected for the coming year: Chairman, Professor O. J. Peterson, State Teachers College, Emporia; Vice-Chairman, Professor W. A. Harshbarger, Washburn College; Secretary-Treasurer, Lucy T. Dougherty, Junior College, Kansas City.

Between sessions the two Associations enjoyed a most delightful luncheon,

served in the Faculty dining room of the palatial new building of the Topeka High School.

At the joint session the following program was presented:

1. "Every day class-room problems" by Amy Irene Moore, State Teachers College, Hays.

2. "Common statistical errors" by Professor Dinsmore Alter, Department of Astronomy, University of Kansas.

3. "Orientation in mathematics" by Professor O. J. Peterson, State Teachers College, Emporia.

At the session in the afternoon, the following papers were presented:

1. "Singular points of polar tangent curves" by Professor W. T. Stratton, State College, Manhattan.

2. "Recent advances in mathematical statistics" by Professor J. A. G. Shirk, State Teachers College, Pittsburg.

3. "On contact of curves in a four-dimensional space" by B. C. Moore, University of Kansas, by invitation.

Abstracts of these papers follow:

1. In this discussion of the general polar tangent curves $\rho = a \tan (p\theta/q) + k$ where p and q are integers and a and k are any constants, Professor Stratton showed that the curves are algebraic and of order $2(p+q)$ if q is odd and of order $(p+q)$ if q is even. He next determined the axes of symmetry, the polar period and the conditions for maxima and minima, and then pointed out certain properties common to all of the curves. The greater part of the paper was devoted to the development of the number of singular points other than the origin. The number of double points can readily be expressed in terms of p and q.

2. Professor Shirk showed large graphs of various distributions, each one being fitted with the Pearsonian equation best adapted. The distributions of the mean and the standard deviation were shown, with a discussion of the wrong use of the probable error of the standard deviation as quite frequently used. Graphs of correlated varities were also shown, together with graphs of the distribution of the correlation coefficient, which is generally quite different from the normal distribution. Fisher's methods of testing the significance of the standard deviation and of the correlation coefficient were presented. Fisher's method of curve fitting by the method of "Maximum likelihood" was illustrated, and its greater efficiency over the "Method of moments" was indicated.

3. Mr. Moore showed that in four-dimensional space two curves which have contact of order n determine a principal tangent plane. In general the curves do not determine a principal line or a principal point. It was shown that the order of contact of the projections of the two curves from a line on the common osculating plane may be made, by proper choices of the line, as high as $n+4$. The method of proof is that devised by E. B. Stouffer in proving analogous theorems in three-dimensional space.

LUCY T. DOUGHERTY, *Secretary*

ON THE PROBLEM OF RUNS

By J. V. USPENSKY, Stanford University

Let us consider a series of n trials independent in regard to a certain event E, the probability of this event being p in each trial. If, in the course of the trials, the event E occurs at least r times in succession, we say that there is a *run of* r successes. For instance, if, tossing a coin 1000 times, heads occur 20 or more times in succession, we have a run of 20 heads in a series of 1000 tosses. The problem of finding the probability that there should be a run of r successes in a series of n trials is an old one propounded and solved by De Moivre. The same problem has been discussed by many other writers on probability, but none of them, so far as I am aware, ever gave an expression for the required probability adapted to the particularly important case of a very long series of trials.

It is the purpose of this note to establish approximate formulae adapted to the case of a long series of trials.

1. De Moivre reduced the problem to an equation in finite differences. Let y_n be the probability of a run of r in n trials. It is easy then to establish the following equation in finite differences:

$$y_{n+1} = y_n + (1 - y_{n-r})p^r q$$

where, as usual, $q = 1 - p$ is the probability of a failure. Since evidently

$$y_0 = y_1 = \cdots = y_{r-1} = 0,\ y_r = p^r,$$

the preceding equation gives a means to determine successively y_{r+1}, y_{r+2}, y_{r+3} and in general y_n for every $n > r$. One can even establish a general expression for y_n, but this expression has little practical value if n is a large number.

It is more convenient to consider not y_n but the complementary probability $z_n = 1 - y_n$ determined by the equation

$$z_{n+1} - z_n + qp^r z_{n-r} = 0 \tag{1}$$

together with the initial conditions

$$z_0 = z_1 = \cdots = z_{r-1} = 1;\ z_r = 1 - p^r. \tag{2}$$

Using (1) and (2) one easily finds the following expression for the generating function of probabilities $z_0, z_1, z_2, \cdots$:

$$\phi(\xi) = \frac{1 - p^r \xi^r}{1 - \xi + qp^r \xi^{r+1}} = \sum_{n=0}^{\infty} z_n \xi^n. \tag{3}$$

The natural way to obtain an appropriate expression of z_n is to resolve the rational function in the left member of (3) into simple fractions corresponding to various roots of the denominator and expand those fractions in power series of ξ. However, to attain definite conclusions following this method, we must first seek information concerning roots of the equation

$$1 - \xi + qp^r\xi^{r+1} = 0.$$

2. Let

$$f(\xi) = \xi - 1 - \alpha\xi^{r+1},$$

where

$$\alpha = p^r(1 - p).$$

When p varies from 0 to 1 the maximum of $p^r(1-p)$ is attained for $p=r/(r+1)$ and is $r^r/(r+1)^{r+1}$, so that $\alpha \leqq r^r/(r+1)^{r+1}$ in all cases. To deal with the most interesting case, we shall assume

$$p < r/(r + 1) \tag{4}$$

which involves

$$\alpha < r^r/(r + 1)^{r+1}$$

and we leave it to the reader to find out how the following discussion should be modified if $p \geqq r/(r+1)$.

When ξ starts to increase from 0 the function $f(\xi)$ steadily increases and attains a positive maximum for $\xi = \xi_0$ where

$$(r + 1)\alpha\xi_0{}^r = 1,$$

after which $f(\xi)$ decreases steadily to negative infinity. Hence there are two positive roots of the equation $f(\xi) = 0$: ξ_1 which is less than $(r+1)/r$ and another root greater than this number. This root is $1/p$ if the condition (4) is fulfilled.

The remaining roots are all imaginary if r is *odd* and there is one negative root among them if r is *even.*

Now we shall prove that the absolute value of every imaginary or negative root is $>1/p$. For let ρ be the absolute value of any such root. We have first

$$f(\rho) = \rho - 1 - \alpha\rho^{r+1} < 0$$

so that ρ belongs either to the interval $(0, \xi_1)$ or to the interval $(1/p, \infty)$, and if we can show that $\rho > \xi_0$ then ρ can be only $>1/p$. If the root we consider is negative, ρ satisfies the equation

$$F(\rho) = 1 + \rho - \alpha\rho^{r+1} = 0$$

and since $F(\rho)$ increases till a positive maximum for $\rho = \xi_0$ is reached, and then decreases, the root of $F(\rho) = 0$ is necessarily $>\xi_0$. If $\xi = \rho e^{i\theta}$ is an imaginary root of $f(\xi) = 0$ we have, equating imaginary parts,

$$\alpha\rho^r \frac{\sin (r + 1)\theta}{\sin \theta} = 1. \tag{5}$$

But whatever θ may be

$$\left|\frac{\sin (r + 1)\theta}{\sin \theta}\right| \leqq r + 1,$$

the equality sign being excluded if $\sin\theta \neq 0$. Hence

$$(r+1)\alpha\rho^r > 1$$

which implies $\rho > \xi_0$. The statement is thus completely proved.

3. The equation

$$\xi - 1 - \alpha\xi^{r+1} = 0$$

can be exhibited in the form

$$\frac{1}{\xi} + \alpha\xi^r = 1$$

Substituting here $\xi = \rho e^{i\theta}$ and again equating imaginary parts, we get

$$\alpha\rho^{r+1}\sin r\theta = \sin\theta,$$

and, combining this with (5),

$$\rho = \frac{\sin(r+1)\theta}{\sin r\theta}, \qquad \alpha = \frac{(\sin r\theta)^r \sin\theta}{[\sin(r+1)\theta]^{r+1}}\cdot$$

If the imaginary part of ξ is positive the argument θ is contained between 0 and π. In this case it cannot be less than $\pi/(r+1)$ nor greater than $\pi - \pi/(r+1)$. For, if $0 < \theta < \pi/(r+1)$,

$$\frac{\sin r\theta}{r\theta} > \frac{\sin(r+1)\theta}{(r+1)\theta},$$

or

$$\frac{\sin r\theta}{\sin(r+1)\theta} > \frac{r}{r+1}\cdot$$

At the same time

$$\frac{\sin\theta}{\sin(r+1)\theta} > \frac{1}{r+1},$$

and hence

$$\alpha = \left\{\frac{\sin r\theta}{\sin(r+1)\theta}\right\}^r \frac{\sin\theta}{\sin(r+1)\theta} > \frac{r^r}{(r+1)^{r+1}}$$

which is impossible. That θ cannot be greater than $\pi - \pi/(r+1)$ follows simply because in this case $\sin(r+1)\theta$ and Sin $r\theta$ would be of opposite signs and ρ would be negative.

As $\pi/(r+1) \leqq \theta \leqq \pi - \pi/(r+1)$, we have

$$\rho\sin\theta \geqq \rho\sin\frac{\pi}{r+1}\cdot$$

On the other hand $\sin x > 2x/\pi$ if $0 < x < \pi/2$, and $\rho > 1/p$. Hence

$$\rho \sin\theta > \frac{2}{(r+1)p} \cdot$$

Thus imaginary parts of all complex roots have the same lower bound

$$\frac{2}{(r+1)p}$$

of their absolute values.

4. Denoting the roots of the equation $f(\xi)=0$ by $\xi_k (k=1, 2, 3, \cdots, r+1)$ we have

$$\phi(\xi) = \sum_{k=1}^{r+1} \frac{1-p\xi_k}{(1-p)\xi_k(r+1-r\xi_k)}\left(1-\frac{\xi}{\xi_k}\right)^{-1}.$$

Hence, expanding each term into power series of ξ and collecting coefficients of ξ^n we find

$$z_n = \sum_{k=1}^{r+1} \frac{1-p\xi_k}{(1-p)\xi_k} \cdot \frac{\xi_k^{-n}}{r+1-r\xi_k} \cdot$$

For every imaginary root we have

$$\left|\frac{(1-p\xi_k)\xi_k^{-n}}{(1-p)\xi_k(r+1-r\xi_k)}\right| < \frac{r+1}{r(1-p)}\, p^{n+2}$$

since

$$|\xi_k|^{-1} < p; \left|\frac{1}{\xi_k} - p\right| < 2p, \quad \frac{1}{|r+1-r\xi_k|} < \frac{(r+1)p}{2r} \cdot$$

If r is *odd* there are $r-1$ imaginary roots and the part in the expression of z_n due to them in absolute value is less than

$$\frac{(r+1)(r-1)}{r(1-p)}\, p^{n+2} < \frac{r}{1-p}\, p^{n+2}.$$

The term corresponding to the root $1/p$ vanishes, so that finally

$$z_n = \frac{1-p\xi_1}{(1-p)\xi_1} \cdot \frac{\xi_1^{-n}}{r+1-r\xi_1} + \theta \frac{r}{1-p}\, p^{n+2}$$

where $|\theta| < 1$ and ξ_1 denotes the least positive root of the equation

$$\xi = 1 + \alpha\xi^{r+1}.$$

If r is *even* there is one negative root. The part of z_n corresponding to this root is less than

$$\frac{2p^{n+2}}{(1-p)r}.$$

The whole contribution due to imaginary and negative roots in absolute value is less than

$$\frac{r^2-r}{r(1-p)}\,p^{n+2} < \frac{r}{1-p}\,p^{n+2}$$

Thus no matter whether r is odd or even, we have

$$z_n = \frac{1-p\xi_1}{(1-p)\xi_1}\cdot\frac{\xi_1^{-n}}{r+1-r\xi_1} + \theta\frac{r}{1-p}\,p^{n+2};\ -1<\theta<1. \tag{6}$$

This is the required expression for z_n, excellently adapted to the case of a large n since then the remainder term involving θ is completely negligible in comparison with the first principal term.

The root ξ_1 can be found either by direct solution of the trinomial equation following Gauss's method, or by application of Lagrange's series. Applying Lagrange's series we have

$$\xi_1 = 1 + \alpha + \sum_{l=2}^{\infty} \frac{(lr+2)(lr+3)\cdots(lr+l)}{l!}\,\alpha^l$$

$$\log \xi_1 = \alpha + \sum_{l=2}^{\infty} \frac{(lr+1)(lr+2)\cdots(lr+l-1)}{l!}\,\alpha^l$$

both series being convergent if $|\alpha| < r^r/(r+1)^{r+1}$, and this condition is satisfied.

5. Let us apply the approximate formula (6) to the case $p=q=\frac{1}{2}$ and $r=10$. Using Lagrange's series we find

$$\xi_1 = 1.0004908$$

and

$$z_n = 1.003937 \times (1.0004908)^{-n} + 5\theta/2^n.$$

Hence for $n=100$, 1000, 10000 respectively

$$z_n = 0.9559;\ 0.6148;\ 0.0074$$

so that, for instance, the probabilities of a run of at least 10 heads in 100, 1000, 10000 throws of a coin are, respectively

$$0.0441;\ 0.3852;\ 0.9926.$$

Thus in 10,000 throws it is quite likely that heads would turn up 10 or more times in succession.

In general for a given r and increasing n the probability y_n tends to 1, so that in a very long series of trials runs of any length are extremely likely to occur, a conclusion which at first sight might seem paradoxical.

THE PROJECTIVE THEORY OF ORTHOPOLES

By SISTER MARY CORDIA KARL, College of Notre Dame of Maryland

Introduction

The term "orthopole of a line with respect to a given triangle" was first used by Neuberg to designate a point constructed as follows:[1] Perpendiculars are drawn from the vertices of the given triangle $A_1A_2A_3$ to the given line l, cutting l in B_1, B_2, B_3, respectively. Then are drawn lines B_1C_1, B_2C_2, B_3C_3, perpendicular respectively to A_2A_3, A_3A_1, A_1A_2. These three perpendiculars are concurrent on a point M which is the orthopole of l with respect to the given triangle. Neuberg showed that while the line l gave rise to a unique orthopole M, yet M was equally well determined by each of two other lines. The three lines that gave rise to a common orthopole he designated as "associated" lines. He also proved that when the orthopole varied on a line, the three associated lines varied on a line parabola.

It is the purpose of this paper to study this (1, 1) correspondence between lines and line parabolas, to deduce therefrom the (1,3) correspondence between orthopoles and their associated lines, and then to prove by means of projective geometry some theorems concerning orthopoles. The theorems, with the exception of 1 and its Corollary, 16, 17, and the construction for finding the lines associated with a given one, have all been given before. They may be found in the works of Neuberg,[2] Cwojdzinski,[3] Goormaghtigh,[4] Ramler.[5] In the proofs as given by these authors, the methods of synthetic and analytical geometry have been used, various types of coordinate systems being employed.

Since the (1,3) correspondence will be defined by means of a net of line conics, some preliminary theorems concerning such nets will be cited.[6]

Theorem A. In a net of line conics, the point pairs that are the centers of the pencils that constitute the degenerate conics of the net, lie on a cubic,[7] which, in the case of a base line, degnerates into that line, l, and a residual conic, C^2.

The following theorems refer to nets of line conics having a base line.

Theorem B. There is a (1,1) correspondence between the points on l and those on C^2, each pair of corresponding points forming a degenerate conic. In

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[1] *Sur une transformation des figures*, Nouv. Corr. Math., vol. 4, p. 379, 1878.

[2] *Die Verwandtschaft zwischen einer Geraden und ihrem Lotpunkt in Bezug auf ein Dreieck*, Archiv der Mathematik und Physik, vol. 3, p. 89, 1902.

[3] *Der Lotpunkt, ein neuer merkwürdiger Punkt des Dreiecks*, Archiv der Mathematik und Physik, vol. 1, p. 175, 1901.

[4] *The Orthopole*, Tohoku Mathematical Journal, 1927, p. 78.

[5] *The orthopole loci of some one-parameter systems of lines referred to a fixed triangle*, American Mathematical Monthly, vol. 37, p. 130. March 1930.

[6] Proofs of these theorems may be found in the original copy of the writer's dissertation on the subject in the Johns Hopkins University library.

[7] Reye, *Geometry of Position* (1898), p. 231.

this correspondence, the points common to l and C^2 correspond doubly, i.e., to each considered as a point on l corresponds the other considered as a point on C^2, and to each considered as a point on C^2 corresponds the other considered as a point on l.

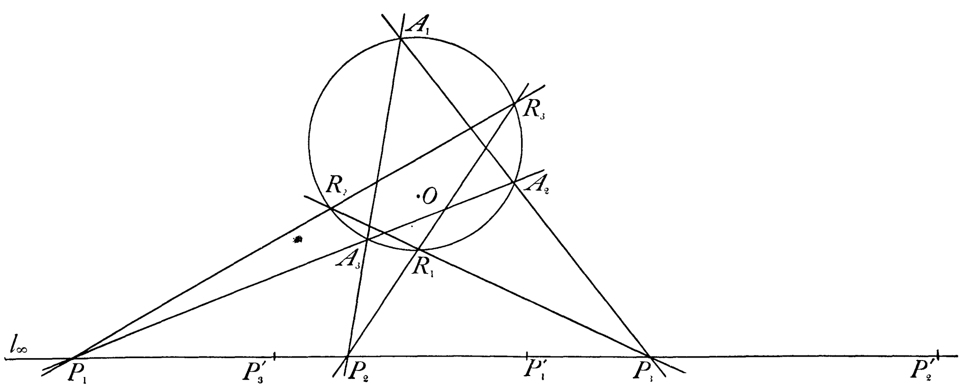
Theorem C. If the centers of the pencils that constitute degenerate conics of the net be joined to any point P on C^2, these lines are conjugate pairs in an involution of lines on P.

Theorem D. If a line l and a conic C^2 be chosen arbitrarily, the net having these for the locus of the centers of the pencils that form the degenerate conics is fixed by assigning arbitrarily one point pair—one point on l, the other on C^2—as one degenerate line conic of the net, the points common to l and C^2 being the centers of a second degenerate conic.

Theorem E. If 1, 2, 3 are three points on l which form with the three vertices of a triangle inscribed in C^2 three degenerate conics, and if $1'$, $2'$, $3'$ are the points in which the sides of the triangle opposite 1, 2, 3, respectively, cut l, then the three pairs $1,1'$; $2,2'$; $3,3'$ are conjugate pairs of an involution.

Basis for Choice of Net and Determining Elements of the Correspondence

Neuberg proved that as the orthopole varies on the side A_1A_2 of the base triangle, the three associated lines vary on the degenerate line parabola having as centers of the two pencils of lines R_3 and P_3'; on A_2A_3, centers R_1, P_1'; on A_1A_3, centers R_2, P_2'; where R_1, R_2, R_3 are points on the circumcircle diametrically opposite to A_1, A_2, A_3; and P_1', P_2', P_3' are conjugates in the absolute involution on l_∞ with respect to the points P_1, P_2, P_3, these being the points of intersection of l_∞ and the sides of the triangle respectively opposite to A_1, A_2, A_3.



The triangles $A_1A_2A_3$ and $R_1R_2R_3$, being perspective from center O of the circle, are also perspective from l_∞. It is readily seen that the preceding is an example of Theorem E, the involution of points on l being the absolute involution.

Neuberg also proved that as a line varies on a point P on the circumcircle

of the base triangle, its orthopole varies on the Simson line of that point with respect to the base triangle. Since l_∞ cuts the circle in the circular points, I and J, its orthopole is the intersection of the Simson lines of I and J. These lines, however, coincide with l_∞. Thus, any point on l_∞ is the orthopole of l_∞. Also, the two lines associated with l_∞ pass one through I and the other through J, the two intersecting at some point K on the circle. It follows that as a point varies on l_∞, the associated lines vary on the degenerate parabola IJ.

In selecting the elements that determine the (1,3) correspondence that will be defined later, the choice is so made that to the three sides of the base triangle $A_1A_2A_3$ will correspond the parabolas as given by Neuberg, and that to l_∞ will correspond IJ. Since four independent pairs of the (1,3) correspondence give rise to the same four pairs in the Neuberg definition, the two definitions must be equivalent.

Determination of the Net and the Correspondence

The net of line conics to be used is set up as follows:

(a) Let l_∞ be chosen as base line. Then all the conics are parabolas.

(b) The absolute involution on l_∞ is fixed by its double points I and J. Let IJ be a degenerate conic of the net. Then, by Theorem B, C^2 must pass through I and J, and consequently is a circle.

(c) Let $R_1R_2R_3$ be any triangle inscribed in this circle, and let R_1R_2, R_2R_3, R_3R_1 cut l_∞ in P_3, P_2, P_1, respectively. Also, let P_1P_1', P_2P_2', P_3P_3' be conjugate pairs in the absolute involution. The last needed element to determine the net, according to Theorem D, is chosen as degenerate conic R_1P_1'. But then, by Theorem E, R_2P_2', R_3P_3' are also degenerate conics.

The base triangle $A_1A_2A_3$, with respect to which the orthopoles of lines of the plane are defined is chosen as the triangle perspective with $R_1R_2R_3$ from center O of the circle. The triangles will also be perspective from l_∞.

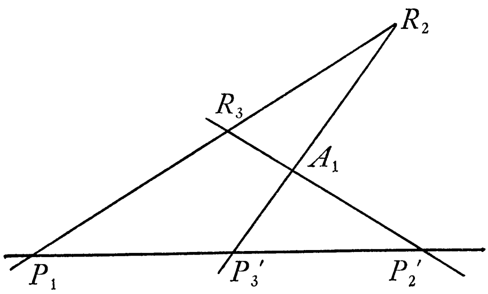
Consider now a projective correspondence between the lines of the plane and the ∞^2 line parabolas of the above linear system. In the (1,1) correspondence, to l_1 and l_2, let correspond parabolas π_1 and π_2, respectively. Then to the intersection of l_1 and l_2 must correspond the range determined by π_1 and π_2. A (1,3) correspondence between points and lines may now be defined. To the point of intersection of l_1 and l_2 correspond the three lines, exclusive of l_∞, common to all parabolas of the range determined by π_1 and π_2. In order that the correspondence be that of orthopoles and associated lines, the determining pairs of homologous elements are chosen as follows: To A_1A_2, let correspond the degenerate parabola R_3P_3'; to A_2A_3, R_1P_1'; to A_1A_3, R_2P_2'; and to l_∞, IJ.

Further Correspondences Derived from the Defining Elements

Since A_1 is common to A_1A_2 and A_1A_3, to A_1 corresponds the range containing degenerate parabolas R_3P_3' and R_2P_2'. The three lines that correspond to A_1 are R_2R_3, R_2P_3', R_3P_2', the two latter intersecting at A_1 on the circle. Also, R_2R_3 cuts l_∞ in P_1. It follows that A_1P_1 is the third degenerate parabola of the range.

In terms of orthopoles, A_1 is the orthopole of R_2R_3, R_2P_3', R_3P_2'; and similarly, A_2 is orthopole of R_1R_3, R_1P_3', R_3P_1'; and A_3, of R_1R_2, R_1P_2', R_2P_1'.

Since P_1 is common to A_2A_3 and l_∞, to P_1 corresponds the range containing degenerate parabolas R_1P_1' and IJ. Thus, P_1 is orthopole of R_1I, R_1J, and l_∞; i.e., l_∞ is a double tangent of parabolas of the range. If in the figure below, two adjacent sides of the quadrilateral become coincident, the points of contact of the inscribed parabolas with those two sides approach each other and also the point of intersection of the two sides. When the two sides coincide, the parabolas will be tangent at that point which forms with the remaining vertex a degenerate parabola. It follows that to P_1 corresponds the range containing degenerate parabolas R_1P_1' and IJ, P_1' being the point of contact; to P_2, R_2P_2' and IJ, P_2' being the point of contact; and to P_3, R_3P_3' and IJ, P_3' being the point of contact.



The point pairs that form the centers of the pencils which are the degenerate conics of the net lie one on the circle and the other on l_∞. Also the three lines that correspond to a given point are always the sides, exclusive of l_∞, of the quadrilateral in which parabolas of the range are inscribed. It follows that the three associated lines always intersect by pairs in three points on the circle. These sets, of three points each, constitute a linear series, a g_3^2, on the circle. It is known that to every g_n^r on a plane algebraic curve, there corresponds a curve of order n, in a space S_r, in (1,1) correspondence with the given curve, on which the image of the g_n^r is the series of ∞^r points cut out by the hyperplanes of that space. Therefore the image of the g_3^2 on the circle is the series of ∞^2 points cut out on a plane cubic curve by the ∞^2 lines of the plane of that cubic, i.e., a g_3^2 on a plane cubic curve.

From this we deduce that there are three types of sets of points on the circle:

(1) Three distinct points, and thus three distinct associated lines—corresponding to the three points in which a generic line cuts the cubic.

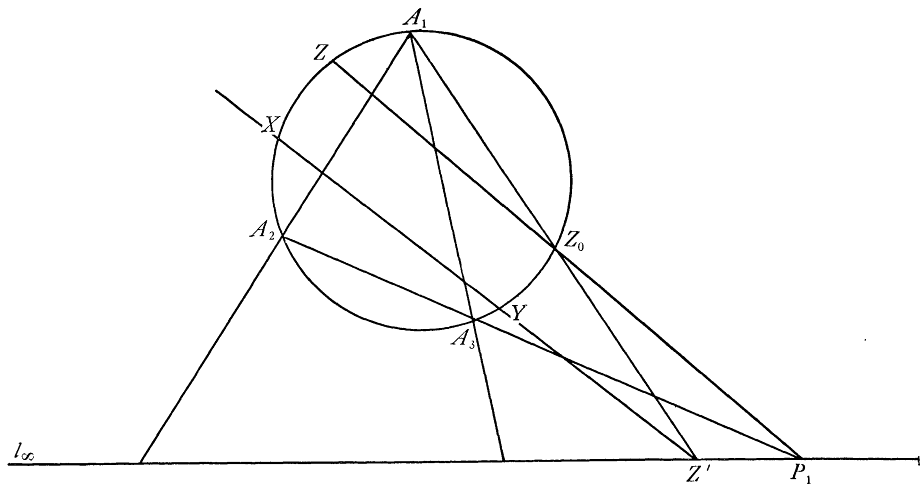
(2) Two of the points coincident, and thus two of the associated lines coincident, the third tangent to the circle at the double point—corresponding to the points in which a tangent line cuts the cubic.

(3) Sets having two points a neutral pair of the g_3^2, in common, and a third

variable point—corresponding to the points in which a line through the double point of the cubic cuts it. It was shown previously that I and J belong to more than one set. They must therefore constitute this neutral pair. For sets of this type, the three associated lines are l_∞ and two others, one on I, the other on J, giving rise to ranges having only two degenerate conics, with a fixed point of contact on l_∞.

Construction

Problem. Given the line XY and the base triangle $A_1A_2A_3$, to construct the two lines XZ and YZ, associated with XY.



Let XY cut l_∞ in Z'. Join A_1 to Z', letting A_1Z' cut the circle again in Z_0. Join Z_0 to P_1, and let the join cut the circle at Z. Then ZX and ZY are the required lines.

Proof. Let π be the projectivity between points on the circle and points on l_∞ that together form degenerate conics of the net (Theorem B). Then

$$\pi(IJA_1A_2A_3R_1R_2R_3) = JIP_1P_2P_3P_1'P_2'P_3'.$$

Let K be any point on the circle. Then

$$KI,\ KJ,\ KA_1,\ \cdots \barwedge KJ,\ KI,\ KP_1,\ \cdots .$$

Since this latter projectivity interchanges the pair KI, KJ, it is an involution. Now, for the pair KA_1, KP_1, A_1P_1 is a degenerate conic. It follows from the involution, that the other points, M, in which KA_1 cuts l_∞, and N, in which KP_1 cuts the circle must also constitute a degenerate conic. And finally, the line A_1N will cut l_∞ in the point L that forms with K a degenerate conic. There is thus an involution of points on the circle, center L.

In the construction as given, $K \equiv A_1$. Then $L \equiv P_1$. Let I_1 be the involution

of points on the circle, center P_1; and I_2, the involution of lines on A_1 passing through conjugate pairs of I_1 on the circle. Then $I_1(Z)=Z_0$ and $I_2(A_1Z_0)=A_1Z$. Therefore, Z and Z' form a degenerate conic of the range inscribed in XY, XZ, YZ, and l_∞. Thus, the lines XZ and YZ are associated with XY.

Some Theorems on Orthopoles

Theorem 1. Every point X on l_∞ is the orthopole of three lines, IJ, IK, JK, where K is the point on the circle that forms with the conjugate of X in the absolute involution a degenerate conic.

It was stated previously that to points on l_∞ correspond ranges having only two degenerate parabolas, one being IJ, the other consisting of a point K on the circle and a fixed point of contact, K', on l_∞, for all parabolas of the range. There is thus a (1,1) correspondence between the orthopole X and the point K on the circle. Let this correspondence be π. Let the correspondence between the point K on the circle and K' on l_∞ that forms with it a degenerate conic be σ. Finally, let the correspondence between X and K' be ω.

Now $\pi(X)=K$ and $\sigma(K)=K'$. But $\omega(X)=K'$. Therefore, $\pi\sigma=\omega$. It is known that $\pi(P_1P_2P_3)=R_1R_2R_3$ and $\sigma(R_1R_2R_3)=P_1'P_2'P_3'$. Thus $\omega(P_1P_2P_3)=P_1'P_2'P_3'$. Therefore, ω must coincide with the absolute involution and thus the theorem is proved.

Remark. Since π transforms three real points on l_∞ into three real points on the circle, the projectivity π must be real.

Corollary. Each of the circular points I and J is the orthopole of l_∞ taken twice and the tangent OJ and OI, respectively, at the other circular point, where O is the center of the circle.

Since $\pi\sigma=\omega$, $\pi=\omega\sigma^{-1}$. Now $\omega(IJ)=IJ$, and $\sigma^{-1}(IJ)=JI$. Therefore $\pi(IJ)=JI$. It follows that I is the orthopole of the lines determined by the set IJJ, while J is the orthopole of the lines determined by the set IIJ. The three associated lines are then l_∞ counted twice and the tangent to the circle at the double point of the set which in each case is the circular point other than the orthopole.

Theorem 2. The orthocenter of the base triangle is the orthopole of its three sides.

Since $\omega(P_1'P_2'P_3')=P_1P_2P_3$ and $\sigma^{-1}(P_1P_2P_3)=A_1A_2A_3$, $\pi(P_1'P_2'P_3')=A_1A_2A_3$. Thus P_1' has corresponding to it the range having IJ and A_1P_1 as degenerate parabolas, P_1 being the fixed point of contact. Similarly, to P_2' corresponds the range having IJ and A_2P_2 as degenerate parabolas, with P_2 the fixed point of contact. And finally, to P_3' corresponds the range having IJ and A_3P_3 as degenerate parabolas, with P_3 as fixed point of contact.

Since the ranges that correspond to the points A_1 and P_1' have in common the degenerate parabola A_1P_1, that parabola must correspond to the line A_1P_1'. Similarly, parabolas A_2P_2 and A_3P_3 correspond respectively to lines A_2P_2' and A_3P_3'. Furthermore, since the three lines A_1P_1', A_2P_2', and A_3P_3' are concurrent in H, the orthocenter of the triangle $A_1A_2A_3$, to H must correspond the range

inscribed in $A_1A_2A_3$ and l_∞. Or, in terms of orthopoles, H is the orthopole of the three sides of the triangle $A_1A_2A_3$.

Theorem 3. To every degenerate conic of the net corresponds the Simson line of a point on the circle with respect to the base triangle.

To every point P of the plane corresponds a range of parabolas. To the three degenerate parabolas of the range correspond three lines on P. As P varies in the plane, these lines envelope a curve of class three. It has already been shown that the three sides and altitudes of the base triangle, also l_∞, have degenerate parabolas corresponding to them. Furthermore, by Theorem 1, Corollary, $\pi(IJ)=JI$. This means that for the set IIJ, having l_∞ as double line, J is the point of contact and hence a point of the envelope. Similarly, I is a point of the envelope.

The envelope of the Simson lines with respect to a given triangle of points on its circumcircle is known to be a three-cusped hypocycloid, having l_∞ as double tangent at the circular points. Each of the seven lines above mentioned is a Simson line with respect to the base triangle. Thus the Simson line envelope and the envelope first considered have in common six tangents, one double tangent and its two points of contact. Dually, there are two cubics having in common six points, one double point, and common tangents at that double point—making a total of twelve intersections. The two cubics, having in common more than nine points, must coincide. Dually, the two envelopes under consideration must be the same. Therefore, every line which has a degenerate parabola corresponding to it is a Simson line, and conversely.

Theorem 4. A Simson line is always perpendicular to the double line of the degenerate parabola to which it corresponds.

Let AB be the double line of the degenerate parabola that corresponds to the Simson line, s, A being on l_∞, B on the circle. Let AA' be a conjugate pair in the absolute involution. By Theorem 1, to A' corresponds the range having IJ and AB as degenerate parabolas. Therefore, all parabolas of the range correspond to lines on A'. Then the Simson line, s, to which AB corresponds must pass through A', and consequently is perpendicular to line AB.

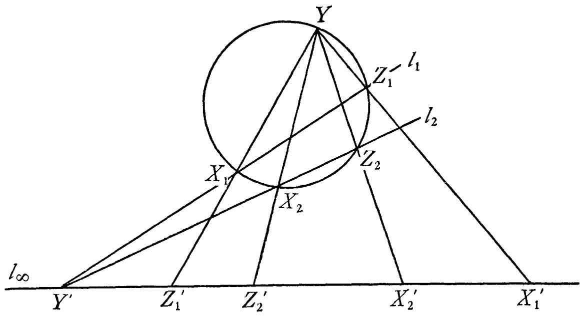
Theorem 5. The orthopole of an altitude of the base triangle is the foot of that altitude.[1]

Theorem 6. When a line l moves parallel to itself, its orthopole moves on a line perpendicular to l, and the two associated lines pass through a fixed point on the circumference of the circle.

Let P be the point to which correspond the lines X_1Z_1, YX_1, YZ_1. To the degenerate parabola YY' corresponds the Simson line perpendicular to the line YY', and the Simson line must also pass through P. For any other position of the line on Y', the range that corresponds to the new orthopole will always contain the degenerate parabola YY'. It follows that the orthopole of any line on Y' lies on the Simson line perpendicular to the line YY'. Also, the two lines

[1] Proofs of Theorems 5, 7, 9 may be found in original copy of writer's dissertation, loc. cit.

associated with the given line will always pass through that point on the circle that forms with Y' a degenerate parabola.



Theorem 7. The orthopole of a line perpendicular to a side of the base triangle is the foot of that perpendicular.

Theorem 8. As a line l varies on a curve of class m, its orthopole M varies on a curve of order $2m$.

As M moves on a line k, l envelopes a line parabola π. As M moves on a curve C, of order x, l envelopes a curve S of class m. To the common points of k and C, will correspond the lines common to π and S. These are $2m$ in number. Therefore the order x of C is $2m$.

Corollary. As a line m varies in a pencil of lines on a point P, the orthopole M varies on a conic, an ellipse.

In this case, m equals one; the locus of M is thus a conic. It remains to show that this is an ellipse, i.e., cuts l_∞ in two imaginary points. Since P is not on l_∞, the only lines on P that can have orthopoles on l_∞ are PI and PJ. For P real, the other point of intersection, Q, of PI with the circle must be imaginary. The projectivity π between l_∞ and points on the circle, by the Remark, Theorem 1, is real. Therefore, the point R that forms a degenerate conic with Q is imaginary. To the range QIJ, then, corresponds the point R', the conjugate of R in the absolute involution. But R', also imaginary, is then the orthopole of QI or PI. Similarly, the orthopole of PJ is imaginary. Thus the conic which is the orthopole locus is an ellipse.

Theorem 9. When a line m varies in a pencil of lines on a point P on l_∞, the orthopole locus degenerates into the Simson line perpendicular to m, and the line l_∞.

Theorem 10. When a line varies on a point P on the circumcircle of the base triangle, the orthopole locus is the Simson line of the point P with respect to the base triangle taken twice.

The feet of the three perpendiculars from P to the sides of the base triangle are their respective orthopoles (Theorem 7). But these points determine the so-called Simson line of the point P with respect to the base triangle. By Theorem 6, there are always two lines on P that give rise to the same orthopole. Thus the Simson line must be traced out twice.

Corollary. The orthopole of a line cutting the circumcircle in points P and Q is the intersection of the Simson lines of P and Q.

Theorem 11. As a point M describes a curve of order n, the three associated lines of which M is orthopole envelope a curve of class $2n$.

As a line l varies on a point P, its orthopole describes a conic (Theorem 8, Corollary). This conic cuts the locus C of M in $2n$ points. To each of these points must correspond a line on P tangent to the envelope described by the lines as M varies on C. Therefore the class of the envelope is $2n$.

Theorem 12. As a line varies on the center O of the circumcircle of the base triangle, its orthopole varies on the nine-point circle of the base triangle.

The locus, a conic, must be a circle since it passes through the circular points I and J, the respective orthopoles of tangents OJ and OI (Theorem 1, Corollary). Also the feet of the three perpendiculars from O to the three sides of the base triangle are on the circle. Thus the required locus is the nine-point circle of the base triangle.

Theorem 13. As a line varies on the hypocycloid which is the Simson line envelope, the orthopole locus is a quartic, having a triple point at the orthocenter H, and passing through the feet of the three altitudes of the base triangle.

The hypocycloid being of class three, the orthopole locus must be of order six. This degenerates into l_∞ counted twice and a residual quartic. That l_∞ is a double line of the locus follows from the fact that l_∞ is a double tangent to the hypocycloid, and has as orthopole any point on l_∞. Since the three sides of the base triangle are tangent to the hypocycloid, and each has H as orthopole, H is a triple point of the quartic. Finally, the altitudes are Simson lines having their respective feet as orthopoles. Thus the quartic also passes through the feet of the altitudes.

Theorem 14. As a line m varies as tangent to the circumcircle of the base triangle, its orthopole varies on the hypocycloid envelope of the Simson lines.

The tangents to the hypocycloid are all Simson lines and thus each gives rise to a degenerate conic of the net. From points not on the hypocycloid, three distinct tangents may be drawn. The corresponding range will thus have three distinct degenerate parabolas, and the associated lines will be distinct. But for points on the hypocycloid, two of the tangents will be coincident and the third distinct. The corresponding ranges will have two coincident degenerate parabolas, and the set of points on the circle has a double point. Of the three associated lines, one will be tangent to the circle at that double point, and the other will be the line joining the double point to the remaining point, this line being counted twice. Thus, for all points on the hypocycloid, one of the three associated lines will be tangent to the circle. Therefore, as a line varies as tangent to the circumcircle, its orthopole varies on the hypocycloid.

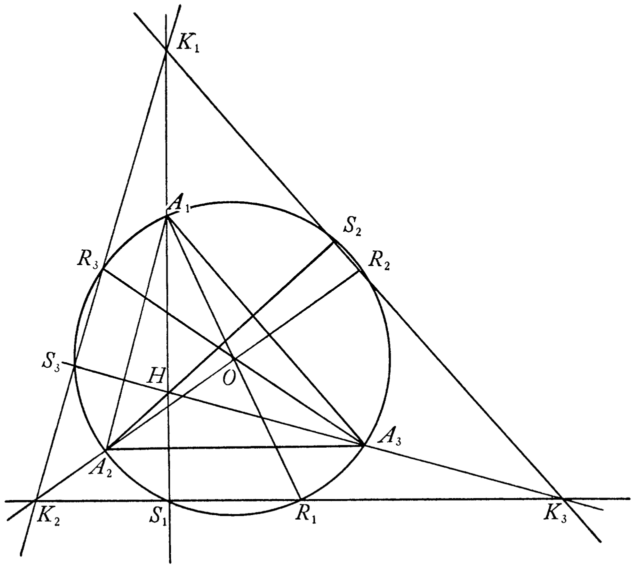
Corollary. If a line is tangent to the circumcircle, its orthopole is the point in which the Simson line of the point of contact touches the hypocycloid.

Theorem 15. Given any two sets of associated lines, the orthopole of either set with respect to the other as base triangle is the midpoint of the join of their orthocenters.

The following is a known property[1] of Simson lines: If the Simson lines of the points B_1, B_2, B_3, with respect to the triangle $A_1A_2A_3$ are concurrent on a point M, then the Simson lines of A_1, A_2, A_3, with respect to triangle $B_1B_2B_3$ are also concurrent on M, and M is midway between the orthocenters of the two triangles. From this property and the Corollary to Theorem 10, the above theorem obviously follows.

Theorem 16. As a point varies on the nine-point circle of the base triangle, its ortholines envelope a curve of class four, which degenerates into a three-cusped hypocycloid and the center of the circumcircle of the base triangle.

As the orthopole varies on a conic, the associated lines envelope a curve of class four. By Theorem 12, the center of the circumcircle is part of the desired locus. Consequently the residual portion is of class three.



Since one of the three associated lines is always on the center O, and thus a diameter, the other two lines—tangent to the envelope—intersect at right angles. Thus the circle is the orthoptic locus of the envelope. It is known that the orthoptic locus is a circle only for central conics and for curves of class three, order four, having l_∞ as double tangent at the circular points.[2] The above envelope must, therefore, be of order four, l_∞ being associated with diameters through both I and J, and touching the envelope at J and I. For all other diameters, the two associated lines cannot possibly coincide. The curve, then, has no other double tangents. Using the Plücker equations:

$$n = m(m-1) - 2t - 3i$$
$$m = n(n-1) - 2d - 3k$$

[1] Johnson, *Modern Geometry*, p. 211.

[2] Hilton, *Plane Algebraic Curves*, pp. 169–174.

and

$$i = 3n(n - 2) - 6d - 8k$$

for $n=4$, $m=3$, and $t=0$, it is found that $i=0$, $d=0$, and $k=3$. Therefore, the envelope is a three-cusped hypocycloid.

It will now be shown which hypocycloid this is. By the method indicated on page 331, the lines associated with A_1R_1 are found to be A_1S_1 and R_1S_1; with A_2R_2, A_2S_2 and R_2S_2; with A_3R_3, A_3S_3 and R_3S_3.

It can readily be proven that the lines R_3S_3, A_1S_1, R_2S_2 are concurrent on K_1; R_2S_2, A_3S_3, R_1S_1 on K_3; and R_1S_1, A_2S_2, R_3S_3, on K_2. Therefore, the required hypocycloid must be tangent to the three sides of the triangle $K_1K_2K_3$, to its three altitudes, and to l_∞ twice at I and J. Then this hypocycloid must coincide with the Simson line envelope of points on the circumcircle of $K_1K_2K_3$ with respect to that triangle.

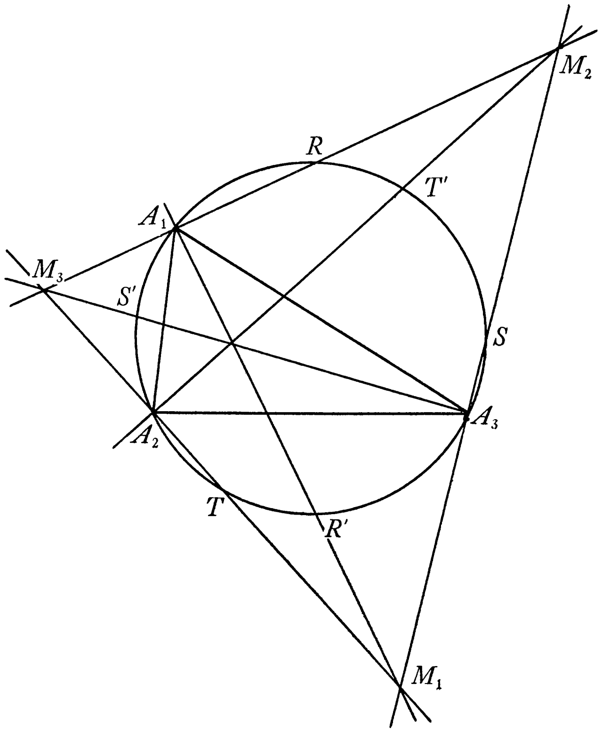
Theorem 17. As a point varies on the hypocycloid envelope of the Simson lines of the base triangle, the ortholines envelope the circumcircle of the base triangle and another three-cusped hypocycloid counted twice.

As the orthopole varies on a curve of order four, the ortholines have as envelope a curve of class eight. By Theorem 14, the circumcircle is part of this envelope. It will now be shown that the residual portion is a three-cusped hypocycloid, counted twice.

As a line moves parallel to itself, the two lines associated with it always pass through a fixed point on the circle. Also, the two lines associated with a tangent are always coincident. Now in every set of parallel lines, there will be two tangents to the circle at points T_1 and T_2, say. Then PT_1 and PT_2, each counted twice, will be associated with the tangents at the respective points, T_1 and T_2. Furthermore, PT_1 and PT_2 are perpendicular.

Since one of the three lines that has a point on the Simson line envelope as orthopole is always tangent to the circle, the other two will always be a certain line counted twice. Also, since l_∞ is associated with the tangents to the circle at both I and J, l_∞ is a double tangent. In a given direction, then, there can be drawn only one tangent. And, from what was said above, any two tangents that are perpendicular must intersect on the circumcircle of the base triangle. The envelope consequently is a curve of class three, counted twice, having the circumcircle as orthoptic locus. The line at infinity being the only double tangent, it follows, as in Theorem 16, that the curve is a hypocycloid of three cusps.

The hypocycloid is determined as follows. The lines associated with a line parallel to any side of the base triangle pass through the opposite vertex. Let a and a' be tangents parallel to A_2A_3; b and b', to A_1A_2; and c and c', to A_1A_3. Let the respective points of contact be R, R'; S, S'; T, T'. Then each of the following lines is tangent to the required hypocycloid: A_1R, A_1R', A_2T, A_2T', A_3S, A_3S'. It may readily be proven that A_1R, A_3S', and A_2T are concurrent on M_3; A_1R, A_2T', and A_3S, on M_2; and A_1R', A_2T, and A_3S, on M_1. The required



hypocycloid must coincide with the Simson line envelope of points on the circumcircle of triangle $M_1M_2M_3$ with respect to that triangle, each of the six lines mentioned above being such Simson lines.

VECTORIAL TREATMENT OF CERTAIN ALGEBRAIC THEOREMS

By T. C. ESTY, Amherst College

I. Consider the homogeneous linear equations

(1) $$a_1x + b_1y + c_1z = 0,$$

(2) $$a_2x + b_2y + c_2z = 0.$$

Let us define three vectors, α, β, ρ, as follows:

(3) $$\alpha = a_1i + b_1j + c_1k, \quad \beta = a_2i + b_2j + c_2k, \quad \rho = xi + yj + zk,$$

where i, j, k form a right-handed system of mutually perpendicular unit vectors.

Equations (1), (2) may then be written as

(1′) $$\alpha\cdot\rho = 0,$$

(2′) $$\beta\cdot\rho = 0.$$

Equations (1′), (2′) are satisfied simultaneously by any vector ρ which is perpendicular to both α and β; that is, when ρ is given by

$$\rho = t\alpha \times \beta, \tag{4}$$

where t is any scalar.

Substituting for ρ, α, β their values from (3), expanding the vector product, and equating coefficients of i, of j, and of k, we obtain

$$x = t(b_1c_2 - b_2c_1), \quad y = t(c_1a_1 - c_2a_1), \quad z = t(a_1b_2 - a_2b_1), \tag{5}$$

showing that

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \tag{6}$$

By putting $z = -1$ in (1) and (2) we obtain the system

$$\begin{aligned} a_1x + b_1y &= c_1, \\ a_2x + b_2y &= c_2, \end{aligned} \tag{7}$$

whose solution, as given by (6) when $z = -1$, is

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \tag{8}$$

II. Consider the homogeneous linear equations,

$$a_1x + b_1y + c_1z = 0, \tag{1}$$

$$a_2x + b_2y + c_2z = 0, \tag{2}$$

$$a_3x + b_3y + c_3z = 0, \tag{3}$$

of which at least two, say (2) and (3), are independent. If we put

$$\begin{aligned} \alpha &= a_1i + b_1j + c_1k, & \beta &= a_2i + b_2j + c_2k, \\ \gamma &= a_3i + b_3j + c_3k, & \rho &= xi + yj + zk, \end{aligned} \tag{4}$$

then the given equations may be written in the form

$$\alpha\cdot\rho = 0, \tag{1′}$$

$$\beta\cdot\rho = 0, \tag{2′}$$

$$\gamma\cdot\rho = 0. \tag{3′}$$

We propose to find a necessary and sufficient condition that the last three equations have a common solution other than $\rho=0$; that is, that the given equations have a common solution other than $x=y=z=0$.

Let us first assume that there *is* such a value of ρ which satisfies (1′), (2′) and (3′). It follows that this ρ is perpendicular to each of the vectors α, β, γ, and hence that α, β, and γ are coplanar. This requires that

$$\alpha\cdot\beta\times\gamma = 0, \tag{5}$$

or

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \tag{6}$$

This is a *necessary* condition.

Next, let us assume that (6), and hence (5), are fulfilled. Solving (2′) and (3′) for ρ in the form

$$\rho = t\beta\times\gamma \quad (\beta \text{ not parallel to } \gamma) \tag{7}$$

as in section I, we find by substitution in (1′),

$$\alpha\cdot\rho = t\alpha\cdot\beta\times\gamma,$$

and hence, by virtue of (5), $\alpha\cdot\rho$ vanishes identically. Thus $\rho=t\beta\times\gamma$, in which t is any scalar whatever, is a common solution of (1′), (2′) and (3′).

This proves that (5) is a *sufficient* condition that the equations have a common solution other than $\rho=0$.

It follows that (6) is a necessary and sufficient condition that equations (1), (2) and (3) have a common solution other than $x=y=z=0$.

One form of the solution may be found directly from (7), namely

$$x = t(b_2c_3 - b_3c_2), \quad y = t(c_2a_3 - c_3a_2), \quad z = t(a_2b_3 - a_3b_2), \tag{8}$$

where t is any scalar whatever.

By putting $z=-1$ in (1), (2), (3) we obtain the system

$$\begin{aligned} a_1x + b_1y &= c_1, \\ a_2x + b_2y &= c_2, \\ a_3x + b_3y &= c_3, \end{aligned} \tag{9}$$

and we see that (6) is a necessary condition that equations (9) have a common solution. If $a_2b_3-a_3b_2\neq 0$ the solution is unique; and it can be found from (8) by putting $z=-1$ and substituting the resulting value of t in the expressions for x and y.

III. Consider the system

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \\ a_3x + b_3y + c_3z &= d_3. \end{aligned} \tag{1}$$

We shall assume, as we may without loss of generality, that the signs have been so adjusted as to make the quantities d_1, d_2, d_3 all positive.

Let us now define three vectors α, β, γ as follows:

$$\alpha = a_1 i + b_1 j + c_1 k, \quad \beta = a_2 i + b_2 j + c_2 k, \quad \gamma = a_3 i + b_3 j + c_3 k. \tag{2}$$

The equations (1) may be written in the equivalent form

$$\alpha\cdot\rho = d_1, \quad \beta\cdot\rho = d_2, \quad \gamma\cdot\rho = d_3. \tag{3}$$

If we regard α, β, γ, ρ as position vectors with a common origin at 0, these equations represent three planes which are perpendicular to α, β, γ, respectively, and at distances from 0 given by d_1/a, d_2/b, d_3/c, where a, b, c are the lengths of α, β, γ.

We now introduce the familiar formula

$$\rho(\alpha\cdot\beta \times \gamma) = \beta \times \gamma(\alpha\cdot\rho) + \gamma \times \alpha(\beta\cdot\rho) + \alpha \times \beta(\gamma\cdot\rho) \tag{4}$$

in which α, β, γ are supposed to be any three non-coplanar vectors. On the supposition, then, that the vectors α, β, γ in (2) are non-coplanar, that is, that $\alpha\cdot\beta\times\gamma\neq 0$, we have from (3) and (4)

$$\rho = \frac{d_1\beta \times \gamma + d_2\gamma \times \alpha + d_3\alpha \times \beta}{\alpha\cdot\beta \times \gamma}. \tag{5}$$

This gives the position vector of the point common to planes (3). Replacing ρ by $xi+yj+zk$ and α, β, γ by their values in (2) we get

$$xi + yj + zk = \frac{d_1\begin{vmatrix} i & j & k \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + d_2\begin{vmatrix} i & j & k \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \end{vmatrix} + d_3\begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}. \tag{6}$$

Expanding the determinants in the numerator it is found that the i-terms are

$$[d_1(b_2c_3 - b_3c_2) + d_2(b_3c_1 - b_1c_3) + d_3(b_1c_2 - b_2c_1)]i = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} i$$

with similar results for the j- and k-terms.

Equating coefficients of like vectors in (6) we obtain

$$(7)\qquad x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Just as (5) furnishes a single value of ρ when $\alpha\cdot\beta\times\gamma\neq 0$, so will equations (1) have a single solution when

$$(8)\qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0.$$

As a numerical example of the foregoing we shall solve the equations

$$\begin{aligned} x + 6y - 5z &= 23, \\ -3x + 8y - 4z &= 1, \\ 7x - 10y + 10z &= 0. \end{aligned}$$

We put

$$\alpha = i + 6j - 5k, \qquad \beta = -3i + 8j - 4k, \qquad \gamma = 7i - 10j + 10k$$

and find

$$\beta\times\gamma = 40i + 2j - 26k,\ \gamma\times\alpha = -10i + 45j + 52k,\ \alpha\cdot\beta\times\gamma = 182.$$

We also have $d_1=23$, $d_2=1$, $d_3=0$.

Substituting these values in (5) we get

$$\rho = xi + yj + zk = \frac{23(40i + 2j - 26k) + (-10i + 45j + 52k)}{182} = 5i + \tfrac{1}{2}j - 3k,$$

Whence $x=5$, $y=\frac{1}{2}$, $z=-3$.

Special cases which can arise under the supposition that $\alpha\cdot\beta\times\gamma$ *is* equal to zero, that is when

$$(9)\qquad \rho(0) = d_1\beta\times\gamma + d_2\gamma\times\alpha + d_3\alpha\times\beta,$$

may be considered under the following heads:

(1) When α, β, and γ are coplanar, without further restrictions.

Here there are two possibilities; (a) when the right side of (9) is *not* equal to zero, and (b) when it *is* equal to zero.

Under (a), there is no solution. In fact equations (3) show that the three planes are perpendicular to the plane of α, β, γ, and these planes intersect by pairs in three parallel lines whose direction is that of $\alpha\times\beta$,

Under (b),

$$d_1\beta \times \gamma + d_2\gamma \times \alpha + d_3\alpha \times \beta = 0. \tag{10}$$

To interpret this result we note that γ is in the plane of α and β; hence we may put

$$\gamma = l\alpha - m\beta. \tag{11}$$

Then $d_1\beta\times\gamma = ld_1\beta\times\alpha = -ld_1\alpha\times\beta$, $d_2\gamma\times\alpha = -md_2\beta\times\alpha = md_2\alpha\times\beta$ and (10) becomes $(-ld_1+md_2+d_3)\alpha\times\beta = 0$, or since $\alpha\times\beta\neq 0$,

$$d_3 = ld_1 - md_2. \tag{12}$$

Substituting from (11) and (12) in the equation $\gamma\cdot\rho = d_3$, the equation of that plane becomes

$$l(\alpha\cdot\rho - d_1) - m(\beta\cdot\rho - d_2) = 0, \tag{13}$$

which is the equation of a plane through the intersection of the two planes $\alpha\cdot\rho = d_1$ and $\beta\cdot\rho = d_2$. Hence planes (3) *meet in one line*, which is parallel to $\alpha\times\beta$, and every point of this line is a solution of equations (3).

The corresponding values of x, y, z are therefore solutions of equations (1), and we say that these equations have an infinite number of solutions. To find the relations between the constants in (1) which exist in this case, we substitute the values of α, β, γ in (10) and expand the vector products. Then collecting the i-, j-, and k-terms separately, we get

$$\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} i + \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} j + \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} k = 0, \tag{14}$$

which requires that the three determinants vanish separately. In addition to these conditions we have, of course, since $\alpha\cdot\beta\times\gamma = 0$:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \tag{15}$$

It is interesting to note that when equations (1) represent three planes with a common line of intersection, the equations are not independent, for by (11)

$$a_3 = la_1 - ma_2,\quad b_3 = lb_1 - mb_2,\quad c_3 = lc_1 - mc_2, \tag{11'}$$

and by (12)

$$d_3 = ld_1 - md_2. \tag{12'}$$

These relations show that the third of the given equations can be obtained by multiplying the first by a number l and subtracting from the result the second

multiplied by a number m. The process of finding the values of l and m will be illustrated in the following example.

Solve

$$\begin{aligned} 2x - 3y + 4z &= 5, \\ 3x + y - 2z &= 1, \\ -5x - 9y + 14z &= 7. \end{aligned}$$

Here $\alpha = 2i - 3j + 4k$, $\beta = 3i + j - 2k$, $\gamma = -5i - 9j + 14k$, and

$$\alpha\cdot\rho = 5,\quad \beta\cdot\rho = 1,\quad \gamma\cdot\rho = 7. \tag{3'}$$

Also $\alpha\cdot\beta\times\gamma = 0$, $\beta\times\gamma = -4i - 32j - 22k$, $\gamma\times\alpha = 6i + 48j + 33k$, $\alpha\times\beta = 2i + 16j + 11k$, so that

$$d_1\beta\times\gamma + d_2\gamma\times\alpha + d_3\alpha\times\beta = 5\beta\times\gamma + \gamma\times\alpha + 7\alpha\times\beta = 0.$$

Since no two of the vectors α, β, γ are parallel and since $d_1\beta\times\gamma + d_2\gamma\times\alpha + d_3\alpha\times\beta = 0$, the planes (3′) must meet in one line, and the given equations have an infinite number of solutions.

From the first of equations (11′) we have $2l - 3m = -5$, and from (12′) $5l - m = 7$. Solving these two equations, we find $l = 2$, $m = 3$. In fact, if we multiply the first of the given equations by 2 and subtract three times the second, we obtain the third.

If the equations of the line of intersection are desired, we may proceed as follows.

Let us first find the point in which the line pierces the plane of i and j. This point of the line may be regarded as the intersection of the three planes $\alpha\cdot\rho = 5$, $\beta\cdot\rho = 1$ and $k\cdot\rho = 0$. Denoting the vector of their common point by σ, and using the formula

$$\rho = \frac{d_1\beta\times\gamma + d_2\gamma\times\alpha + d_3\alpha\times\beta}{\alpha\cdot\beta\times\gamma}\,.$$

we have

$$\sigma = \frac{5\beta\times k + k\times\alpha}{\alpha\cdot\beta\times k} = \frac{8i - 13j}{11}\,.$$

Since the line is parallel to $\alpha\times\beta$, its equation will have the form $\rho = \sigma + t\alpha\times\beta$, where t is a variable scalar. Thus

$$xi + yj + zk = \frac{8}{11}i - \frac{13}{11}j + t(2i + 16j + 11k)$$

whence

$$\frac{x - \frac{8}{11}}{2} = \frac{y + \frac{13}{11}}{16} = \frac{z}{11}\,.$$

(2) When α, β, γ are coplanar, as before, but in addition, two of them, β and γ, (say) are parallel.

Here $\beta\times\gamma=0$, and from (9)

(16) $$\rho(0) = d_2\gamma \times \alpha + d_3\alpha \times \beta.$$

We have two possible cases; (*a*) when the right side is *not* equal to zero, and (*b*) when the right side *is* equal to zero.

Under (*a*) there is no solution. Two of the planes are parallel and are cut by the third plane in two lines which are parallel to $\alpha\times\beta$. Furthermore, since $\beta\times\gamma=0$, we have

(17) $$\frac{a_2}{a_3} = \frac{b_2}{b_3} = \frac{c_2}{c_3}\cdot$$

Under (*b*),

(18) $$d_2\gamma\times\alpha+d_3\alpha\times\beta=0$$

To interpret this, we first note that β and γ are supposed to be parallel. Now if they agree in sense, the vectors $\gamma\times\alpha$ and $\alpha\times\beta$ differ in sense. On the other hand, if β and γ differ in sense, then $\gamma\times\alpha$ and $\alpha\times\beta$ agree in sense. Under the latter supposition (18) is impossible, since d_2 and d_3 are positive. But under the former supposition, (18) becomes

(19) $$d_2c - d_3b = 0, \text{ or } d_2/b = d_3/c.$$

Now d_2/b and d_3/c are the distances from 0 of the planes $\beta\cdot\rho=d_2$ and $\gamma\cdot\rho=d_3$, respectively, and since these distances are equal, the two planes coincide. All points of the line in which these coincident planes are cut by the third plane are solutions of (3), and hence equations (1) have in this case an infinite number of solutions.

It is easy to determine the conditions under which equations (1) present this case, for, since β is parallel to γ, and of the same sense, we may write

(20) $$\beta = n\gamma, \text{ whence } n = b/c = d_2/d_3.$$

Also

(21) $$a_2i + b_2j + c_2k = n(a_3i + b_3j + c_3k),$$

whence by (20) and (21)

(22) $$a_2/a_3 = b_2/b_3 = c_2/c_3 = d_2/d_3.$$

(3) When all three vectors α, β, γ are parallel. Then, separately,

(23) $$\beta \times \gamma = 0,\ \ \gamma \times \alpha = 0,\ \ \alpha \times \beta = 0.$$

In this case all of the planes (3) are parallel and in general there is no solution. We see from (23) that this situation arises when

(24) $$a_2/a_3 = b_2/b_3 = c_2/c_3,\ a_3/a_1 = b_3/b_1 = c_3/c_1, \text{ and } a_1/a_2 = b_1/b_2 = c_1/c_2.$$

If, however, α, β, and γ agree in sense, and, in addition, $d_1/a = d_2/b = d_3/c$, then all three planes coincide, and, as in 2(*b*) the groups (24) of equal ratios include as a fourth member d_2/d_3, d_3/d_1, and d_1/d_2 respectively.

IV. To express the square of a determinant of the third order as a determinant of the same order.

Consider

$$(1)\qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 .$$

Introducing the vectors

$$\alpha = a_1 i + a_2 j + a_3 k,\ \beta = b_1 i + b_2 j + b_3 k,\ \gamma = c_1 i + c_2 j + c_3 k.$$

we may write (1) as

$$\begin{aligned} (\alpha\cdot\beta\times\gamma)(\alpha\cdot\beta\times\gamma) &= \alpha\cdot\beta\times[\gamma(\alpha\cdot\beta\times\gamma)], \\ &= \alpha\cdot\beta\times[\gamma(\alpha\cdot\beta\times\gamma) - \alpha(\gamma\cdot\beta\times\gamma)], \end{aligned}$$

since $\gamma\cdot\beta\times\gamma = 0$. Then

$$(2)\qquad (\alpha\cdot\beta\times\gamma)(\alpha\cdot\beta\times\gamma) = \alpha\cdot\beta\times[(\beta\times\gamma)\times\gamma\times\alpha],$$

by the formula for a vector triple product; or

$$\begin{aligned} (\alpha\cdot\beta\times\gamma)(\alpha\cdot\beta\times\gamma) &= \alpha\times\beta\cdot[(\beta\times\gamma)\times(\gamma\times\alpha)] \\ &= (\alpha\times\beta)\cdot(\beta\times\gamma)\times(\gamma\times\alpha) \end{aligned}$$

which is equivalent to

$$(3)\qquad \begin{vmatrix} a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \\ c_2a_3 - c_3a_2 & c_3a_1 - c_1a_3 & c_1a_2 - c_2a_1 \end{vmatrix}.$$

V. To express the product of two determinants of the third order as a single determinant of the same order.

Consider the product

$$(1)\qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \times \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}.$$

Introduce the vectors

$$(2)\qquad \begin{aligned} &\alpha = a_1 i + a_2 j + a_3 k, \quad \beta = b_1 i + b_2 j + b_3 k, \quad \gamma = c_1 i + c_2 j + c_3 k, \\ &\lambda = l_1 i + l_2 j + l_3 k, \quad \mu = m_1 i + m_2 j + m_3 k, \quad \nu = n_1 i + n_2 j + n_3 k. \end{aligned}$$

Then the product (1) may be expressed as

$$
\begin{aligned}
(3)\quad & (\alpha\cdot\beta\times\gamma)(\lambda\cdot\mu\times\nu)\\
&= \alpha\cdot\beta\times\left[\gamma(\lambda\cdot\mu\times\nu)\right]\\
&= \alpha\cdot\beta\times\left[\mu\times\nu(\gamma\cdot\lambda)+\nu\times\lambda(\gamma\cdot\mu)+\lambda\times\mu(\gamma\cdot\nu)\right]\\
&= \alpha\cdot\beta\times\mu\times\nu(\gamma\cdot\lambda)+\alpha\cdot\beta\times\nu\times\lambda(\gamma\cdot\mu)+\alpha\cdot\beta\times\lambda\times\mu(\gamma\cdot\nu)\\
&= \alpha\cdot\left[\mu(\beta\cdot\nu)-\nu(\beta\cdot\mu)\right](\gamma\cdot\lambda)+\alpha\cdot\left[\nu(\beta\cdot\lambda)-\lambda(\beta\cdot\nu)\right](\gamma\cdot\mu)\\
&\quad +\alpha\cdot\left[\lambda(\beta\cdot\mu)-\mu(\beta\cdot\lambda)\right](\gamma\cdot\nu)\\
&= (\alpha\cdot\mu)(\beta\cdot\nu)(\gamma\cdot\lambda)-(\alpha\cdot\nu)(\beta\cdot\mu)(\gamma\cdot\lambda)+(\alpha\cdot\nu)(\beta\cdot\lambda)(\gamma\cdot\mu)-(\alpha\cdot\lambda)(\beta\cdot\nu)(\gamma\cdot\mu)\\
&\quad +(\alpha\cdot\lambda)(\beta\cdot\mu)(\gamma\cdot\nu)-(\alpha\cdot\mu)(\beta\cdot\lambda)(\gamma\cdot\nu).
\end{aligned}
$$

Or

$$
(4)\qquad (\alpha\cdot\beta\times\gamma)(\lambda\cdot\mu\times\nu)=\begin{vmatrix}\alpha\cdot\lambda & \alpha\cdot\mu & \alpha\cdot\nu\\ \beta\cdot\lambda & \beta\cdot\mu & \beta\cdot\nu\\ \gamma\cdot\lambda & \gamma\cdot\mu & \gamma\cdot\nu\end{vmatrix}
$$

$$
=\begin{vmatrix}a_1l_1+a_2l_2+a_3l_3 & a_1m_1+a_2m_2+a_3m_3 & a_1n_1+a_2n_2+a_3n_3\\ b_1l_1+b_2l_2+b_3l_3 & b_1m_1+b_2m_2+b_3m_3 & b_1n_1+b_2n_2+b_3n_3\\ c_1l_1+c_2l_2+c_3l_3 & c_1m_1+c_2m_2+c_3m_3 & c_1n_1+c_2n_2+c_3n_3\end{vmatrix}.
$$

QUESTIONS AND DISCUSSIONS

EDITED by R. E. GILMAN, Brown University, Providence, Rhode Island.

The Department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE SOLUTIONS OF THE EQUATION $\binom{x+1}{y}=\binom{x}{y+1}$.

By W. RANDOLPH CHURCH, Hamden, Conn.

In the June–July issue of this Monthly, F. Underwood[1] solved the problem of determining all integral values of x and y which satisfy the equation

$$
(1)\qquad \binom{x}{y+1}=\binom{x+1}{y}.
$$

It is the purpose of the present note to bring out certain aspects of the subject which may be of interest. In particular, a different solution will be outlined in which it will not be necessary to distinguish different cases, and which will be

[1] This Monthly, vol. 38 (1931), pp. 351–354. The problem was proposed by Norman Anning, ibid., vol. 37 (1930), p. 508.

better adapted to the present needs than the one referred to. The relationship between the solutions of (1) and the well known numbers of Fibonacci will be brought out. And finally, it will be shown that the solutions form a recurring series, and the recurrence relations will be given.

Solution of Equation (1).

The equation (1) is equivalent to

$$x^2 - 3xy + y^2 - 2y - 1 = 0. \tag{2}$$

Making the substitution

$$\begin{aligned} x &= (3\xi + 4\eta - 6)/5 \\ y &= (2\xi + \eta - 4)/5 \end{aligned} \tag{3}$$

we obtain the equation

$$\xi^2 + \xi\eta - \eta^2 = -1. \tag{4}$$

The expression on the left hand side of (4) is the norm of an integer in the quadratic field $K(\sqrt{5})$:

$$N[\xi + \eta\omega] = \xi^2 + \xi\eta - \eta^2,$$

where $\omega = (1+\sqrt{5})/2$. The fundamental unit in this field is ω, and its norm is -1. From these facts it follows that the solutions of (4) are coextensive with the odd powers of ω. Thus, the complete solution of (4) is given by $\xi = \xi_m$, $\eta = \eta_m$, where

$$\omega^m = \xi_m + \eta_m\omega, \quad m = 2n + 1, \tag{5}$$

and n is any integer.

Expanding $\omega^m = (1+\sqrt{5})^m/2^m$ in (5) and equating rational and irrational parts, we find

$$(2\xi_m + \eta_m)2^{2n} = \sum_{i=0}^{n} \binom{2n+1}{2i} 5^i \tag{6}$$

and

$$\eta_m 2^{2n} = \sum_{i=0}^{n} \binom{2n+1}{2i+1} 5^i. \tag{7}$$

Setting $5^i = (2^2+1)^i$, expanding, reducing and solving these relations we obtain[1]

$$\xi_m = \sum_{i=0}^{n-1} \binom{n+i}{2i+1}, \tag{8}$$

[1] The reduction leading to (9) can be accomplished conveniently by using the fourth formula of (36) on p. 253 of Netto, *Combinatorik*, 1927. Expression (8) may then be obtained by using $\xi_{i+1} = \eta_i$, which is proved below.

$$\eta_m = \sum_{i=0}^{n} \binom{n+i}{2i}. \tag{9}$$

These expressions (8) and (9) give all integral solutions of (4), where $m=2n+1$; we will show that all integral solutions of (2) correspond to odd values of n.

For, from (6)

$$(2\xi_m + \eta_m)4^n \equiv 1 \pmod 5$$

or

$$2\xi_m + \eta_m \equiv (-1)^n \pmod 5$$

so that

$$2\xi_m + \eta_m - 4 \equiv (-1)^n + 1 \pmod 5.$$

If we denote the values of x and y corresponding to $\xi=\xi_m$ and $\eta=\eta_m$ as x_m and y_m respectively, comparing the last congruence above with (3) we have the condition

$$0 \equiv (-1)^n + 1 \pmod 5;$$

whence y_m is integral if and only if $n=2k+1$. Adding the two equations (3), we get $x+y=\xi+\eta-2$, so that x_m is integral if and only if y_m is integral.

The complete solution of (1) is then given by (8) and (9) where $m=2n+1$, $n=2k+1$. It can be shown without difficulty that the solutions of (1) correspond to the positive values of m. Therefore, unless otherwise specified, m will henceforth be a positive integer, congruent to 3, modulo 4.

Relations to the Numbers of Fibonacci.[1]

It can be shown that

$$\begin{aligned} \xi_i &= u_{i-1} \\ \eta_i &= u_i \end{aligned} \quad i = 1, 2, \cdots, \tag{10}$$

where u_i is the ith Fibonacci Number. For, from (5)

$$\omega^{i+1} = \xi_{i+1} + \eta_{i+1}\omega,$$

and multiplying (5) by ω and reducing by $\omega^2=\omega+1$,

$$\omega^{i+1} = \eta_i + (\xi_i + \eta_i)\omega.$$

These give, upon eliminating ω^{i+1} and equating powers of ω:

$$\begin{aligned} \xi_{i+1} &= \eta_i \\ \eta_{i+1} &= \xi_i + \eta_i \end{aligned}$$

[1] The facts regarding the numbers of Fibonacci made use of in this paper may be found in Lucas, *Théorie des Nombres*, vol. 1, Paris, 1891, pp. 3 f., 127.

Using these relations together, we have

$$\xi_{i+1} = \xi_i + \xi_{i-1} \tag{11}$$

$$\eta_{i+1} = \eta_i + \eta_{i-1}. \tag{12}$$

But the numbers of Fibonacci may be defined by

$$u_0 = 0,\ u_1 = 1,\ u_{i+1} = u_i + u_{i-1}$$

and using $m=0, 1, 2$ in (5) we obtain

$$\begin{array}{ll} \xi_0 = 1 & \eta_0 = 0 \\ \xi_1 = 0 & \eta_1 = 1 \\ \xi_2 = 1 & \eta_2 = 1 \end{array}$$

from which (10) follows.

All solutions of (1) can then be written

$$\begin{aligned} x_m &= (3u_{m-1} + 4u_m - 6)/5, \\ y_m &= (2u_{m-1} + u_m - 4)/5. \end{aligned} \tag{13}$$

By applying the relations

$$u_i = u_{i-1} + u_{i-2} \tag{14}$$

and

$$u_{k+j} = u_{j+1}u_k + u_j u_{k-1} \tag{15}$$

these can be brought into the form

$$\begin{aligned} x_m &= (u_{m+3} + u_{m+1} - 1)/5 - 1, \\ y_m &= (u_{m+1} + u_{m-1} + 1)/5 - 1. \end{aligned} \tag{16}$$

From (14) the relation

$$u_{i+3} + u_{i+1} = 5u_i + u_{i-1} + u_{i-3} \tag{17}$$

easily follows. Using this with (16) the following can be shown by induction to hold:

$$\begin{aligned} x_m &= u_m + u_{m-4} + \cdots + u_3 - 1 \\ y_m &= u_{m-2} + u_{m-6} + \cdots + u_1 - 1. \end{aligned} \tag{18}$$

If we denote by $U_{k,n}$ the sum of the first $n+1$ Fibonacci numbers whose subscripts are congruent to k, modulo 4, that is

$$U_{k,n} = u_k + u_{4+k} + \cdots + u_{4n+k},\ k = 0, 1, 2, 3,$$

then

$$U_{k,n} = u_{2(n+1)}u_{2n+k},\ k = 0, 1, 2, 3. \tag{19}$$

For, by using (15) we have

$$u_{4n+k+4} + u_{2(n+1)}u_{2n+k} = u_{2n+3}u_{2n+2+k} + u_{2n+2}u_{2n+1+k} + u_{2n+2}u_{2n+k}$$

and the right hand side of this reduces to

$$u_{2(n+2)}u_{2(n+1)+k}$$

upon combining and applying (14). Hence (19) may be easily established by induction; in fact, if

$$U_{k,n} = u_{2(n+1)}u_{2n+k}$$

then

$$U_{k,n} + u_{4n+k+4} = u_{2(n+1)}u_{2n+k} + u_{4n+k+4}$$

or, by the preceding and (14),

$$U_{k,n+1} = u_{2(n+2)}u_{2(n+1)+k}.$$

Referring now to (18) it is clear that

$$\begin{aligned} x_m &= U_{3,(m-3)/4} - 1 \\ &= u_{(m+1)/2}u_{(m+3)/2} - 1, \end{aligned} \tag{20}$$

and

$$\begin{aligned} y_m &= U_{1,(m-3)/4} - 1 \\ &= u_{(m+1)/2}u_{(m-1)/2} - 1. \end{aligned} \tag{21}$$

Also, using (18) and (14),

$$\begin{aligned} x_m - y_m &= u_m - u_{m-2} + u_{m-4} - u_{m-6} + \cdots + u_3 - u_1 \\ &= u_{m-1} + u_{m-5} + \cdots + u_2 \\ &= U_{2,(m-3)/4} \\ &= u^2{}_{(m+1)/2}. \end{aligned} \tag{22}$$

Again, using (18) and (14)

$$\begin{aligned} x_m + y_m &= u_m + u_{m-2} + u_{m-4} + u_{m-6} + \cdots + u_3 + u_1 - 2 \\ &= 3u_{m-2} + u_{m-3} + 3u_{m-6} + u_{m-7} + \cdots + 3u_1 + u_0 - 2 \\ &= 3y_m + U_{0,(m-3)/4} + 1, \end{aligned}$$

so that

$$\begin{aligned} x_m - 2y_m &= U_{0,(m-3)/4} + 1 \\ &= u_{(m+1)/2}u_{(m-3)/2} + 1 \\ &= u^2{}_{(m-1)/2}, \end{aligned} \tag{23}$$

since $u_i^2 = u_{i-1}u_{i+1} + (-1)^i$.

Using (3) we have

$$(24)\qquad \begin{aligned} x_m + y_m &= \xi_m + \eta_m - 2 \\ &= u_{m-1} + u_m - 2 \\ &= u_{m+1} - 2. \end{aligned}$$

Recurrence Relations.

The recurrence relations are

$$(25)\qquad \begin{aligned} x_m &= 8(x_{m-4} - x_{m-8}) + x_{m-12} \\ y_m &= 8(y_{m-4} - y_{m-8}) + y_{m-12}. \end{aligned}$$

By using (14) successively it can easily be shown that

$$8u_{i-4} = u_i + u_{i-4} + u_{i-8}.$$

The relations (25) follow immediately upon substituting from (18) in the right hand side and using the relation just mentioned; for example

$$\begin{aligned} 8(x_{m-4} - x_{m-8}) + x_{m-12} &= 8u_{m-4} + (u_{m-12} + \cdots + u_3 - 1) \\ &= u_m + u_{m-4} + u_{m-8} + \cdots + u_3 - 1. \\ &= x_m. \end{aligned}$$

The following relation connects the y_i with the x_i of the same and one lower rank:

$$(26)\qquad 3y_m = x_m + x_{m-4}.$$

Writing (26) as

$$3y_m - x_m = x_{m-4}$$

and substituting in the left hand side of this from (18), the result reduces to x_{m-4} upon applying (14) successively.

The following table contains the first few values of x and y which satisfy (1).

m	3	7	11	15	19	23	27
x	1	14	103	713	4894	33551	229969
y	0	5	39	272	1869	12815	87840

A Geometric Paradox

By C. H. Rowe, Trinity College, Dublin

Consider the quadric surface that is represented by the general equation of the second degree in four homogeneous coordinates. It is well known that we must impose three independent conditions on the coefficients in order that the

quadric should reduce to a pair of planes. However, the following argument *seems* to show that two conditions are sufficient.

By imposing the condition that the discriminant of the equation should vanish we ensure that the quadric is degenerate, and that we can transform to a new system of homogeneous coordinates so that the transformed equation contains only three of the four variables. One further condition is sufficient to ensure that this transformed equation represents a pair of planes. The total number of conditions that we need is therefore two.

A Note by the Editor

Professor Rowe writes: "Encouraged by the interest shown in Professor Coolidge's *Paradox* in the American Mathematical Monthly, I venture to send you another paradox which contains a different kind of fallacious argument about the number of conditions that are required for a certain geometrical property."

Now will some one furnish the explanation for Professor Rowe's paradox?

R.E.G.

Rational Right Triangles

By H. T. R. Aude, Colgate University

This paper presents a method of finding integral solutions of the equation $x^2+y^2=z^2$, or of obtaining rational right triangles. It uses one of the acute angles of the triangle as a function and a few trigonometric formulas. These help to show the necessary and sufficient conditions and assist in establishing a systematic procedure.

The problem may be limited to finding only the primitive right triangles where x, y and z, the integers for the sides, are relatively prime. Let A be the angle opposite to y, then it will be possible to find integers for x, y, and z, only if $\sin A$ and $\cos A$ are both rational. From the formula $\tan A/2=\sin A/(1+\cos A)$ it is seen that $\tan A/2$ will also be rational; and since A is an acute angle of the triangle, $A/2<\pi/4$. Conversely, if $\tan A/2$ is a positive proper fraction, then A is acute and belongs to a right triangle, whose sides are the integers x, y, z, where y/z and x/z are respectively equal to $2 \sin A/2 \cos A/2$ and $\cos^2 A/2-\sin^2 A/2$. If $\tan A/2$ is taken equal to n/m, where n and m are integers relatively prime and $n<m$, then a set of values for x, y, z are

$$m^2 - n^2, \quad 2mn, \quad m^2 + n^2. \tag{1}$$

In order to obtain all rational primitive right triangles with a hypotenuse h less than a certain integer p we select systematically values for $\tan A/2=n/m$ such that the integers m and n, relatively prime, satisfy the conditions $m^2+n^2<p^2$. All the proper fractions thus formed may be arranged in two groups:

(I) those for which m and n are not both odd numbers, and (II) those for which m and n are both odd. If A and B denote the two acute angles of the right triangle and if $\tan A/2 = n/m$, then $\tan B/2 = (m-n/(m+n)$. And, furthermore, it becomes evident upon examination that if n/m is a fraction of group (I) n and m not both odd, then $(m-n)/(m+n)$ is a fraction of group (II) with numerator and denominator both odd; and the converse also holds. In fact this relation between the fractions of the two groups is involutory. The fractions in group (I) when selected for $\tan A/2$ lead by formulas (1) to the same right triangles as the fractions in group (II) which represent in some order corresponding values of $\tan B/2$. Therefore, by omitting entirely group (II) and choosing for $\tan A/2 = n/m$ only those proper fractions where m and n are not both odd, there is obtained once and once only all the primitive right triangles with $h < p$.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y. and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Nomographie. By H. Fréchet and H. Roullet. Paris, Librarie Armand Colin, 1928. Paper, 208 pages. 10.50 fr.

Handbook of Statistical Nomographs, Tables, and Formulas By J. W. Dunlap and A. K. Kurtz. Yonkers, N. Y., World Book Company, 1932. x+164 pages.

The Physical Significance of the Quantum Theory. By F. A. Lindemann. Oxford, the Clarendon Press, 1932. viii+148 pages. $2.50.

National Council of Teachers of Mathematics. The Fifth Yearbook: The Teaching of Geometry. xii+206 pages. The Sixth Yearbook: Mathematics in Modern Life. xii+126 pages. The Seventh Yearbook: The Teaching of Algebra. xii+180 pages. New York, Teachers College, 1930, 1931, 1932.

Mathematical Nuts for Lovers of Mathematics. By Samuel I. Jones. Nashville, Published by the Author, 1932. xii+340 pages.

Partial Differential Equations of Physics. By H. Bateman. New York, Macmillan, 1932. xxii+522 pages.

Modern Junior Mathematics. By W. W. Hart. Three books, with teacher's manuals. New York, D. C. Heath, 1931.

C-D Machine Correlation Chart. Designed by E. E. Cureton and J. W. Dunlap. 25 sheets, $16\frac{1}{2}'' \times 11''$. New York, Macmillan, 1932. $1.00.

Die Liesche Theorie der partiellen Differentialgleichungen Erster Ordnung. By F. Engel and K. Faber. Leipzig, B. G. Teubner, 1932. xii+368 pages. 28 marks.

Freshman Mathematics; a course in the essentials of Algebra, Trigonometry, and Analytic Geometry. By H. L. Slobin and W. E. Wilbur. New York, Long and Smith, 1932. xviii+438 pages.

Algebra. By Oskar Perron. I. Die Grundlagen. Göschens Lehrbucherei, I Gruppe, Band 8. Berlin, Walter de Gruyter, 1932. viii+302 pages. 12.80 marks.

Dynamics of Engine and Shaft. By Ralph E. Root. New York, John Wiley, 1932. viii+184 pages. $3.00.

Le Calcul des Différences finies et ses applications. By Alfred Henry. Paris Hermann, 1932. 212 pages. 50 francs.

Outline of the History of Mathematics. By R. C. Archibald. Bulletin 18 of the Society for Promotion of Engineering Education, 1932. 54 pages.

Catalog of an Exhibition at Columbia University to commemorate the one hundredth anniversary of the birth of Lewis Carroll. New York, Columbia University Press, 1932. 154 pages. 35 cents.

Philosophische Versuch über die Wahrscheinlichkeit. By P. S. Laplace. Ostwalds Klassiker der Exakten Wissenschaften, bo. 233. Leipzig, Akademische Verlagsgesellschaft M. B. H., 1932. viii+212 pages, 9.60 marks.

College Algebra. By H. P. Pettit and P. Luteyn. New York, John Wiley and Sons, 1932. viii+284 pages. $1.90.

The Calculus. By H. H. Dalaker and H. E. Hartig. Second Edition. New York, McGraw Hill Book Company, 1932. viii+276 pages. $2.25. (From the preface to the Second Edition: This edition is substantially the same as the first. A considerable number of exercises and a chapter on differential equations have been added.)

REVIEWS

The Foundations of Mathematics. By F. P. Ramsey. New York, Harcourt, Brace, & Co., 1931. xviii+292 pages.

This is a collection of essays in mathematics, mathematical logic, and philosophy, including the most important of the author's previously published works, and a number of papers, some of them fragmentary, which were still unpublished at the time of the author's death in 1930.

The first of these essays, which was originally published in the *Proceedings of The London Mathematical Society* in 1926, sets forth the author's proposal for revision of the system of logic contained in Whitehead and Russell's *Principia Mathematica.* Three important changes are advocated, namely, abandonment of the principle that x and y are identical (or equal) when every propositional function satisfied by x is also satisfied by y, abandonment of the principle that every class is determined by a propositional function, and a modification of the theory of types designed to avoid the much discussed axiom of reducibility.

Of these proposals, the first seems to be open to serious objection. For if x and y are any two things which have all their properties in common, and if we allow that x has the property of being identical with x, then we must allow that y also has the property of being identical with x, that is, that $y=x$. The possibility that x may not be a definable object, and therefore that the property of being identical with x may not be a definable property, is clearly irrelevant.

And the second proposal is open to an entirely similar objection, for if α is a class, then the propositional function $\hat{x}\epsilon\alpha$ determines α, and also the propositional function, "$\hat{x}$ has all the elementary properties which are common to all the members of α," determines α. Of course, there may be such things as undefinable classes, and if α is an undefinable class then there is no definable propositional function which determines α. But this, as we understand it, is not denied by the authors of *Principia Mathematica*.

The third proposal, however, is worthy of more serious consideration. Mr. Ramsey observes that the contradictions with which the theory of types is intended to deal fall into two classes, firstly what we may call the mathematical paradoxes (for example the paradox of Burali-Forti) which can be expressed entirely by means of the symbols of formal logic, and secondly what we may call the epistemological paradoxes (for example Richard's paradox) which require the use of the verb "means." Two hierarchies of types, of rather different sorts, are required, one of them for the sake of avoiding the mathematical paradoxes, and the other to avoid the epistemological paradoxes. And it is the second of these two hierarchies which renders the axiom of reducibility necessary. Mr. Ramsey's proposal is to replace this second hierarchy of types by another hierarchy of types, which he defines, and under which an axiom of reducibility is not necessary. And he then gives, in terms of this new hierarchy, a solution of the epistemological paradoxes which depends on an ambiguity in the meaning of "means."

Distrust of the axiom of reducibility is, of course, widespread, being shared even by the authors of *Principia Mathematica*, and there seems to be no doubt of the desirability of a theory which avoids this axiom. But we cannot agree with Mr. Ramsey, that the reason for the desirability of avoiding it is that the axiom is not a tautology in the sense of Wittgenstein, or that it is desirable or necessary that all the axioms of logic should be tautologies. For the notion of a tautology in this sense depends for its intuitive significance on the identification of $(x).\phi x$ with the logical product of all possible values of ϕx, and this identification seems to be doubtful, because the nature of a logical product is such that its meaning cannot be understood without first understanding the meaning of each proposition which enters into the logical product.

And certainly the notion of a tautology loses much of its connotation of "necessary" when we discover that the axiom of infinity is a tautology if it be true, but a contradiction if it be false.

It is worth remarking that the proof that the axiom of infinity is a tautology (if true) depends, not, as the author seems to imply, on his proposed revision of

the notion of identity, but wholly on the introduction of propositional functions in extension. This notion of a propositional function in extension is certainly legitimate, but it seems doubtful whether the distinction can successfully be maintained between ordinary propositional functions and propositional functions in extension.

In the second essay of the book, the author defends his treatment of mathematical logic against the formalism of Hilbert and the intuitionism of Brouwer and Weyl. The third essay, which is devoted to the solution of a particular case of the Entscheidungsproblem, contains also a theorem of the theory of classes which is not without interest on its own account. And the remaining papers are, for the most part, on subjects which are philosophical, rather than mathematical, in character. All of them are well worth reading, for the sake of the author's insight into the questions of which he treats, and for their power of stimulating thought on the part of the reader.

ALONZO CHURCH

Vectorial Mechanics. By Louis Brand. New York, John Wiley and Sons, Inc., 1930. xvii+544 pages.

This is a very carefully planned textbook in which the fundamental problems of statics, kinematics and dynamics of a particle and of rigid bodies are treated methodically by the vector method. An introductory chapter of thirty-eight pages gives a clear account of the vector algebra necessary for problems of statics. Four chapters (132 pages) are devoted to statics and following this a chapter of fourteen pages is devoted to vector analysis. Chapters 7 to 10, inclusive (154 pages) are devoted to flexible cables and kinematics and these are followed by four chapters (174 pages) on dynamics. A final chapter treats problems of impact. Every chapter has examples fully and carefully worked out and closes with a summary of the important results. The typography is excellent and there seem to be very few misprints. It is clear that the book will be very useful to students of mechanics.

The author states in his preface that the book has been written with a view to engineering applications, but the mathematics involved has not been shirked. An engineer mastering this book will know a lot of geometry and algebra. It is perhaps inevitable that certain compromises have to be made. Thus in the very beginning a vector quantity is defined as one involving the idea of direction, but this is a vague idea. It is necessary for the author to say later that the zero vector is not a proper vector because it does not have a determinate direction The zero vector is in some respects the most fundamental vector of all since every vector equation expresses that a combination of vectors is the zero vector and it is very unfortunate to have, or to convey, the impression that there is something improper about this fundamental vector. Again in vector analysis it might perhaps be emphasized somewhat more that a vector can not be differentiated without considering a reference frame. It is very important for a student

to grasp the obvious fact that such things as velocity, momentum, energy are not properties of a system, but expressions of the relation of the system to a reference frame. However, your reviewer has sufficiently expressed his opinions on these matters in the book "Theoretical Mechanics" (written in collaboration with President Ames) and he need say no more than that he can strongly recommend the present work to beginning students and teachers alike. The wise teacher will tell his students about other good textbooks, such as Routh's five volumes or Hamel's fine book "Elementare Mechanik."

F. D. MURNAGHAN.

Fourier'sche Reihen. By J. Wolff. Groningen, P. Noordhoff N. V., 1931. 60 pages.

This booklet has for its purpose the exposition of certain well known properties of Fourier Series. The text is divided into three sections each supplemented by exercises which total over a hundred. The first two sections deal with Riemann integrable functions and are intended for students of pure as well as of applied mathematics, while the last section deals with summable functions and is intended for students interested primarily in pure mathematics.

The first section contains proofs of the usual type of pointwise and of uniform convergence under simple hypotheses on the function; nothing is said about the degree of the convergence. In the second section, a discussion of the properties of the first arithmetic mean of Féjer is followed by more general theorems which have for their basis the least square property of the partial sum of the Fourier Series. Theorems previously proved are extended to the case of summable functions in the first part of the third section. These are followed by general propositions on sequences of measurable sets and functions which form a basis for theorems concerning the Fourier coefficients and the validity of the Fourier expansion of suitably restricted summable functions. Misprints which occur are few in number and easily recognizable as such.

J. M. EARL

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL. OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3557. *Proposed by Robert E. Moritz, University of Washington*

Given a free flow of water from a vertical circular aperture, flowing partially full. The radius of the aperture is r, the depth of the water above the center of the aperture is a, the coefficient of entry is k. Required the rate of discharge under the action of gravity.

3558. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Of a plane triangle given one side, the length of the bisector of the opposite angle, and the altitude on the given side; state and prove the ruler-compass construction of the triangle.

3559. *Proposed by George A. Yanosik, New York University.*

Variable circles are drawn having any point on a central conic and one of its foci as ends of a diameter. Prove that the envelope of these circles is the auxiliary circle of the central conic.

3560. *Proposed by Frank Morley, Johns Hopkins University.*

In a Euclidean space, perpendiculars from the vertices of a regular tetrahedron to a plane meet the plane at the points represented by the complex numbers x_i, $i=1, 2, 3, 4$. Show that the four points obey the relation

$$\sum^{6}(x_i - x_j)^2 = 0.$$

3561. *Proposed by Emmanuel Wad, Baltimore, Md.*

The number 12345678 is not divisible by 11, but by placing the eight figures in different orders we can form other numbers which are divisible by 11. Determine how many such numbers can be formed.

SOLUTIONS

3233. [1927, 45]. *Proposed by A. A. Bennett, Lehigh University.*

Describe geometrically the continuous function represented by the Fourier series, $\Sigma(\cos mt)/m^2$, where m is restricted to positive integers prime to 6. (This question arose in the theory of three-phase alternating currents.)

Solution by Robert E. Moritz, University of Washington

The series is obviously absolutely convergent and will remain so if it is augmented by the terms corresponding to values of m which are not prime to 6.

Denote the sum of the given series by y, then we have

$$(1) \qquad y = y_1 - y_2 - y_3 + y_6,$$

where, m taking in succession the values 1, 2, 3, etc.,

$$(2)\ y_1 = \sum\frac{\cos mt}{m^2},\quad y_2 = \sum\frac{\cos 2mt}{(2m)^2},\quad y_3 = \sum\frac{\cos 3mt}{(3m)^2},\quad y_6 = \sum\frac{\cos 6mt}{(6m)^2},$$

and we may obviously write,

$$(3)\qquad \begin{cases} y_2 = \dfrac{1}{4}\sum\dfrac{\cos mu}{m^2}, & u = 2t; \\ y_3 = \dfrac{1}{9}\sum\dfrac{\cos mv}{m^2}, & v = 3t; \\ y_6 = \dfrac{1}{36}\sum\dfrac{\cos mw}{m^2}, & w = 6t. \end{cases}$$

Now the sum y_1 is well-known.[1] Its value is

$$(4)\qquad y_1 = \frac{1}{12}[3(t - \overline{2k+1}\pi)^2 - \pi^2], \qquad 2k\pi \leqq t \leqq 2(k+1)\pi;$$

hence also

$$(5)\qquad \begin{cases} y_2 = \dfrac{1}{4\cdot 12}[3(2t - \overline{2k+1}\pi)^2 - \pi^2], & k\pi \leqq t \leqq (k+1)\pi; \\ y_3 = \dfrac{1}{9\cdot 12}[3(3t - \overline{2k+1}\pi)^2 - \pi^2], & 2k\pi/3 \leqq t \leqq 2(k+1)\pi/3; \\ y_6 = \dfrac{1}{36\cdot 12}[3(6t - \overline{2k+1}\pi)^2 - \pi^2], & k\pi/3 \leqq t \leqq (k+1)\pi/3. \end{cases}$$

Let us consider the positive t-axis divided into intervals of $\pi/3$ each, beginning at the origin. The values of k in y_1, y_2, y_3, y_6, depend on the interval in which t lies and may be readily computed. Let n be one less than the number of the interval in which t lies. The value of k in y_1 will then be the quotient obtained on dividing n by 6, the values of k in y_2 and y_3 will be the integral parts of the quotients obtained on dividing n by 3 and 2 respectively, and the value of k in y_6 will be n itself. If we denote by n_i the integral part of the quotient of n divided by i, we have from (1), (4), and (5)

$$y = \frac{1}{12}[3(t - \overline{2n_6+1}\pi)^2 - \pi^2] - \frac{1}{4\cdot 12}[3(2t - \overline{2n_3+1}\pi)^2 - \pi^2]$$

$$- \frac{1}{9.12}[3(3t - \overline{2n_2+1}\pi)^2 - \pi^2] + \frac{1}{36.12}[3(6t - \overline{2n+1}\pi)^2 - \pi^2]$$

which on reducing gives

$$(6)\qquad y = A\pi t + A'\pi^2,$$

[1] Weber, H., *Die Partiellen Differentialgleichungen*, 6th Edition, Vol. 1, p. 76.

where

$$A = (-6n_6 + 3n_3 + 2n_2 - n - 1)/6,$$
$$A' = [36n_6(n_6 + 1) - 9n_3(n_3 + 1) - 4n_2(n_2 + 1) + n(n + 1) + 4]/36.$$

The quantities A and A' may be further simplified. By definition

$$n_6 = (n - r_6)/6, \quad n_3 = (n - r_3)/3, \quad n_2 = (n - r_2)/2,$$

where r_6, r_3, r_2, are the remainders resulting from the division of n by 6, 3, and 2 respectively. These values substituted in A and A' give

$$y = A\pi t + (Bn + C)\pi^2, \tag{7}$$

where

$$A = (r_6 - r_3 - r_2 - 1)/6, \quad B = -A/3,$$
$$C = [r_6^2 - r_3^2 - r_2^2 + 4 - (6r_6 - 3r_3 - 2r_2)]/36,$$

and n is one less than the interval in which t lies.

Since each remainder r_i is limited to 0 or an integer less than i, we see that A, B, and C, admit each of at most 6 different values which are repeated cyclically for each successive set of six intervals.

Let us tabulate the successive values of r_6, r_3, r_2, in the first six intervals, and compute the corresponding values of A, B, and C.

Interval	1	2	3	4	5	6
$n =$	0	1	2	3	4	5
$r_2 =$	0	1	0	1	0	1
$r_3 =$	0	1	2	0	1	2
$r_6 =$	0	1	2	3	4	5
$6A =$	-1	-2	-1	$+1$	$+2$	$+1$
$18B =$	$+1$	$+2$	$+1$	-1	-2	-1
$18C =$	$+2$	$+1$	-1	-2	-1	$+1$
$18(Bn + C) =$	$+2$	$+3$	$+1$	-5	-9	-4

We can now write down the value of the sum of the given series for any given value of t.

If

$$0 \leqq t \leqq \pi/3, \qquad y = -\pi t/6 + \pi^2/9;$$
$$\pi/3 \leqq t \leqq 2\pi/3, \qquad y = -\pi t/3 + \pi^2/6;$$
$$2\pi/3 \leqq t \leqq \pi, \qquad y = -\pi t/6 + \pi^2/18;$$
$$\pi \leqq t \leqq 4\pi/3, \qquad y = +\pi t/6 - 5\pi^2/18;$$
$$4\pi/3 \leqq t \leqq 5\pi/3, \qquad y = +\pi t/3 - \pi^2/2;$$
$$5\pi/3 \leqq t \leqq 2\pi, \qquad y = +\pi t/6 - 2\pi^2/9; \text{ etc.}$$

Geometrically considered the series represents an infinite number of line-segments whose slopes are limited to four values of A and whose intercepts vary indefinitely with t.

3298. [1927, 537]. *Proposed by W. A. Thompson, Supt. of Schools, Webster, South Dakota.*

Given two circles of unequal sizes, such circles intersecting each other, to draw a secant cutting both circles, such that the chords of the circles are equal to each other and equal to the segment of the line between the two chords.

Solution by Otto Dunkel, Washington University.

Let the two circles have the points O_1, O_2 as centers; let their radii be R, r; let $O_1 O_2 = a$; and let $2x$ be the length of the required chord. We find at once by inspection of a figure

$$[(R^2 - x^2)^{1/2} \pm (r^2 - x^2)^{1/2}]^2 + 16x^2 = a^2,$$

where the plus sign refers to the case where the required secant cuts within the segment O_1O_2, and the minus sign to the case where the secant cuts the extension of O_1O_2. This equation is easily solved for x^2, and it will be shown how to determine the number and character of the real constructions given by this solution for x.

The given problem will be considered as including equal circles as well as non-intersecting circles. If the given circles are tangent, the common tangent gives a trivial construction; if a required secant passes through the center of the smaller circle, we shall call this a diameter secant; if it cuts within the segment O_1O_2 we shall call it a W secant; if outside, an E secant. Two constructions for which the secants are symmetrically placed with respect to O_1O_2 will be considered as a single construction.

If we set $x^2 = y$, $a^2 = z$, then

$$z = 14y + R^2 + r^2 \pm 2[(R^2 - y)(r^2 - y)]^{1/2}, \quad R \geqq r, \tag{1}$$

where the plus sign gives a W construction, the minus sign, an E construction. With a horizontal y-axis and a vertical z-axis, the above equation is that of an hyperbola, for which the part above the diameter $z = 14y + R^2 + r^2$ gives the W solutions, while the part below gives the E solutions. If $R = r$, the curve degenerates into the two straight lines

$$z = 12y + 4R^2, \quad z = 16y, \tag{2}$$

which meet in the point $(R^2, 16R^2)$. Since r^2 and R^2 enter symmetrically in (1) and in the given diameter, we obtain the equations of the asymptotes and the center

$$z = 12y + 2(R^2 + r^2), \quad z = 16y, \quad [(R^2 + r^2)/2, \ 8(R^2 + r^2)] \tag{3}$$

by replacing $2R^2$ in the above results by R^2+r^2. The z-intercepts of (1) are $(R-r)^2$ and $(R+r)^2$, while those of the asymptotes are 0 and $2(R^2+r^2)$. Since $0 \leqq (R-r)^2 < (R+r)^2 \leqq 2(R^2+r^2)$, the position of the curve between the asymptotes is located. One branch of the curve is beyond $y=(R^2+r^2)/2 \geqq r^2$, and this part can give no real construction. Moreover, the part on the negative side of $y=0$ gives no real construction. Thus the real constructions are confined to that part of one branch which lies within the triangle formed by the two straight lines in (3) and the z-axis.

There is a vertical tangent at $[r^2, R^2+15r^2]$, the point of separation of the W and E parts of the curve. By setting $dz/dy=0$, we find a horizontal tangent at (y_1, z_1), where

$$y_1 = [(12 - 7\sqrt{3})R^2 + (12 + 7\sqrt{3})r^2]/24,$$
$$z_1 = 4[(2 - \sqrt{3})R^2 + (2 + \sqrt{3})r^2],$$

and this fixes the greater limit for a^2 in certain cases. In the following separation of the cases, the single letter E denotes a single real construction of this type, while E, W denotes two real constructions, one of each type, etc.

Case I. $R \leqq 7r$.

$$0 \leqq (R - r)^2 \leqq a^2 < (R + r)^2,\ E;\ (R + r)^2 \leqq a^2 < R^2 + 15r^2,\ E,\ W;$$
$$R^2 + 15r^2 \leqq a^2 \leqq z_1,\ W,\ W.$$

If $R=r$, the last interval reduces to a point.

Case II. $7r<R<(7+4\sqrt{3})r$.

$$0 < (R - r)^2 \leqq a^2 < R^2 + 15r^2,\ E;\ R^2 + 15r^2 \leqq a^2 < (R + r)^2, W;$$
$$(R + r)^2 \leqq a^2 \leqq z_1,\ W,\ W.$$

Case III. $R \geqq (7+4\sqrt{3})r$.

$$0 < (R - r)^2 \leqq a^2 < R^2 + 15r^2,\ E;\ R^2 + 15r^2 \leqq a^2 \leqq (R + r)^2,\ W.$$

The trivial constructions are easily recognized in the above scheme; when $a^2=R^2+15r^2$, there is a diameter construction; when $a^2=z_1$, the two W constructions coincide in the same secant.

We may make use of a different unknown. Let M be the middle point of O_1O_2; N, the intersection of the radical axis of the two circles with this line, $a=4u$, $MN=m$. If the secant cuts the radical axis in the point P, then MP is perpendicular to the secant and P is the mid-point of the segment between the circles (see the solutions of problem 3288 [1927, 491] in this Monthly, vol. 36 (1929), pp. 108–110). Let the absolute value of the power of N with respect to either circle be p^2; the power is positive if the circles do not intersect and negative if they do. Then the secant can be constructed from $NP=t$, which satisfies the equation

$$t^4 + (m^2 - 3u^2 \pm p^2)t^2 \pm p^2m^2 = 0,$$

where

$$2am = R^2 - r^2, \ \pm 4a^2p^2 = (a^2 - R^2 - r^2)^2 - 4r^2R^2, \quad 3x^2 = t^2 \pm p^2.$$

The equation in x may be written in the form

$$3x^4 + (m^2 - 3u^2 \mp p^2)\, x^2 \pm u^2p^2 = 0,$$

or

$$48x^4 + [8(R^2 + r^2) - 7a^2]x^2 \pm a^2p^2 = 0,$$

3482. [1931, 170]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

If the circumscribed and the inscribed spheres of a tetrahedron are concentric, the sum of the face angles of each trihedral angle of the tetrahedron is equal to two right angles.

Solution by Wallace Smith, New River State College, Montgomery, W. Va.

Sphere S is circumscribed about and sphere s is inscribed in the tetrahedron $ABCD$, each having the center O. Since planes ABD, BDC, ADC, and ABC are tangent to sphere s, they are at equal distances from the center of sphere S. Planes equidistant from the center of a sphere intersect the sphere in equal circles. Therefore, triangles ABD, ABC, ADC, and BCD are inscribed in equal circles. The chords AD, AB, and BD are each common to two equal circles. It follows at once that $\angle ACD = \angle ABD$, $\angle BCA = \angle BDA$, and $\angle BCD = \angle BAD$. Therefore, $\angle BCD + \angle BCA + \angle ACD$ equals $\angle BAD + \angle BDA + \angle ABD$ equals two right angles.

A similar proof applies to the other three vertices.

Therefore, the sum of the face angles of each trihedral angle of the tetrahedron is equal to two right angles.

Also solved by V. F. Ivanoff, A. Pelletier, J. Rosenbaum, Otto J. Ramler, and F. Underwood.

A Note by Otto Dunkel. In the above proof the fact that pairs of angles such as ACD and ABD are equal rather than supplementary follows from the fact that each of such face angles is acute. For, the center O of the inscribed sphere s is the intersection of the three planes bisecting the dihedral angles at AB, BC, CA. Thus, if the point of tangency of s with the face ABC is T_d, this point must lie within each angle of the triangle ABC, and, consequently, it must lie within the area of that triangle. Moreover, since O is also the center of S, T_d is the center of the circle circumscribing ABC and lying upon S. Since the center of the circumcircle lies within ABC, all its angles must be acute; and, similarly,

all the face angles of the tetrahedron must be acute. It was proved above that the four circumcircles are equal.

Let the other three faces be rotated into the plane of ABC so that the vertex D falls at D', D'', D''', where A and D', B and D'', C and D''', lie, respectively, on opposite sides of BC, CA, AB. As shown above the interior angles at D', D'', D''' are equal, respectively, to the angles A, B, C of triangle ABC. Also $\angle D''AC=C'$, $\angle D'''AB=B'$, where $A+B'+C'=180°$; thus $D''AD'''$ is a straight line. Similarly, $D'''BD'$ and $D'CD''$ are straight lines. Hence $D'D''D'''$ is a triangle with C, A, B as the mid-points of its sides. Therefore BC is parallel to $D'''D''$, and $B'=B$, $C'=C$. Continuing in this way we see that the four triangles are congruent. Thus the tetrahedron $ABCD$ is such that its opposite edges are equal.

Such tetrahedrons may be constructed by taking any triangle $D'D''D'''$ which has only acute angles and reversing the above process.

3491. [1931, 339]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

The internal bisectors of the angles subtended by the six edges of a tetrahedron at the centroid of the tetrahedron are such that the three lines joining the points on the pairs of opposite edges are concurrent.

Solution by F. Underwood, University College, Nottingham.

Let G be any point in space, and let E be the intersection with AB of the internal bisector of the angle at G of the triangle GAB. Let F, H, K, L, M be points on CD, AC, BD, AD, BC, respectively, determined with respect to G in a similar manner. Denote AG, BG, CG, DG by a, b, c, d, respectively. Then

$$\frac{AE}{a}=\frac{EB}{b},\quad \frac{CF}{c}=\frac{FD}{d}\,. \tag{1}$$

Consider now the centroid of a system of particles of masses m/a, m/b, m/c, m/d, attached to A, B, C, D, respectively. Then by (1) E is the centroid of the particles at A and B, and F is the centroid of the particles at C and D. Hence the centroid of the four particles lies on EF. Similarly, it lies on HK and on LM; and therefore EF, HK, and LM are concurrent.

A Note by Otto Dunkel: This problem is closely related to the system of barycentric coordinates with $ABCD$ as the tetrahedron of reference, in which the coordinates α, β, γ, δ of a point P are defined as equal, or proportional, to the volumes of the tetrahedrons $PBCD$, $APCD$, $ABPD$, $ABCP$. The coordinates may be considered as positive if P is inside $ABCD$. If the plane of PCD cuts AB in P_{ab}, then

$$\frac{AP_{ab}}{P_{ab}B}=\frac{\text{vol. } APCD}{\text{vol. } PBCD}=\frac{\beta}{\alpha},$$

since the two tetrahedrons have the base PCD in common. The remaining edges are divided in a similar manner. The lines AP_{bc}, BP_{ca}, CP_{ab} meet in a point P_d, the intersection of DP with the face ABC. The planes PCD, PAB have obviously the points P, P_{ab}, P_{cd} in common and hence these three points lie in a straight line. There are similar results for the remaining two such pairs of points.

Conversely, it is easily seen that, if α, β, γ, δ are given and if the edges of $ABCD$ are divided as above, a point P is determined with these numbers as coordinates. It is also easily shown that if particles of masses α, β, γ, δ are attached to the vertices of $ABCD$, the center of mass of the particles at A and B is P_{ab}; the center of mass of those at A, B, C is P_d; the center of mass of the whole system is P.

As shown in the above solution, the point G determines a point P with the barycentric coordinates a^{-1}, b^{-1}, c^{-1}, d^{-1}. These coordinates are defined even in the case where G lies on an edge, but not at a vertex. In the latter case it is usual to take all the coordinates but one as zero: the geometric theorem is trivial.

If P_{ab}' is taken on AB so that it is the harmonic conjugate of P_{ab} with respect to A, B, we obtain, proceeding similarly, six new points, which are easily shown to lie in a plane. This is called the polar plane of P with respect to $ABCD$. In the problem P_{ab}' is determined by the external bisector of $\angle AGB$. The equation of the polar plane of P in this case is

$$a\alpha + b\beta + c\gamma + d\delta = 0,$$

and the coordinates of this plane may be taken as a, b, c, d.

Also solved by A. Pelletier, and J. Rosenbaum.

3493. [1931, 339]. *Proposed by Paul Wernicke, Washington, D.C.*

An opaque circular circumference with a small transparent gap G rotates about its center with uniform velocity. A luminous point L outside illuminates through G a wire soldered to the circumference at two points and forming a curve going through C. The point of this curve illuminated by the ray LG is always to lie on a perpendicular to CL at C. (a) Give the equation of the curve formed by the wire and the points where it is soldered to the circumference, and (b) Give the velocity of the illuminated point on CL.

Solution by Wm. B. Campbell, Rangoon, Burma.

Since the curve (P) to be derived passes through C, it will be found that the light source L lies in the plane of the circumference. Set $CL=b$, $CG=a$, $CP=r$, and $\angle GCP=\theta$; then it is easily seen from a figure that

$$r = \frac{ab\cos\theta}{b - a\sin\theta}\,. \tag{1}$$

This is the equation of (P) with reference to CG. As θ increases from zero, r increases from a at G to its maximum value of $ab(b^2-a^2)^{-1/2}$ for which $\theta_1 = \text{arc sin}\,(a/b)$. It then decreases to a for which $\theta_2 = 2$ arc tan (a/b); it continues to de-

crease from this point and becomes zero when $\theta=\pi/2$. The part of the curve for $\pi/2<\theta\leqq\pi$ is symmetrical with respect to CG to the part already described The part described so far resembles a cardioid; it is perpendicular to CG at C and the upper part has the slope b/a at G. The part of the curve inside the circle and illuminated by the ray of light through G is given by

$$\theta_2 \leqq \theta \leqq \pi - \theta_2, \quad \tan(\theta_2/2) = a/b; \tag{2}$$

and the remaining part of the curve for the interval $0\leqq\theta\leqq\pi$ lies outside the circle. For the interval $\pi<\theta\leqq 2\pi$ there is a loop inside the circle and also inside the part described above, tangent to it at C and beginning at G with a slope of $-b/a$. This loop is also symmetrical with respect to CG.

The velocity v of the spot of light on (P) is $(ds/d\theta)(d\theta/dt)$, or

$$v^2 = (\omega ab)^2[\cos^2\theta(b - a\sin\theta)^{-2} + (a - b\sin\theta)^2(b - a\sin\theta)^{-4}], \tag{3}$$

where ω is the angular velocity of rotation.

Also solved by Frank L. Wilmer and the Proposer.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Secretary W. D. Cairns of the Mathematical Association of America, with Mrs. Cairns, will spend July and early August in Scotland, England and France, the remainder of August in Belgium, Holland, the Rhine country and Switzerland, attending the International Congress of Mathematicians at Zurich in September. Their mail address will be in care of the American Express Company (10, Frederick St., Edinburgh, until July 6; 84 Queen St. E. C. 4, London until July 24; 11 Rue Scribe, Paris, until August 10) and then in care of the Congress at Zurich. The address for Association business will be Oberlin, Ohio, as usual.

On April 19, 1932, Professor Robert Courant of Göttingen delivered the first William Lowell Putnam Lecture of 1932 at Harvard University on the subject "Hyperbolic partial differential equations and wave mechanics."

On April 27, 1932, Professor R. L. Moore of Texas, delivered the second William Lowell Putnam Lecture of 1932 at Harvard University on the subject "The foundations of point set theory."

Professor Arnold Dresden of Swarthmore College gave a lecture at Oberlin College April 25, 1932, on "Logic, Mathematics and Reality" under the auspices of the departments of philosophy and mathematics.

Dr. Cornelius Lanczos, on leave from the University of Frankfort, who has been visiting professor of mathematics at Purdue University during 1931–32, has received a permanent appointment at the latter institution as professor of

mathematical physics. Professor Lanczos will be in residence only during the first semester of each year.

Dr. Hassler Whitney, National Research Fellow, will conduct a Seminar in Metric Spaces at Harvard University during the first term of 1932–33.

Professor Frederick Wood, of Hamline University, has been appointed head of the department of mathematics at the University of Nevada.

Dr. John J. Gergen has been reappointed a Benjamin Pierce Instructor at Harvard University for the year 1932–33.

Dr. Vladimir Seidel has been appointed a Benjamin Pierce Instructor at Harvard University for the year 1932–33.

The following men have been appointed Part Time Instructors at Harvard University for the year 1932–33. P. A. Adams, T. L. Downs, J. S. Frame, A. S. Galbraith, Z. I. Mosesson, L. T. Moston, F. H. Steen.

Dr. L. A. Bauer, Director Emeritus of the Department of Terrestrial Magnetism, Carnegie Institution of Washington, died on April 12, 1932. He was a charter member of the Association.

Professor C. G. Simpson of the department of mathematics, College of Engineering, Milwaukee, Wis., a charter member of the Association, died February 5, 1932.

The following seventy-six doctorates with mathematics or mathematical physics as major subject were conferred during 1931 in universities in the United States and Canada; the major subject is mathematics unless otherwise specified. The university, month in which the degree was conferred, minor subject (other than mathematics), and title of dissertation are given in each case if available.

D. B. Ames, Yale, June, *A resonance problem in celestial mechanics.*

F. S. Beale, Michigan, *On the solutions of systems of linear difference equations with polynomial coefficients.*

H. F. Bohnenblust, Princeton, April, *The absolute convergence of Dirichlet series.*

Samuel Borofsky, Columbia, April, *Expansion of analytic functions into infinite products.*

Sister Leonarda Burke, Catholic, June, minors in education and physics, *On a case of the triangles in- and circumscribed to a rational quartic curve with a line of symmetry.*

L. E. Bush, Ohio State, August, *Some properties of algebras without moduli.*

S. S. Cairns, Harvard, June, *The cellular division and approximation of regular spreads.*

Helen Calkins, Cornell, November, *Some implicit functional theorems.*

A. B. Cardwell, Wisconsin, June, major in physics, minor in mathematical physics, *Photoelectric and thermionic properties of metals.*

Evelyn T. Carroll, Cornell, October, *Systems of involutorial birational transformations contained multiply in special linear line complexes.*

Emily M. Chandler (Mrs. H. H. Pixley), Chicago, August, *Waring's theorem for fourth powers.*

Max Coral, Chicago, August, *The Euler-Lagrange multiplier rule for double integrals.*

A. T. Craig, Iowa, June, *On the distribution of certain statistics derived from small random samples.*

J. H. Dillon, Wisconsin, June, major in physics, minor in mathematics, *Photoelectric properties of zinc single crystals.*

J. E. Donohue, Columbia, March, *Concerning the geometry of the second derivative of a polygenic function.*

H. L. Dorwart, Yale, June, *Certain types of criteria for the irreducibility of polynomials.*

R. D. Douglass, Massachusetts Institute of Technology, June, minor in physics, *Stirling expansions derived by means of finite de la Vallee-Poussin summation.*

Ben Dushnik, Michigan, *On the Stieltjes integral.*

J. N. Eastham, Catholic, June, minors in civil engineering and physics, *The triangles in- and circumscribed to the tacnodal rational quartic curve with residual crunode.*

H. C. T. Eggers, Michigan, *The graduation of experimental data.*

J. M. Feld, Columbia, November, *Projective pedal transformations and birational contact transformations.*

N. C. Fisk, Michigan, *An investigation of surfaces in euclidean four-space by means of three-vectors.*

A. L. Foster, Princeton, January, *A continuous development of formal logic in finite terms.*

Sister Mary de Lellis Gough, Catholic, June, minors in education and physics, *On the condition for the existence of triangles in- and circumscribed to certain types of the rational quartic curve and having a common side.*

Eli Gourin, Columbia, May, *On irreducible polynomials in several variables which become reducible when the variables are replaced by powers of themselves.*

M. L. Hartung, Wisconsin, June, minor in mathematical physics, *On a family of integral equations with discontinuous kernels.*

T. W. Hatcher, Cornell, August, minor in engineering, *Symmetric strain in an infinite plate with a circular hole.*

R. A. Hefner, Chicago, August, *The condition of Mayer for discontinuous solutions of the Lagrange problem.*

C. T. Holmes, Harvard, June, *The approximation of harmonic functions in three dimensions by harmonic polynomials.*

H. N. Hubbs, Cornell, February, minor in physics, *Rational quintic surfaces without double curves.*

W. R. Hutcherson, Cornell, June, minor in physics, *Maps of certain cyclic involutions on two dimensional carriers.*

D. R. Inglis, Michigan, *Atomic problems in the perturbation theory of quantum mechanics.*

Sister Mary Cordia Karl, Johns Hopkins, June, *The projective theory of orthopoles.*

Donald McDonough, Pennsylvania, June, *On the expansion of a certain type of determinant.*

W. O. Menge, Michigan, *A direct method of obtaining transformations to canonical forms.*

H. L. Miller, Cincinnati, June, *On the summability of non-integral orders of the double Fourier series.*

G. E. Moore, Illinois, June, minor in astronomy, *The four-termed theta identities arising from a generalization of the Weddle surface in* S_4.

C. B. Morrey, Harvard, June, *Invariant functions of conservative surface transformations.*

Sister Charles Mary Morrison, Catholic, June, minors in philosophy and physics, *The triangles in- and circumscribed to the biflecnodal rational quartic curve.*

M. E. Mullings, Cincinnati, June, minor in physics, *On Gibb's phenomenon in the double Fourier series.*

C. V. Newsom, Michigan, *On the behavior of entire functions in distant portions of the plane.*

E. G. Olds, Pittsburgh, June, *The nature of the distributions in small samples.*

A. L. O'Toole, Michigan, *On symmetric functions and symmetric functions of symmetric functions.*

L. J. Paradiso, Cornell, October, minor in economics, *Solutions of bounded variation of the Fredholm Stieltjes integral equation.*

W. V. Parker, Brown, June, *The addition formulas for hyperelliptic functions.*

Jesse Pierce, Michigan, *Logarithmic solutions of systems of differential equations and solutions in the vicinity of isolated singular points.*

H. H. Pixley, Chicago, August, *A problem in the calculus of variations suggested by a problem in economics.*

Saul Pollock, California (Berkeley), May, *Graphic representation in enumerative geometry.*

J. E. Powell, Chicago, August, *Edge conditions for multiple integrals in the calculus of variations.*

H. R. Pyle, California (Berkeley), May, *The conditions for conformality in the elliptic and hyperbolic geometries.*

Mina S. Rees, Chicago, March, *Division algebras associated with an equation whose group has four generators.*

B. J. Roberts, Iowa, August, minor in physics, *On continua in vectorial spaces.*

H. A. Robinson, Johns Hopkins, March, *Case of planar motion in which one centrode and one path are circles.*

S. L. Robinson, Iowa, August, minors in physics and philosophy, *A study of simple and multiple transitivity in topological spaces.*

A. E. Ross, Chicago, August, *On representation of integers by indefinite ternary quadratic forms.*

T. A. Rouse, Wisconsin, June, major in physics, minor in mathematics, *Characteristic X-ray absorption.*

R. G. Sanger, Chicago, June, *Functions of lines and the calculus of variations.*

M. G. Scherberg, Minnesota, June, minor in physics, *The degree of convergence of a series of Bessel functions.*

Mabel F. Schmeiser, Ohio State, August, *Properties of arbitrary functions concerning approach to a straight line.*

C. Grace Shover, Ohio State, December, *On the class number and ideal multiplication in a rational linear associative algebra.*

James Singer, Princeton, April, *Three-dimensional manifolds and their Heegaard diagrams.*

P. K. Smith, Illinois, June, minor in physics, *On solutions of linear partial differential equations near singular points.*

T. L. Smith, Harvard, June, *The Birkhoff fluid theory of electricity.*

I. S. Sokolnikoff, Wisconsin, June, *On a solution of Laplace's equation with an application to the torsion problem for a polygon with reentrant angles.*

F. W. Sparks, Chicago, August, *Universal quadratic zero forms in four variables.*

R. C. Stephens, Iowa, August, minor in physics, *Continuous functions on abstract sets.*

W. T. Stratton, Washington (Seattle), August, minor in physics, *A study of general polar tangent curves.*

Mildred E. Taylor, Illinois, August, minor in physics, *A determination of the types of planar Cremona transformations with not more than 9 F-points.*

C. C. Torrance, Cornell, June, minor in physics, *On plane Cremona triadic characteristics.*

Sister Mary Felice Vaudreuil, Catholic, June, minors in education and physics, *Two correspondences determined by the tangents to a rational cuspidal quartic curve with a line of symmetry.*

C. C. Wagner, Michigan, *A statistical study of the dependence for four chance variables.*

W. G. Warnock, Illinois, May, minor in astronomy, *On the geometry of groups of line configurations.*

H. E. Wheeler, Chicago, June, *An application of symbolic methods to frequency arrays.*

S. S. Wilks, Iowa, June, minor in education, *On the distribution of statistics in samples from a normal population of two variables with matched sampling of one variable.*

R. P. Winch, Wisconsin, June, major in physics, minor in mathematics, *Photoelectric properties of silver.*

Yue-Kei Wong, Chicago, December, *Spaces associated with non-modular matrices with application to reciprocals.*

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teachers' agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

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CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Sixteenth Summer Meeting of the Association, Los Angeles, Calif., Aug. 29-30, 1932.

Seventeenth Annual Meeting of the Association, Atlantic City, N.J., Dec. 27-30, 1932.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1932 and reported to the Secretary.

ILLINOIS, Urbana, May 6-7.
INDIANA, Indianapolis, May 6-7.
IOWA, Cedar Falls, April 29-30.
KANSAS, Topeka, Feb. 13.
KENTUCKY, Lexington, May.
LOUISIANA-MISSISSIPPI, Oxford, Miss., March 11-12.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7.
MICHIGAN, Ann Arbor, March 19.
MINNESOTA, River Falls, Wis., May 7.
MISSOURI.
NEBRASKA, Omaha, May, 6-7.
OHIO, Columbus, Ohio, April 7.
PHILADELPHIA, Philadelphia, Pa., Nov. 26.
ROCKY MOUNTAIN, Laramie, Wyo., April 15-16.
SOUTHEASTERN, Gainesville, Fla., Mar. 18-19.
SOUTHERN CALIFORNIA, San Diego, March 26.
TEXAS, Austin, Jan. 30.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

EDITED BY

WALTER BUCKINGHAM CARVER, Editor-in-Chief

HERBERT ELLSWORTH SLAUGHT

AUBREY JOHN KEMPNER

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXIX, 1932

NUMBER 7, AUGUST-SEPTEMBER

PUBLISHED BY THE ASSOCIATION

MENASHA, WIS., AND ITHACA, N.Y.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

\$4.00 a Year, Single Copies 45 cents, to Members

\$5.00 a Year, Single Copies 60 cents, to Others

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twelfth regular meeting of the Southern California Section was held at State Teachers College, San Diego, on Saturday, March 26, 1932. Professor H. C. Van Buskirk presided.

The attendance was forty-five including the following twenty members of the Association: L. D. Ames, Harry Bateman, Clifford Bell, P. H. Daus, A. S. Gale, Raymond Garver, J. M. Gleason, J. D. Hill, G. H. Hunt, C. G. Jaeger, L. M. Klauber, G. R. Livingston, W. E. Mason, G. F. McEwen, G. E. F. Sherwood, H. M. Showman, S. E. Urner, H. C. Van Buskirk, Morgan Ward, H. C. Willett.

The meeting began with a luncheon at the Aztec restaurant on the new campus of the San Diego State Teachers College, after which it adjourned to Science Hall for a short business meeting and the program. The following officers were elected for the year 1932–33: Chairman, L. D. Ames, University of Southern California; Vice-chairman, O. W. Albert, University of Redlands; Program Committee, C. G. Jaeger, Pomona College, and Clyde Wolfe, California Institute of Technology. The next meeting was tentatively scheduled for March 18, 1933, at Pomona College.

The following program was presented:

1. "Mathematics and finance" by Julius Wangenheim, President of the Southern Title and Trust Company of San Diego, by invitation.

2. "A triangular arrangement of numbers leading to the graphical determination of Euler sequences" by L. M. Klauber, San Diego.

3. "The application of the law of mathematical probability to the behavior of gases in their pressure-volume-temperature relations" by Dr. A. Linhart, Riverside Junior College, by invitation.

4. "Personal reminiscences of Josiah Willard Gibbs" by Dean A. S. Gale, University of Rochester.

5. "Function-theoretic properties of a simple logarithmic transcendent" by Professor Morgan Ward, California Institute of Technology.

6. "A class of surfaces applicable to the sphere" by Professor C. G. Jaeger, Pomona College.

7. "On the approximate solution of certain equations" by Professor Raymond Garver, University of California at Los Angeles.

8. "On representation of integers by certain quadratic forms" by Dr. A. E. Ross, California Institute of Technology, by invitation.

Abstracts of certain of these papers follow, the numbers corresponding to the numbers of the titles.

2. The integers are arranged in a triangular order with 1 at the apex, the second line containing numbers 2 to 4, the third 5 to 9, and so forth. When the primes have been indicated, it is found that there are concentrations in certain vertical and diagonal lines, and amongst these the so-called Euler sequences

with high concentrations of primes are discovered. In particular the orderly arrangement and character of the composites in the series x^2+x+41 are pointed out.

3. A revision of the gas laws is based upon a combination of the law of mathematical probability and principles of Piezo-dynamics. The resulting equation, when changed into logarithmic form, gives a straight line, and thus permits an easy interpretation of the pressure-volume-temperature relation of gases.

5. The function in question is one of Spence's logarithmic transcendents defined for $|z|<1$ by $L(z)=z-z^2/2^2+z^3/3^2-z^4/4^2+\cdots$. The function $L(z)$ is one-valued in the entire z plane if a cut is made from -1 to $-\infty$ along the negative axis of reals, and has only the one zero, $z=0$. By virtue of certain functional equations which it satisfies, $L(z)$ is known in the entire z plane if it is known in the half-segment bounded by the x axis, the circle $|z+1|=1$ and the line $x=1/2$. For points on the unit circle whose angles are commensurable with π, $L(z)$ may be expressed in terms of the second derivative of the logarithm of the Γ function. $L(z)$ may be expressed in numerous ways as a definite integral, and satisfies a number of interesting integral relations.

6. A two parameter class of surfaces applicable to the sphere of unit radius is found by solving Gauss's and Codazzi's equations. The general form of these equations presented such difficulties that certain restrictions were put upon the form of the three fundamental quantities, D, D', D''.

7. Professor Garver pointed out that the method of solving $x^x=c$ given by E. C. Kennedy in this MONTHLY (vol. 38, 1931, pp. 449–450) is equivalent to applying Newton's method to $x \log x = \log c$. He then applied the same device to other examples, and pointed out its superiority, in certain cases, to the ordinary use of Newton's method.

8. In this paper we study representation of integers by indefinite quaternary quadratic forms $\phi=f(x, y, z)+gn^2$, where g is an arbitrary integer and the determinant of f is free from square factors. We find necessary and sufficient conditions on g in order that ϕ should be universal (that is, should represent all integers) and determine all integers represented in case ϕ is not universal. Unlike the case of positive quaternary forms, there is an infinity of classes of indefinite quaternary quadratic forms which represent all integers.

P. H. DAUS, *Secretary*

SEVENTEENTH ANNUAL MEETING OF THE OHIO SECTION

The seventeenth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, April 7, 1932. An afternoon session, dinner, and evening session were held, with the chairman of the Section, Professor W. D. Cairns, presiding.

Fifty-nine persons registered attendance, of whom the following forty-two were members of the Association: R. B. Allen, C. E. Amos, W. E. Anderson,

F. R. Bamforth, Grace M. Bareis, I. A. Barnett, P. E. Baur, H. A. Bender, L. T. Black, M. G. Boyce, C. E. Buell, C. T. Bumer, R. S. Burington, W. D. Cairns, E. H. Clarke, Rufus Crane, O. L. Dustheimer, T. M. Focke, B. C. Glover, Harris Hancock, F. C. Jonah, L. C. Knight, H. W. Kuhn, R. H. MacCullough, Florentina Mathias, G. M. Merriman, C. C. Morris, J. R. Musselman, R. L. Newlin, Rufus Oldenburger, Jesse Pierce, H. S. Pollard, C. E. Rhodes, Hortense Rickard, S. A. Rowland, H. E. Stelson, C. F. Thomas, H. A. Toops, M. O. Tripp, J. H. Weaver, R. B. Wildermuth, C. O. Williamson.

The following officers were elected for the coming year: Chairman, O. L. Dustheimer; Secretary-Treasurer, Rufus Crane; Member of Executive Committee, J. H. Weaver; Member of Program Committee, C. O. Williamson.

It is expected that the next meeting of the Section will be held on Thursday, April 6, 1933, at the Ohio State University.

At the afternoon session the following seven papers were presented:

1. "An undergraduate course leading to the study of wave mechanics" by the Chairman of the Section, Professor W. D. Cairns, Oberlin College.

2. "An application of elliptic integrals to a problem in geometry" by C. E. Rhodes, University of Cincinnati.

3. "Canonical multilinear forms with rank as the only invariant" by Rufus Oldenburger, Case School of Applied Science.

4. "A proof of the rule for evaluating an indeterminate form" by C. L. Weaver, Kent State College, by invitation.

5. "Equilateral triangles and squares" by Professor J. R. Musselman, Western Reserve University.

6. "Psychological notions expressed in mathematical form" by Professor H. A. Toops, Ohio State University, department of psychology.

7. "The trihedron and its genesis" by Dr. E. S. Loomis, Professor emeritus, Baldwin Wallace College, by invitation.

Abstracts of these papers follow:

1. Professor Cairns described a two-hour year seminar, conducted this year by the departments of mathematics and physics at Oberlin College, and adapted to senior and first-year graduate students and, as well, selected students who had had six semester hours in calculus and four semester hours in advanced analytic geometry the preceding year. The staff chooses topics in college and first-year graduate mathematics and physics, developing and correlating these with the purpose of giving the students those topics which are needful in beginning the study of wave mechanics. An outline of the topics, with illustrations of assigned problems and drawings (e.g., wave trains), was presented. The course makes it possible for undergraduate students to be ready for an intensive graduate course in wave mechanics.

2. In this paper Mr. Rhodes obtains the geometric interpretation of Landen's transformation given on page 74 of Hancock's Elliptic Integrals from a simple comparison of different methods for finding the volume between a cylinder and a cone.

3. The algebraic theorem on the equivalence of bilinear forms $a_{\alpha\beta}x_\alpha y_\beta$ of rank r to a form $x_\alpha' y_\alpha'$ ($\alpha = 1, 2, \cdots, r$) is generalized in this paper to a theorem on the equivalence of multilinear forms under linear non-singular transformations with coefficients in the field of these forms to canonical types. Similarly to the bilinear case, the matrices of these canonical forms are either identity matrices for certain operations on these forms, or identity matrices bordered by zeros. They are completely characterized by a generalized rank set which insures a factorization property for the associated forms.

4. An indeterminate form expressed as the quotient of two power series can be reduced by synthetic division and the use of the remainder theorem to the derivative form.

5. Professor Musselman discussed equilateral triangles connected with two or three given equilateral triangles or squares. In particular, he proved the two following theorems believed to be new: (a) If $A_1B_1C_1$ and $A_2B_2C_2$ are two directly similar triangles in the plane, if we construct triangles $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ directly similar to the two given triangles, then $A_3B_3C_3$ itself is similar to the others; (b) If $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, $C_1C_2C_3C_4$ are three positively ordered pseudo-squares, i.e., quadrilaterals whose diagonals are perpendicular and equal, if we construct the positively ordered pseudo-squares $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, $A_3B_3C_3D_3$, $A_4B_4C_4D_4$, then $D_1D_2D_3D_4$ is itself a pseudo-square.

6. Professor Toops believes that the teaching of mathematics generally, and of the applied branches especially, is particularly susceptible of improvement through the employment of such tested techniques as: (a) Student practice in "insight," such as seeing the possible permutations of binomials or trinomials in $a+B+C_1+\lambda/\pi$; in analyzing problems without solving them; and in seeing functions plotted in the third dimension as in stereoscopic presentation:[1] (b) Inducing faith in the inevitability and correctness of the end result, and simultaneously making a frontal attack on the initial mental blocking that sometimes affects a student confronted with a new problem, through the several expedients of solving the same problem in many ways in preference to solving many problems each in only one way; of writing out, in the initial attack on a problem, the entire series of steps or operations to be followed, divorced from the concrete numerical application to the problem in question, as a job-analyzed skeleton formula; of estimating numerical results in advance of solving the problem; of employing checking formulae in routine statistical computations, operator checks in tabular substitutions, and degeneracy checks in derivations: (c) Freeing the student from the sheer drudgery of computation through maximal use of calculating machines and computational aids, to the end that the time set free and the time gained through lack of necessity for recomputation because of the accuracy thereby secured, will be such as to enable the student to spend a maximal amount of time in an effort to generalize his results, e.g., as by plotting whole families of nomographic curves while varying the constants systemati-

[1] Vuibert, H., *Les Anaglyphes Géométriques*, Paris, Libraire Vuibert, 1912, 32 pp.

cally, or by rectifying and plotting such rectified systems on log paper or on specially devised coordinates: (d) Intense motivated drill (e.g., by paired comparisons, or permutations of terms, device) aiming at generalizing a specific operation independent of letters and orders, say the squaring of any binomial, as a series of indicated operations, and at freeing the mind to the maximal extent of all rules and crutches by establishing the "verbal set" instead. With the better insight, better background and more time thereby secured, the student also may be encouraged to coin and solve many problems more difficult than those presented in the text or assigned by the instructor: (e) Specific drill in generalization and in application. General transfer of mathematics probably takes place with such great difficulty that the attempt to transfer mathematical thinking specifically is the only remaining hope of efficacious learning, remembering and utilization of mathematics on the part of the average student. Nevertheless, the inductive psychological approach herein advocated will surely improve the attainments of even the sub-average student.

7. Dr. Loomis related the circumstances under which he conceived of an interesting geometric figure, the properties of which he described. Let BAC be the plane angle of a dihedral angle, where $AB = AC$; through B and C draw lines parallel to the edge of the dihedral angle and on these lines take points equally spaced and opposite to each other; these form with A a series of isosceles triangles. Now place each triangle so that its plane is perpendicular to the edge of the dihedral angle and contains the pair of points which determined it, and so that it is symmetrical with the bisecting plane of the dihedral angle. The solid thus formed is bounded by three faces, one generated by the base of the isosceles triangle, being a portion of a hyperbolic cylinder, and two generated by its legs, being conoidal surfaces. It has two planes of symmetry and is infinite in extent in two directions. The consideration of the properties of this figure led Dr. Loomis into some of the properties of space as considered in non-euclidean geometry. It is believed that this figure, which Dr. Loomis calls a "trihedron," does not appear anywhere in the literature of mathematics.

The evening session was given to a study of the work in mathematics that is being done in our secondary schools, as evidenced by the preparation of students entering our colleges. Tests were given on the first day of the fall semester or quarter in September, 1931, in eleven of the colleges and universities of the state to freshman students registered for mathematics. Students to the number of 1446 took this test in this manner, and 642 of these were unable to secure a 50% mark on it. The Section plans to continue these tests and extend the scope of the study, and to make public the results, in an effort to improve the situation.

RUFUS CRANE, *Secretary*

THE ANNUAL MEETING OF THE NEBRASKA SECTION

The annual meeting of the Nebraska Section was held in conjunction with the annual session of the Nebraska Academy of Sciences at the Fontenelle Hotel in Omaha, Friday afternoon May 6th, 1932, with Professor A. K. Bettinger of Creighton University as chairman.

The attendance was about thirty with several students and instructors of both Creighton University and the University of Omaha as guests. Members of the Association present were: A. K. Bettinger, C. C. Camp, A. L. Candy, J. M. Earl, A. L. Hill, J. M. Howie, T. A. Pierce, B. C. Zimmerman.

At the close of the program the following officers were elected for the coming year: Chairman, T. A. Pierce, University of Nebraska; Secretary, A. L. Hill, Peru State Teachers College; Treasurer, J. M. Howie, Nebraska Wesleyan University.

The following papers and reports were presented:

1. "Formulae and algorisms in arithmetic" by B. C. Zimmerman, Creighton University.
2. "A class of non-linear approximating functions" by Professor J. M. Earl, University of Omaha.
3. "Magic squares" by Professor A. L. Candy, University of Nebraska.
4. "Numerical evaluation of infinite series and products" by Professor C. C. Camp, University of Nebraska.
5. "A note on n-ic residues" by Professor T. A. Pierce, University of Nebraska.

Abstracts of some of these papers follow:

1. Mr. Zimmerman presented a variety of algorisms or arrangements in arithmetic operations, showing the great advantage of a left to right procedure in these operations, both in the use of computation formulae, and in the determination of the position of the decimal point in a product or in a quotient.

2. This paper considers the implications of linearity in certain convergence proofs to a class of non-linear functions of approximation.

3. Professor Candy's problem was to construct a magic square of order $2n$ from a given square of order n by what is known as the "Method of current groups." In order to make the process—which is very simple—more conspicuous, heavy lines are used to form the cells of the given square, and the small numbers written at the center of these large cells form the given magic square of order n. Now divide each of these large cells into four small cells thus giving $4n$ cells for the required square of order $2n$. Then in the four cells thus formed within the large cell that contains the number 1, write the first four consecutive numbers 1, 2, 3, 4, and in the four cells within the large cell containing the number 2 write the second four consecutive numbers 5, 6, 7, 8, in large cell No. 3 write the third consecutive group 9, 10, 11, 12. Proceed in like manner until all the cells are filled. These "current groups" are not necessarily "consecutive integers." They may be consecutive terms of any arithmetic series.

But these groups can not be written in these 2×2 squares in any order at random. Here lies the only difficulty in the process. Particular care must be given to the order in which these groups are written, especially if the desired square is to be perfect. There is, however, no necessary order for any group or combination of groups. For this reason it is possible to construct many squares from any given square and with any given set of groups. The orders to be used so as to balance the rows and columns of the required square cannot be made clear in a brief statement.

4. The object of Professor Camp's paper was to present a simple method for finding the numerical value of a series for practical purposes with a minimum of arithmetic. The geometric basis of the inequality given in a former paper[1] was discussed together with further results in the determination of the accuracy by higher analysis of the deviation ratio. For any $p>1$ it was found that the maximum error for $n=10$ in $\sum n^{-p}$ is $<2.3\times 10^{-6}$ and is asymptotic to $p(3-p)/60(n+1)^{p+3}$. By this for $p=2$ the previous error of 2.070×10^{-7} was checked; for $p=3$ when 9 decimals were used an accidental error of 1×10^{-9} was found; and for $p=4$ with 12 decimals and k_{10} to 7 the result was true to 10 decimals. The product $\prod(1-1/n^2)$ was evaluated with 7-place logarithms, k_{10} to 6, and the deviation ratio for the smallest value of p in $\log(1-1/n^2)$, namely $p=2$; and the result 3.676081, obtained instead of the true value, $3.676078=\sinh \pi/\pi$.

5. It is proved in this note that all n-ic residues of an odd prime p are given by $g^{\delta}, g^{2\delta}, \cdots, g^{p-1} \pmod p$ where δ in the greatest common divisor of n and $p-1$ and g is a primitive root of p. This result is used to prove that all n-ic residues of p form a cyclic group.

A. L. HILL, *Secretary*

THE ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at the State Teachers College, River Falls, Wis., on Saturday, May 7, 1932. Sessions were held at 11:00 o'clock and at 2:15 o'clock with a luncheon at 12:30 o'clock.

Professor Margaret C. Eide, chairman of the Section, presided, except when relieved by Professor Dunham Jackson and Professor R. W. Brink. Ninety persons attended the meeting including the following twenty-four members of the Association: R. W. Brink, W. H. Bussey, Elizabeth Carlson, J. O. Chellevold, H. H. Dalaker, Margaret C. Eide, Grace Erwin, Gladys Gibbens, C. H. Gingrich, W. L. Hart, H. E. Hartig, Dunham Jackson, C. M. Jensen, W. H. Kirchner, Marie M. Ness, A. L. O'Toole, Ole Schey, R. R. Shumway, H. I. Tanjerd, F. J. Taylor, Ella Thorp, A. L. Underhill, Marion B. White, Marian A. Wilder.

At the afternoon session a vote of thanks was adopted as a sign of appreciation of the cordial hospitality of the River Falls Teachers College, and the

[1] Bulletin of the American Mathematical Society, vol. 38 (1932) abstract no. 77, p. 181.

$$
\begin{cases}
a_i^k = a^{k-n}(x)\dfrac{d^k\{a^n(x)\}}{dx^k}, & i,\ k \neq 0 \\
a_0^k = a^k(x)\dfrac{d^k}{dx^k} & \\
a_i^0 = 1 &
\end{cases}
$$

A proof of the above theory by mathematical induction was given.

2. This paper consisted in some introductory remarks on a problem of Enriques with some extensions and illustrations. In particular it discussed the conditions under which the rational curves admit a parametric representation by means of polynomials.

3. Professor Taylor commented upon certain methods in the presentation of work in the theory of equations chapter in a college algebra course.

4. If sufficient hypotheses be assumed for $F[x, v(x)]$, then the solutions of the differential equation $d^2u/dx^2+F[x, v(x)]u=0$ can be shown to converge asymptotically to the solution of $\partial^2u/\partial x^2+F(x, 0)u=0$ as x becomes negatively infinite. On applying this theorem to the well-known Bessel and Fourier transformed equations, $\partial^2u/\partial x^2+[e^{2x}-n^2]u=0$ and $\partial^2u/\partial x^2+e^{2x}u=0$ respectively, a straightforward and simple method can be found for finding the solutions of the above equations, whether n be an integer or not.

5. This paper was concerned with the time, the path of totality, and the duration of the eclipse. The features which are of essential interest to the casual observer at the time of a total eclipse of the sun were referred to. The problems which engage the attention of the professional astronomer were briefly reviewed. The fact was mentioned that, after this eclipse, there would be only four more during this century which are total within the United States.

6. This paper is concerned with the coefficient of correlation between the $2n$ coordinates of a set of n points in a plane and the $2n$ coordinates of another such set of n points. If each set is supposed to have its center of gravity at the origin, the coefficient is $+1$ if the configurations are similar and oriented alike, -1 if they are similar but oriented oppositely, and otherwise intermediate between $+1$ and -1. In particular, if one configuration is obtained from the other by rotation through an angle α, the coefficient of correlation is $\cos\alpha$. The analysis is naturally capable of generalization in various ways.

7. The student of engineering spends about two years studying mathematics. College algebra, trigonometry and analytic geometry are taught during the freshman year, differential and integral calculus during the sophomore year. If we analyze the contents of those courses as they are represented by some well known and good text-books, we can see that a great amount of knowledge is accumulated. A junior or senior engineering student will be inclined to go further in theoretical mathematical studies and will be interested to see some applications of the acquired science to engineering. It is obvious, therefore, that elective courses in applied mathematics will be welcomed by juniors and seniors.

This is confirmed by the practical experience which the writer has had in the University of Minnesota. The courses, accompanied by actual exercises in a mathematical laboratory, have been willingly attended by students and have been recommended to them also by some other departments. It is hardly possible to nominate the abundance of topics fitted for presentation in such courses. A few of them are, for instance: Graphical methods for differentiation and integration; graphical treatment of differential equations; computation with approximate values; interpolation and approximation of functions; computation of definite integrals and surfaces by approximate methods; trigonometric series and harmonical analysis; nomography; mathematical instruments. A great many interesting and important topics might be still added to the few just mentioned.

8. Mathematical instruction in the high school seeks to insure facility and accuracy in computation, to consider quantitative aspects of the environment in such a manner that background and meanings are created, and to develop dispositions of and powers in accurate and precise quantitative thinking. It is a platitude to say that the process must be gradual and that the materials through which the aims may be achieved must be presented at the time when they can be learned most economically. For several years both college algebra and trigonometry, per se, have been offered in the twelfth grade of the University of Minnesota High School. The students in these classes have been of the superior type in intelligence. At the completion of the algebra course each class was given the same final examination as that administered in the Arts College of the University. The students in the high school classes occupied, without exception, the tail end of the distribution of scores. Basing the marks upon the classifications used in the college, 3.2 per cent would have received C's, 20 per cent D's, and 76.8 per cent F's. H. W. Bailey,[1] University of Illinois, reports unfavorable results for trigonometry. Some schools, however, have reported favorable data. In some of these instances the replies to inquiries seem to justify the statement that it is one thing to have students learn to be manipulators of symbols and it is an entirely different thing to require them to think in terms of these symbols. Mathematics is a mode of thought. The available data seem to be insufficient to warrant the offering of college algebra and trigonometry in the high school. The high school should do thoroughly what it attempts to do. It should scrutinize its procedures and direct its efforts accordingly. It takes maturity to learn college mathematics, and Bailey contends that repetition means wastefulness as well as poor pedagogy.

9. The problem of determining the voltage and current at every point of a uniform electric line connected to a single frequency sine voltage generator was sketched. The mathematical formulation of the problem and its solution in terms of complex hyperbolic functions were stated. Charts of the complex func-

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[1] H. W. Bailey, *Trigonometry in the High School.* The Mathematics Teacher, vol. 25 (1932), pp. 303–308.

tions were shown and their suitability for the solution of the given problem was pointed out.

10. In the development of the mathematical theory of sampling one comes in contact with the tedious work of expressing more or less complicated symmetric functions of the variates in terms of the power sums. And what is worse, one not only must deal with symmetric functions of the variates but with symmetric functions of symmetric functions of the variates. Sets of differential operators which will do the work with a minimum of labour and without the necessity of knowing the expressions for the symmetric functions of lower weight are offered in this paper. See papers by the present author in the Annals of Mathematical Statistics, vol. 2 (1931), pp. 102–149; same journal, vol. 3 (1932), pp. 56–63.

A. L. UNDERHILL, *Secretary*

QUASI-CYCLOTOMIC POLYNOMIALS

D. H. LEHMER,[1] Stanford University

In this paper we consider polynomials of the form

$$f(x) = x^n - a_1x^{n-1} + - \cdots + (-1)^n a_n, \tag{1}$$

where the a's are integers. The polynomial $f(x)$ is said to be quasi-cyclotomic if all its roots lie on a circle[2] of radius r in the complex plane and if the arguments of these roots are commensurable with 2π. In other words each root is r times a root of unity. In case $r=1$, we have the familiar cyclotomic polynomials. Kronecker[3] has shown that cyclotomic polynomials are the only polynomials of the form (1) whose roots lie on the unit circle. For $r\neq 1$, quasi-cyclotomic polynomials are not the only polynomials whose roots lie on a circle of radius r. Further restrictions on the coefficients of $f(x)$ are found to be necessary and sufficient to render $f(x)$ quasi-cyclotomic when its roots lie on a circle.

We shall first give a few theorems for the case $r=1$. Kronecker stated his theorem as follows.

THEOREM 1. *If all the roots of f lie on the unit circle, then they are roots of unity.*

From this theorem Kronecker deduced

THEOREM 1′. *If all the roots of f are real and do not exceed* 2 *in absolute value, then each root is twice the cosine of a rational multiple of* 2π.

Although Kronecker's proof of Theorem 1 leaves nothing to be desired, it is perhaps of interest to give an independent proof of Theorem 1′ and to deduce Theorem 1 from it.

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[1] National Research Fellow.

[2] All circles in this paper have their centers at the origin.

[3] Kronecker: *Crelle's Journal* 53, 173–175, Werke 1, 105–108. Netto: *Vorlesungen über Algebra* 1, 357–358. Pólya and Szegö, *Aufgaben und Lehrsätze*, II, 149, 368.

The extreme simplicity of Kronecker's proof of Theorem 1 adds not a little to the interest of the theorem. Our proof of Theorem 1′ is quite different from Kronecker's proof. We have tried to make it as elementary as possible, as is also the case with the theorems that follow.

PROOF OF THEOREM 1′. Let m be any integer. From the indentity

$$2 \cos m\theta = (2 \cos \theta)(2 \cos (m-1)\theta) - 2 \cos (m-2)\theta$$

follows the well known fact that

$$2 \cos m\theta = V_m(2 \cos \theta)$$

where $V_m(x)$ is a polynomial of the type (1).

Now suppose that Theorem 1′ is false. Then $f(x)$ has a root

$$\rho_1 = 2 \cos 2\pi\omega_1$$

where ω_1 is irrational. Let $h(x)$ be the irreducible[1] factor of $f(x)$ for which $h(\rho_1)=0$ and let

$$\rho_\nu = 2 \cos 2\pi\omega_\nu \qquad (\nu = 2, 3, \cdots \mu)$$

be the other roots of $h(x)=0$. Finally consider the product

$$I_m = \prod_{\nu=1}^{\mu} 2 \cos 2\pi m\omega_\nu = \prod_{\nu=1}^{\mu} V_m(2 \cos 2\pi\omega_\nu) = \prod_{\nu=1}^{\mu} V_m(\rho_\nu).$$

This product is a rational integral symmetric function of the roots of $h(x)$ and hence an integer. Also

$$|I_m| \leqq 2^\mu |\cos 2\pi m\omega_1|.$$

Since ω_1 is irrational we may choose m so that the fractional part of $m\omega_1$, i.e.

$$m\omega_1 - [m\omega_1]$$

is arbitrarily close to $1/4$, so close in fact that

$$|\cos 2\pi m\omega_1| < 2^{-\mu}.$$

With such a value of m, $|I_m| < 1$. But I_m is an integer. Hence $I_m = 0$. This means that some factor $V_m(2 \cos 2\pi\omega_\nu)$ of I_m must vanish so that $h(x)$ and $V_m(x)$ have a root in common. Since h is irreducible, all its roots belong to V_m. In particular

$$V_m(2 \cos 2\pi\omega_1) = 2 \cos 2\pi m\omega_1 = 0.$$

But this contradicts the assumption that ω_1 is irrational.

PROOF OF THEOREM 1. Let us designate the roots of f by

$$e^{i\theta_1}, e^{i\theta_2}, \cdots e^{i\theta_n},$$

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[1] In this paper "irreducible" refers to the rational field. The coefficients b_ν of $h(x)=x^\mu+b_1x^{\mu-1}+\cdots+b_\mu$ are integers by Gauss's Lemma (see for instance, Weber, *Lehrbuch der Algebra* vol. 1, p. 27, or Serret, *Cours d'Algèbre Supérieure* (1885), vol. 1, p. 243.

The coefficients of the polynomial

$$f_1(x) = \prod_{\nu=1}^{n}(x - (e^{i\theta_\nu} + e^{-i\theta_\nu}))$$

are symmetric functions of the roots of f and hence are integers. Its roots, $2 \cos \theta_\nu$, do not exceed 2 in absolute value. Therefore by Theorem 1′, θ_ν are rational multiples of 2π. Hence the roots of f are roots of unity.

THEOREM 2. *Let the roots of $f(x)$ lie on the unit circle. Writing each root $\rho_\nu = e^{2\pi i\omega_\nu}$ we suppose further that*

$$\tfrac{1}{8} < | \omega_\nu | < \tfrac{3}{8}. \tag{2}$$

Then $\omega_\nu = \pm 1/6$, $\pm 1/4$ or $\pm 1/3$ and $f(x)$ has the form

$$f(x) = (x^2 + 1)^a(x^2 + x + 1)^b(x^2 - x + 1)^c.$$

PROOF. Let $h(x)$ be any irreducible factor of $f(x)$. By Theorem 1, the roots of h are roots of unity. Let $\rho = e^{2\pi ik/m}$, where k is prime to m, be a root of $h(x) = 0$. Finally let

$$Q_m(x) = \prod_{\nu}(x - e^{2\pi i\nu/m})$$

(where ν runs over the positive integers $< m$ and prime to m) be the irreducible factor of $x^m - 1$ which has for roots the primitive mth roots of unity. Then, since h and Q_m are irreducible and have the common root ρ, $h(x) = Q_m(x)$. But for $m \geqq 8$, Q_m and hence h has the root $e^{2\pi i/m}$ which contradicts the first inequality (2). The roots of Q_m for $m = 1$ and 2 are real. Therefore m is reduced to 3, 4, 5, 6 or 7. But Q_5 has the root $e^{2\pi i2/5}$ and Q_7 has the root $e^{2\pi i3/7}$ and

$$3/7 > 2/5 > 3/8,$$

so that for $m = 5$ or 7 the second inequality (2) is violated. Hence $m = 3, 4, 6$ are the only possible values, and

$$Q_4 = x^2 + 1, \ Q_3 = x^2 + x + 1, \ Q_6 = x^2 - x + 1.$$

As an analogue of Theorem 2 we have

THEOREM 2′. *Let all the roots of $f(x)$ be real and less than $\sqrt{2}$ in absolute value. Then these roots must be* 0 *or* ± 1 *so that*

$$f(x) = x^a(x - 1)^b(x + 1)^c$$

PROOF. Let $\rho_\nu = 2 \cos 2\pi\omega_\nu$ $(\nu = 1, 2, \cdots, n)$ be the roots of $f(x)$. Since

$$-\sqrt{2} < \rho_\nu < \sqrt{2}$$

it follows that

$$\tfrac{1}{8} < | \omega_\nu | < \tfrac{3}{8}.$$

Now the polynomial

$$f_1(x) = \prod_{\nu=1}^{n}(x^2 - \rho_\nu x + 1) = \prod_{\nu=1}^{n}(x - e^{2\pi i\omega_\nu})(x - e^{-2\pi i\omega_\nu})$$

has integer coefficients since they are symmetric functions of ρ_ν. Hence the hypothesis of Theorem 2 is satisfied so that $|\omega_\nu| = \frac{1}{6}, \frac{1}{4}$ or $\frac{1}{3}$. That is $\rho_\nu = 0, \pm 1$.

Before proceeding to generalize Theorem 1 to the case where the roots lie on the circle of any radius it is first desirable to consider what this radius could be. To avoid circumlocution it is helpful to make the following

ASSUMPTION. *$f(x)$ is not a polynomial in x^k where $k>1$.* In the opposite case we can simplify matters by replacing x^k by x thus replacing the roots by their kth powers. This transformation throws a set of roots having the same absolute value into another such set without altering the rational or irrational character of the arguments of the roots. Having agreed on this assumption we proceed to prove

THEOREM 3. *Let the roots of f lie on a circle of radius r. Then r is an integer or the square root of an integer according as n is odd or even.*

PROOF. It follows from our assumption that f has a coefficient $a_h \neq 0$ for which h is prime to n. Writing the roots of f in the form $\rho_\nu = re^{i\theta_\nu}$ (where θ_ν may be zero) we consider the symmetric function

$$\begin{aligned} a_{n-h} &= a_n \sum(\rho_1\rho_2 \cdots \rho_h)^{-1} = r^{n-h}\sum e^{-i(\theta_1+\theta_2+\cdots+\theta_h)} \\ &= r^{n-h}\sum e^{i(\theta_1+\theta_2+\cdots+\theta_h)} = r^{n-2h}a_h. \end{aligned}$$

That is

$$r^{2h}a_{n-h} = a_n a_h. \tag{3}$$

Raising both sides to the nth power we get

$$a_n^{2h} a_{n-h}^{n} = a_n^{n} a_h^{n}.$$

Since $a_h \neq 0$, a_n^{2h} is a perfect nth power. If n is odd, $2h$ is prime to n. Hence $a_n = r^n$ is the nth power of an integer. If n is even h is prime to n. Hence $a_n^2 = (r^2)^n$ is the nth power of an integer. From this the theorem follows.

In order to show that r need not be an integer we have only to exhibit the case $f(x) = x^2+2x+2$.

The question now arises: If the roots of f lie on a circle, are their arguments necessarily commensurable with 2π as in the case when the circle is the unit circle? The example $f(x) = x^2 - x + 4$ shows that the answer is no. Our problem is to find a necessary and sufficient condition on f in order that its roots be proportional to roots of unity. We consider first the case in which r is an integer.

THEOREM 4. *If all the roots of $f(x)$ lie on the circle of integer radius r, then in order that these roots be proportional to roots of unity, it is necessary and sufficient that a_ν be divisible by r^ν.*

PROOF. Define b_ν by $a_\nu = b_\nu r^\nu (\nu = 1, 2, \cdots, n)$. Then the polynomial whose roots are $1/r$ times the roots of $f(x)$ is

$$g(x) = f(rx)/r^n = x^n - b_1 x^{n-1} + \cdots + (-1)^n b_n$$

and has all its roots on the unit circle. If the b's are integers these roots are roots of unity by Theorem 1. Conversely if the roots are roots of unity, the b's must be integers since roots of unity satisfy irreducible equations with integer coefficients. This proves the theorem.

In case r is not an integer, the situation is slightly more complicated. It is convenient to write $r = s\sqrt{R}$, where s and R are integers and R has no square factor >1. We have then the following counterpart of Theorem 4.

THEOREM 5. *Let all the roots of f lie on a circle of radius $s\sqrt{R}$. Then in order that these roots be proportional to roots of unity it is necessary and sufficient that a_ν be divisible by*[1] $s^\nu R^{[(\nu+1)/2]}$.

PROOF OF NECESSITY. Let the roots of f be $s\sqrt{R}\epsilon_\nu$ where ϵ_ν are roots of unity. We consider the polynomial

$$g(x) = f(sx)/s^n = x^n - \frac{a_1}{s} x^{n-1} + \cdots + (-1)^n \frac{a_n}{s^n}$$

whose roots are $\sqrt{R}\epsilon_\nu$. Our first task is to show that the coefficients of $g(x)$ are integers. Let $a_\nu/s^\nu = b_\nu$ be its νth coefficient. Consider next the polynomial $g_2(x)$ whose roots are the squares of the roots of $g(x)$, so that $g_2(x^2) = g(x)g(-x)$.

$$g_2(x) = \prod_{\nu=1}^{n}(x - R\epsilon_\nu^2) = \sum_{\nu=0}^{n}(-1)^\nu x^{n-\nu} A_\nu$$

where[2]

$$A_\nu = (-1)^\nu b_\nu^2 + 2\sum_{i=0}^{n}(-1)^i b_{2\nu-i} b_i (\nu > 0),\ A_0 = 1. \tag{4}$$

Hence A_ν are rational numbers. Finally the coefficients of the polynomial $g_3(x) = g_2(xR)/R^n$ are rational while its roots, ϵ_ν^2, are roots of unity. Hence its coefficients as well as those of $g_2(x)$ must be integers. Thus $g(x)g(-x) = g_2(x^2)$ has integer coefficients. Hence by Gauss' lemma the coefficients b_ν of $g(x)$ are integers. To complete the proof it is sufficient to show that b_ν is divisible by $R^{[(\nu+1)/2]}$.

By Theorem 3, n is even. Let $n = 2k$. Also (3) can be written

$$a_{n-\nu} = R^{k-\nu} a_\nu. \tag{5}$$

Now by Theorem 4

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[1] Here $[x]$ denotes, as usual, the greatest integer $\leqq x$.

[2] In (4) $b_j = 0$ for $j<0$ and $j>n$; $b_0 = 1$.

$$A_\nu = R^\nu B_\nu. \tag{6}$$

If in (4) we set $\nu = k$ and make use of (5) and (6) we get

$$A_k = R^k B_k = (-1)^k a_k^2 + 2\sum_{i=0}^{k-1}(-1)^i a_i^2 R^{k-i}. \tag{7}$$

Hence R divides a_k^2. But R has no square factor. Therefore R divides a_k. Writing $\nu = k-1$ in (4) we have

$$A_{k-1} = R^{k-1}B_{k-1} = (-1)^{k-1}a_{k-1}^2 + 2\sum_{i=0}^{k-2}(-1)^i a_i a_{i+2} R^{k-i-2}.$$

Since R divides a_k, R divides the summation. Hence a_{k-1} is divisible by R. Similarily we may show that R divides $a_{k-2}, a_{k-3}, \cdots, a_1$. Fortified by this knowledge we return to (7). The summation is now divisible by R^3 and hence by R^4. Hence R^2 divides a_k. As before $a_{k-1}, a_{k-2}, \cdots, a_3$ are divisible by R^2. We now return again to (7) and repeat the reasoning until we find that for $\nu \leqq k$, a_ν is divisible by $R^{[(\nu+1)/2]}$. Finally we use (5) and find that $a_{n-\nu}$ is divisible by R^λ, where

$$\lambda = [(\nu+1)/2] + k - \nu = [(n-\nu+1)/2].$$

Hence our condition is necessary.

Proof of Sufficiency. Let a_ν be divisible by $s^\nu R^{[(\nu+1)/2]}$. Then b_ν is divisible by $R^{[(\nu+1)/2]}$. Hence from (4), we see that A_ν contains R^μ where

$$\mu = \min_{i \leqq \nu}\left\{\left[\frac{2\nu - i + 1}{2}\right] + \left[\frac{i+1}{2}\right]\right\} = \nu.$$

Theorem 4 then tells us that the roots of $g_2(x)$ and hence those of $f(x)$ are proportional to roots of unity. This proves the sufficiency of our condition.

The totality of all polynomials of degree n with integer coefficients whose roots are roots of unity multiplied by the square root of an integer, is, of course, infinite, since we may multiply the roots by any integer. A polynomial f whose roots lie on a circle of radius $r = \sqrt{R}$, where R has no square factor may be thought of as the representative of the class of all those polynomials whose roots are integer multiples of the roots of f. The following theorem shows that the number of such classes is actually finite.

Theorem 6. *Let the $n = 2k$ roots of $f(x)$ be of the form $\sqrt{R}\,\epsilon_\nu$ where ϵ_ν are roots of unity, and where R has no square factor. Then*

$$\sqrt{R} \leqq \frac{(2k)!}{(k+1)!(k-1)!}.$$

Proof. Since f is not a polynomial in x^2 or x^k it has a coefficient $a_\nu \neq 0$ such that ν is odd and $< k$. Then by Theorem 5

$$a_\nu = b_\nu R^{[(\nu+1)/2]} = \sum R^{\nu/2}\epsilon_1\epsilon_2 \cdots \epsilon_\nu.$$

Hence

$$| a_\nu | = | b_\nu | R^{[(\nu+1)/2]} \leqq R^{\nu/2}\binom{n}{\nu}.$$

That is

$$| b_\nu | \sqrt{R} \leqq \binom{n}{k-1} = \frac{(2k)!}{(k-1)!(k+1)!}.$$

Since $|b_\nu| \geqq 1$, the theorem follows. A rather crude estimate for R is

$$R < \frac{4^n}{\pi(n+4)}.$$

Examples of quasi-cyclotomic polynomials whose roots lie on a circle of radius $\sqrt{R}$ are

$$R = 2:\ x^2 + 2x + 2$$
$$R = 2:\ x^4 + 2x^3 + 2x^2 + 4x + 4$$
$$R = 5:\ x^4 + 5x^3 + 15x^2 + 25x + 25$$
$$R = 6:\ x^4 + 6x^3 + 18x^2 + 36x + 36$$
$$R = 7:\ x^6 + 7x^5 + 21x^4 + 49x^3 + 147x^2 + 343x + 343\ .$$

THE INDIAN ORIGIN OF THE MODERN PLACE-VALUE ARITHMETICAL NOTATION. PART II

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It has been shown in previous issues[1] of this MONTHLY that the works written by the elder Āryabhaṭa and some of his successors contain clear evidence of the employment in India of the modern place-value arithmetical notation at least as early as 499 A.D. Further examination of the works confirms this conclusion. Varāhamihira speaks of the addition and subtraction of the zero.[2] Brahmagupta gives the following rules for different operations with the zero:[3]

(a) The sum of a negative quantity and the zero is negative; the sum of a

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[1] Vol. 34, p. 409, and vol. 39, p. 251.

[2] *Pañca-siddhāntikā*, III. 17; IV. 7, 8, 11, 12; XVIII. 35, 44, 45, 48, 51.

[3] The reference is to Sudhākar Dvivedī's edition (Benares). In the preface the editor writes that he prepared this edition after consulting three manuscript copies of the work. Colebrooke's edition is based only on one copy found in the India Office Library. The serial numbers of the verses giving these rules in Colebrooke's edition are the numbers immediately following those in Dvivedī's edition.

positive quantity and the zero is positive; and the sum of two zeros is zero (*Brāhma-sphuṭa-siddhānta*, XVIII.30).

(b) On being diminished by the zero, a negative quantity becomes negative, a positive quantity becomes positive, and the zero becomes zero (*Ibid.*, XVIII. 32).

(c) The product of the zero and a negative quantity or of the zero and a positive quantity or of two zeros is zero (*Ibid.*, XVIII.33).

(d) The zero divided by the zero is zero (*Ibid.*, XVIII.34).[1]

(e) Khodhṛtamṛṇaṃ dhanaṃ vā tacchedaṃ khamṛṇadhanavibhaktaṃ vā (*Ibid.*, XVIII.35).

When a positive or negative quantity is divided by the zero, the quotient is *taccheda*[2] (*khaccheda*)—an indeterminate quantity of which the zero is the divisor. When the zero is divided by a positive or negative quantity, the quotient is *taccheda* (*khaccheda*)—a fraction of the zero, *i.e.*, zero.

(f) The square root of zero is zero; the square of zero is the same as its square root (*Ibid.*, XVIII.35).

If a place-value notation with the zero to indicate a vacant place had not existed at the time of Varāhamihira and Brahmagupta, there would have been no necessity for the former to speak of addition and subtraction of the zero and for the latter to give the rules quoted above. Consider the numbers 'five hundred and seven' and 'four hundred and two.' In finding the sum of these two numbers hundreds and units are naturally added separately so that the result becomes 'nine hundred and nine.' The question of addition of zeros can arise only when the numbers are expressed in a place-value decimal notation with the zero indicating a vacant place. Similar is the case with most other arithmetical operations with the zero. The above-mentioned rules, therefore, presuppose the existence of the modern arithmetical notation in India at the time of Varāhamihira and Brahmagupta.

The above conclusion is further supported by the use of the word *aṅka* for the number *nine* in Varāhamihira's *Pañcasiddhāntikā* (XVIII.35), Brahmagupta's *Brāhma-sphuṭa-siddhānta* (II.7) and Lalla's (7th century A.D.) *Śiṣyadhi-vṛddhida*.[3] As there is only one moon or one earth, the word *śaśī* (the moon) or *bhū* (the earth) has been used to denote the number *one*. As a man has two hands and two eyes, the word *kara* (hand) or *nayana* (eye) has been used to denote the number *two*. Similar considerations have restricted the choice of words

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[1] This rule shows that Brahmagupta did not take the zero to mean an infinitesimal quantity. In deducing this rule from the rule $0\times 0=0$ he has ignored the rule $0\times a=0$.

[2] The compound word *taccheda* ($=tat+cheda$) has two different meanings according to the kind of *samāsa* (compounding of words) used to form it. If the *samāsa bahuvrīhi* is used, it means a quantity of which the zero is the divisor; for, *tat* (*i.e.*, it) refers to the zero. If the *samāsa tatpuruṣa* is used, it means *tasya* (of the zero) *chedaḥ* (a fraction), *i.e.*, a fraction of the zero. In the former sense Brahmagupta's *taccheda* is the same as Bhāskara's *khahara* (infinity).

[3] Chapter I (Graha-gaṇitādhyāya), Section I, Verses 53 and 59; Section II, Verses 2, 5, 8, 9, and 28; Section III, verse 9; etc.

for other numbers. Hence the use of the word *aṅka*, for the number *nine*, points to the fact that the number of *aṅkas* or figures used at the time of these writers must have been nine. Previous Indian notations required many more signs or figures. It is, therefore, concluded that the notation employed by these writers required nine significant figures with the zero and was, therefore, the modern place-value decimal notation. When, therefore, Severus Sebhokt in the year 662 A.D. speaks admiringly of the Hindu method of computation by means of nine signs,[1] he means the employment of the modern place-value decimal notation.

Oriental scholars (whether American, European, or Indian) who had sufficient knowledge of Sanskrit to be able to examine for themselves astrological and astronomical works written by Varāhamihira and his successors were definitely of the opinion that the modern notation was in use in India as early as the sixth century A.D. Even G. R. Kaye—the most uncompromising supporter of the theory of the non-Indian origin of the modern notation—did not doubt the authenticity of the Sanskrit texts of the works of Varāhamihira and Brahmagupta as handed down to us. It will thus be seen that Cajori is not justified in remarking that the manuscript of Brahmagupta's work used by Colebrooke "cannot be accepted as evidence that Brahmagupta himself used the zero and the principle of local value,"[2] unless by 'the manuscript' he means the numbers expressed in the modern notation by means of decimal figures, which occurred in the manuscript by the side of the corresponding number-expressions in terms of word-numerals based on the modern notation.

It may be pointed out here that, in the case of numbers and dates included in verse, word-numerals had to be used instead of decimal figures. But the underlying notation was the modern place-value notation. In arithmetical operations signs are far more advantageous than words. As there is indisputable evidence to show that numeral signs have been in use in India from before the Christian era, there is no reason to suppose that in performing arithmetical operations the Indians from the sixth century onwards did not use the modern notation with decimal figures. It is well-known that there have always existed in India two parallel systems of expressing numbers, arising from the same notation—one by means of words or letters so as to be capable of being used in verse and the other by means of figures so as to be most suitable for calculation. Cajori is quite right when he observes, "There appear to have been several notations in use in different parts of India, which differed, not in principle, but merely in the forms of the signs employed."[3] The genealogical table given below will demonstrate the truth of this observation and show how the different Indian systems of expressing numbers by means of figures, word-numerals, or letter-numerals, have arisen, by successive natural stages, from the ordinary method of stating numbers in words.

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[1] Cajori, *A History of Mathematical Notations*, vol. I (1928), p. 48.

[2] *Ibid.*, Article 76.

[3] *A History of Mathematics* (1922), p. 89.

A genealogical table of different Indian systems of expressing numbers

(a) "Nava śatāni ṣaṭpañcāśatkāni" (956)
(9 100 56)

- (b) Brāhmī notation (left-to-right) (100) 9 (50) 6
  - (d) (100) 9 (10) 5 6
    - (e) Modern notation 9 5 6
      - (f) A letter-numeral system which seems to ignore value of position (System of Āryabhaṭa I) Jhi-la-ca 900 50 6 Or, Jhi-ra-ta 900 40 16
      - Systems which recognise value of position
        - (g) With word-numerals Rasa-śara-nava 6 5 9
        - With letter-numerals
          - (h) With left-to-right order (System of Āryabhaṭa II) Dha-ṇi-ta 9 5 6
          - (i) With right-to-left order Ta-nma-dhu 6 5 9
- (c) Kharosṭhi notation (right-to-left) 6 (50) (100) 9

In the above table it should be understood that (100), (50), and (10) mean single symbols for these numbers; (100)9 means a single compound symbol for 900 obtained by prefixing the symbol for 100 to that for 9.

The expression "nava śatāni ṣaṭpañcāśatkāni" for the number 956 has been taken from the Caraka[1] *Saṃhita* (Śārīrasthānam, Ch. VII, 13) where the number 29956 has been expressed in words as "ekonatriṃśat sahasrāṇi nava ca śatāni ṣaṭpañcāśatkāni." The notations (b), (c), (e), (g), and (i), occur in Indian inscriptions. Kaye says that he has not found epigraphical examples of the system (f).[2] I do not know whether the notations (d) and (h) may be found in in-

[1] Caraka was the official physician of King Kaniṣka in the first century A.D. (Macdonell's *History of Sanskrit Literature*, (1917), pp. 435, 436.)

[2] I think that this sytem was never used in any inscription. It is very doubtful whether its author used it except in the *Daśagītikā*. Bhaṭṭotpala (966 A.D.) quotes a verse from the elder Āryabhaṭa in his commentary on Varāhamihira's *Vṛhat-saṃhitā*. The first half of the verse is found in the *Āryabhaṭīya* but not the second half. A later work of Āryabhaṭa has been lost (*Bulletin* of the Calcutta Mathematical Society, Vol. XXII, pp. 115–120). It is very probable that Bhaṭṭot-

scriptions. Notations (f) and (h) have been explained respectively by the elder[1] and the younger Āryabhaṭa.[2] The notation (d) is certainly the connecting link between (b) and (e), and is easily obtained as shown in the above table.[3] No development of the Kharosṭhi notation could take place as, according to Marshall,[4] the Kharosṭhi script disappeared from India in the fifth century A.D.

At first sight there appears to be no connection between the modern notation (e) and the letter-numeral system (f) of the elder Āryabhaṭa. The case of this system is somewhat similar to that of the arithmetical operation 'multiplication' which "is not necessarily addition, but an operation self-contained, self-interpretable—springing originally out of addition, but, when full-grown, existing apart from its parent."[5] Just as the relation between multiplication and its parent, *viz.*, addition, is apparent only when the multiplier is a positive integer, so the relation between the alphabetic system of the elder Āryabhaṭa and its parent, *viz.*, the modern place-value notation, is noticeable only in expressions like *jhi-la-ca* for 956 and *yu-gu-śi-khi-la-ga* for 337253.

It will appear from the above genealogical table of different Indian notations that the abacus is not an essential pre-requisite to the modern place-value arithmetical notation.

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pala quotes from this lost work. The second half of the verse states the number 364224 in word-numerals as *jina-yama-veda*+*ṛtu-havyabhuk* (24, 2, 4, 6, 3). If this view be correct, the elder Āryabhaṭa must be given the credit of also first showing the method of constructing number-expressions which occur in the works of subsequent Indian authors.

[1] *The Āryabhaṭīya*, Chapter I (*Daśagītikā*), Verse 2.

[2] *Mahāsiddhānta*, Chapter I, Verse 2.

[3] This suggestion seems to be supported by the Tamil numeral figures found in inscriptions and manuscripts of the 16th century. These figures "are remarkable as forming the stage of development between the W. Cave numerals and the modern systems, and are, therefore, relics of a system that became more or less obsolete in the sixth century A.D.; we find here separate figures for *ten*, *hundred* and *thousand* nearly identical with the W. Cave forms; but the figures for *twenty*, etc. are rejected, and tens, hundreds or thousands are expressed by prefixing the sign for the units to the left side of the figure representing the order." (Burnell, *South Indian Palæography*, p. 68). Also see Humboldt quoted by Cajori in his *History of Mathematical Notations*, Vol. I (1928), Art. 88. In the Brāhmī notation hundreds as well as thousands used to be expressed by affixing the sign for the units to the right side of the figure representing the order. There is no reason why the order in which these symbols occurred in the notation should not have been retained when the symbols were separated.

[4] *Memoirs of the Archæological Survey of India*, No. 7, p. 10.

[5] Kelland and Tait, *Quaternions*, Introduction.

NOTES ON CODE WORDS

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Introduction

The following notes are believed to be of interest in connection with the present regulations governing the construction of code words, as adopted in 1928 by the International Telegraph Conference of Brussels.[1] Briefly stated, two types of code language are permissible under the present regulations. We shall proceed first to a consideration of words of the first type, called *Category A* words, and conclude with a brief discussion of words of the second type, called *Category B* words, which are, it may as well be stated at once, of far less importance than the Category A words.

That part of the protocol adopted by the 1928 Telegraph Conference dealing with the requirements which words of Category A must fulfill reads as follows:[2]

Category A. Telegrams the text of which contains code words formed of a maximum of 10 letters and in which there is at least one vowel when they have a maximum of 5 letters, at least 2 vowels when they have 6, 7, or 8 letters, and at least 3 vowels when they have 9 or 10 letters. In words of more than 5 letters there must be at least one vowel in the first 5 letters, and at least one vowel in the remainder of the word, it being understood that words of 9 or 10 letters must contain at least a total of 3 vowels. The vowels are a, e, i, o, u, y.

A number of queries suggest themselves in connection with words of this category. In order to understand the origin and significance of these queries, a brief history[3] of code language as employed in international telegrams must first be presented.

It is well known that the basis of all or practically all modern cable and telegraph codes is the 5-letter code word. Through a loophole unforseen by those who participated in formulating the rules drawn up by the International Telegraph Conference of London, in 1903, it became possible to combine two 5-letter code words to form a 10-letter word chargeable as a single word in cablegrams, thus cutting the cost of messages in half. Code compilers and code users were very quick to find and take advantage of this loophole, with the result that

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[1] The present regulations went into effect on October 1, 1929. That they are inadequate and that they have failed to accomplish the reforms intended is sufficiently attested to by the fact that proposals for their further modification will constitute one of the most important subjects on the agenda for the forthcoming International Telegraph Conference, which is scheduled to open in September of this year at Madrid.

[2] Code words of this type must conform to other requirements not mentioned herein, but they are of no interest in the present discussion.

[3] For a more detailed history see Friedman, William F., *The history of codes and code language, the international telegraph regulations pertaining thereto, and the bearing of this history on the Cortina report*, Government Printing Office, Washington, 1928.

within a short time 5-letter codes became very widespread and have by this time practically superseded all other types of codes in international cable and radio communications. Thus, while theoretically the 1928 regulations governing Category A code language are applicable to codes based upon words of a length up to 10 letters, in practice they are aimed at codes based upon 5-letter words which, as every code user now knows, are to be combined in pairs to form singly-charged-for 10-letter words. Now let us suppose that in a given code there are some 5-letter words with only one vowel. (With one or two exceptions, this is actually the case in the 5-letter codes constructed up to 1929.) It follows that in a certain number of cases there will be brought together, in the pairing of two 5-letter words to form a single 10-letter word, two words each containing a single vowel, resulting in the formation of a 10-letter word with only two vowels. According to the present regulations, such a word would have to be charged for as two words, thus increasing the cost of messages to an extent more or less dependent upon the number of code words containing only one vowel.[1] The only absolutely sure way of avoiding this source of surcharge is to arrange that *every* word in the code contain at least two vowels. Thus, while the regulations do not specifically state that each 5-letter code word of Category A must contain at least two vowels, it is clear that, in order to insure that the pairing of two 5-letter groups will in *no case* result in the formation of a 10-letter word with only two vowels, *each 5-letter word must contain at least two vowels.* The new codes (those published since October 1, 1929) take this indirectly imposed requirement into account.

Thus far we have been dealing with certain limitations imposed upon codes by the regulations themselves. We come now to the consideration of a very important limitation imposed upon modern codes as a result of practical difficulties inherent in telegraphic communication.

From the earliest days of codes based upon the 5-letter artificial word, it was recognized that some method or device is necessary whereby errors of transmission can be more or less automatically detected and corrected by the recipient of a message; otherwise communication by means of such artificial words becomes impractical. Of the methods elaborated for this purpose, that based upon the principle of including within a given code only such words as differ from one another in at least 2 letters has proved to be the most satisfactory, and has practically superseded all other methods. The 2-letter difference may consist in:

(1) A difference in the identity of 2 letters. For example, if the code contains the word ABABA, it must not contain any word differing from it in only one letter, such as ABABE, ABACA, ABEBA, ACABA, or EBABA. But words

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[1] It is incorrect to assume that the number of 10-letter words that will be subject to the double charge is directly dependent upon the actual number of single-vowel, 5-letter words present in the code. Only an actual test made upon many bonafide messages, all prepared with the same code, can satisfactorily determine the number of doubly-charged-for words to be expected for that code, because the words and phrases of any code are employed with greatly varying frequencies.

differing from ABABA in at least two letters, such as ABACE, ABEBE, ACEBA, ECABA, ABECA, etc., are permissible.

(2) A difference in the position of 2 letters. For example, if the code contains the word ABECI, it may contain words such as EBACI, ACEBI, ABICE, BAECI, AEBCI, etc. In the early codes, no attempt was made to eliminate or to suppress one or the other member of a pair of words differing simply in the positions occupied by two adjacent letters. Many of them contained groups, such as ABECI and ABEIC, which are easily converted one into the other by a common type of psychological *lapsus calami* referred to in code work as "transposition." But in the better codes constructed up to about the year 1925, the authors have usually recognized the necessity of avoiding the possibility of errors introduced by a transposition of adjacent letters, and, to a very large extent, have succeeded in eliminating or almost completely suppressing this source of error. Something will be said of this later in these notes. In the more modern codes serious attempts have been made to eliminate errors due to transpositions of both adjacent and alternate letters, and it may be said that the problem is rather a difficult one.

(3) A difference in the identity of one letter and in the position of another. For example, if the code contains the word ABECK, the following would be legitimate words in the same code: ABERC, AERCK, etc.

In short, when at least two homologous letters in a pair of code groups differ in their identities, the code words are said to present a 2-letter difference.

In what follows we shall refer to *classes* and *subclasses* of words. By the designation *class*, we refer to a set of words merely by indicating the *number* of their constituent vowels and consonants. Using the symbol C to represent any of the 20 consonants, and the symbol V to represent any of the 6 vowels, the designation "$3V/2C$" merely indicates that words of this class contain 3 vowels and 2 consonants, without indicating the positions occupied by any of these constituent elements. Under each class, except the first and last, there are different subclasses of words with respect to the exact arrangement or *position* of the vowels and the consonants composing them. In the accompanying Table I there are shown the six classes and the 32 subclasses of words which can be constructed by taking vowels and consonants in groups of five letters.

If we wish to compose 5-letter words imposing no limitations other than that they must show a 1-letter difference, we may place any one of 26 letters in each of the five positions and shall thus have a total of $26\times26\times26\times26\times26=$ 11,881,376 words. We may consider that we are here concerned with a simple multiplication involving five factors the numerical value of each of which is equal to the number of elements available for permutation.

If, however, we now impose the limitation that the words shall show a difference of two letters, the number of words that can be composed will be considerably reduced, because, in the parlance of code compilers, it is necessary to "sacrifice one of the letters." Just what this means will now be explained.

TABLE I

Class No.	Class Description	Subclass No.	Subclass Description	Class No.	Class Description	Subclass No.	Subclass Description
I	$5V$	1	*VVVVV*	IV	$2V/3C$	17	*CCCVV*
II	$4V/1C$	2	*VVVVC*			18	*CCVCV*
		3	*VVVCV*			19	*CVCCV*
		4	*VVCVV*			20	*VCCCV*
		5	*VCVVV*			21	*CCVVC*
		6	*CVVVV*			22	*CVCVC*
III	$3V/2C$	7	*VVVCC*			23	*VCCVC*
		8	*VVCVC*			24	*CVVCC*
		9	*VCVVC*			25	*VCVCC*
		10	*CVVVC*			26	*VVCCC*
		11	*VVCCV*	V	$1V/4C$	27	*VCCCC*
		12	*VCVCV*			28	*CVCCC*
		13	*CVVCV*			29	*CCVCC*
		14	*VCCVV*			30	*CCCVC*
		15	*CVCVV*			31	*CCCCV*
		16	*CCVVV*	VI	$5C$	32	*CCCCC*

Consider for example, code words of two letters. Obviously, with an alphabet of 26 letters a total of 26×26, or 676, 1-letter difference pairs can be constructed. Such pairs will represent *all* the permutations of the 26 letters taken 2 at a time, such as AB, BA, AC, CA, etc. But if a 2-letter difference is desired, then only a total of 26 different pairs can be constructed, whether the pairs be simple doublets, such as AA, BB, $\cdots$, ZZ, or permutations of 2 different letters, such as AB, BC, $\cdots$, ZA. Thus, the formula for 1-letter difference pairs, 26^2, becomes modified to $26^{(2-1)}$ in order to take care of a 2-letter difference. Of the two factors in the multiplication 26×26, one will be reduced to unity, which is what the code compiler really means when he says that "one of the letters must be sacrificed." Similarly, in the case of 3-letter words, the total number of 1-letter difference words is 26^3 or 17,576 words; but if a 2-letter difference is desired the total number becomes reduced to $26^{(3-1)} = 26^2 = 676$ words. For 4-letter groups with a 2-letter difference the number becomes 26^3 words; and for 5-letter groups with a 2-letter difference, 26^4 or 456,976 words. A general formula may be derived from the foregoing:

$$\text{Number of 2-letter difference words} = \lambda^{(n-d+1)},$$

where $\lambda =$ the number of elements in the alphabet, $n =$ the number of characters per word, $d =$ the differential.

So far we have imposed no restrictions on the *kind* of letter that may occupy any given position in a word: any letter may be a vowel or a consonant as we happen to take it. If we restrict the number of vowels or the number of consonants allowed to a word, the case is altered. When we demand a consonant we are limited to 20 letters instead of 26, and when we demand a vowel, we are

limited to 6 letters instead of 26. If we wish to form *all* the words that can be composed containing, say 1 consonant followed by 4 vowels, i.e., words of the form $CVVVV$, we shall have $20\times6\times6\times6\times6=25{,}920$ words each differing from all the others in at least one letter.

If, in addition to limiting ourselves to words consisting exclusively of one consonant followed by four vowels, we make the further demand that our words shall differ among themselves in at least two letters, we shall further lessen their number. We have already seen that the requirement of a 2-letter difference entails the sacrifice of one factor, or rather the reduction of one factor to unity. Just what this sacrifice will mean when we limit ourselves to a certain number of vowels and a certain number of consonants in each word will become plain as we proceed.

Query I

What is the theoretical maximum number of 5-letter code words that can be constructed with a minimum 2-letter difference, each code word containing a minimum of 2 vowels?

Solution

1. Table II shown below sets forth in brief form all the classes of 5-letter code words and the maximum number of 2-letter difference words in each class. Some explanation as to how the totals for each class of words are derived is added.

TABLE II

Based upon an Alphabet of 20 Consonants and 6 vowels

	Description		*Maximum number of 2-letter difference words*
(1)	Class I—With $5V$	$=6\times\ 6\times\ 6\times\ 6\times1=\ 1{,}296$	$=\ 1{,}296$
(2)	Class II—With $4V/1C$	$=\ 6\times\ 6\times\ 6\times\ 6\times1=\ 1{,}296;$	$1{,}296\times\ 5=\ 6{,}480$
(3)	Class III—With $3V/2C$	$=\ 6\times\ 6\times\ 6\times20\times1=\ 4{,}320;$	$4{,}320\times10=\ 43{,}200$
(4)	Class IV—With $2V/3C$	$=\ 6\times\ 6\times20\times20\times1=\ 14{,}400;$	$14{,}400\times10=144{,}000$
(5)	Class V—With $1V/4C$	$=\ 6\times20\times20\times20\times1=\ 48{,}000;$	$48{,}000\times\ 5=240{,}000$
(6)	Class VI—With $5C$	$=20\times20\times20\times20\times1=160{,}000$	$=160{,}000$

(1) In Class I (words composed of 5 vowels) the number of words is the same as if we were forming our words from a 6-letter alphabet. If we allow for a 2-letter difference, which means that we must reduce one of the five factors to unity, we shall, accordingly, have $6\times6\times6\times6\times1=1{,}296$ words.

(2) Class II consists of words composed of 1 consonant and 4 vowels. There are 5 subclasses according to the position occupied by the consonant. If we take the consonant as the first letter of each word, we can divide our 5 letters into two groups—one of 2 letters, CV, and one of 3 letters, VVV. If we use all the VVV groups, we shall have $6\times6\times6=216$ VVV groups. Each of these must now be combined with as many CV groups as possible to make 5-letter words. Each

of the CV groups with which any one VVV group is combined must differ from the other groups combined with the same VVV group in 2 letters. (We could not, for example, use both BAAAA and BEAAA.) Despite the availability of 20 consonants, since there are only 6 vowels in the alphabet we can form only 6 CV groups differing each from all the other CV groups in 2 letters. We can, for example, combine AAA with BA, CE, DI, FO, GU, HY, but with no more CV groups. Having now associated 6 CV groups with AAA, we can associate 6 others with AAE, say BE, CI, DO, FU, GY, HA, and so on with our remaining VVV groups. Thus we may obtain $6\times 216=1{,}296$ groups of the type $CVVVV$. Since our single consonant can occupy any one of 5 positions in the word, the grand total of $1C/4V$ words will be $5\times 1{,}296=6{,}480$.

(3) It is not possible to increase this number. It is true that we have used only 6 different consonants in all, but the remaining 14 consonants are of no value, for if we use them to replace the single consonant already used in the words we have formed, we shall obtain new words differing in only one letter from those we already have. If, for example, we have BAAAA, CAAAE, DAAAI, FAAAO, GAAAU, HAAAY, we cannot use JAAAA, KAAAE, LAAAI, MAAAO, NAAAU or PAAAY. The case is the same no matter what position in the word the single consonant occupies. And it is manifestly impossible to add to our words by additional vowel variations as those variations have already been exhausted.

(4) It is important to note in the foregoing explanation that the number of permutations of the complete set of 20 consonants suffers a serious reduction as a result of the association of the consonants with a much more limited number of vowels. It is clear, in fact, that if there were but six consonants available instead of 20 the total number of Class II words would still remain the same, 6,480. Fourteen consonants are wholly valueless in composing these words.

(5) Perhaps a concrete example employing a miniature alphabet will be useful in demonstrating this point conclusively. For this purpose, let us take an alphabet composed of three vowels, A, E, I, and of five consonants, B, C, D, F, G, to form 4-letter words with a 2-letter difference. If we require that all four letters be vowels, we shall be able to form, with three vowels, $3\times 3\times 3\times 1=27$ 4-letter words with a 2-letter difference. If we set up a further limitation and demand that the first three letters be vowels and the last a consonant ($VVVC$), we can produce no more words than when all four letters were vowels, viz., 27, because, despite the availability of five different consonants, we are unable to make use of them all, as will now be shown.

(6) Let us assume that our $VVVC$ words are divided into two sections, VV and VC. For VV we have $3\times 3=9$ possibilities. For VC we have $3\times 5=15$, if we are forming only 1-letter difference words; but, as we have already seen, in order to obtain a 2-letter difference we must "sacrifice" one factor, that is, reduce it to unity. Now we cannot sacrifice the vowel of the VC section with its factor value of 3, and keep the consonant with its factor value of 5, because we are unable to use the full factor value of the consonant *unless the consonant has*

another letter with a factor value of 5 with which it can be permutatively associated. Since under the conditions of the problem this consonant must be associated with vowels, of which there are only three, our single consonant loses its factor value of 5 and becomes reduced to 3. It is now immaterial whether the factor that must be "sacrificed" in making the final calculation be considered as that pertaining to the vowel or to the consonant: the full number of words is $3\times3\times3\times1=27$. We may actually form a set of *VVVC* words under the conditions given:

AAAB	EAAC	IAAD	AEAC	EEAD	IEAB	AIAD	EIAB	IIAC
AAEC	EAED	IAEB	AEED	EEEB	IEEC	AIEB	EIEC	IIED
AAID	EAIB	IAIC	AEIB	EEIC	IEID	AIIC	EIID	IIIB

We have here our full quota of 27 words, and have not used the consonants F and G at all. We can use them if we will, but only to replace B, C, or D, and that will give us merely *alternative* words, not additional ones. (For example, we might have AIAG or IEAF, but each of these will show only a 1-letter difference from some words we already have.)

(7) In computing the total number of possible words with a 2-letter difference, composed of a mixture of vowels and consonants (always assuming that the consonants outnumber the vowels in the alphabet employed) if the words have one consonant each, the factor value of the set of consonants is exactly the same as that of the set of vowels with which the consonants must be associated; and then, as regards the factor which must be reduced to unity or "sacrificed," it is immaterial which is the one considered to have been sacrificed, that pertaining to the vowels or that pertaining to the consonants. If the words contain more than one consonant each, one of the factors pertaining to the consonant positions must be reduced to unity while all other factors will retain their full value.

(8) From the foregoing this general rule may be stated: In a set of unequal factors which are used as a basis for computing the total number of d-letter difference words that may be constructed, there must be at least d factors of equal and maximum value, and $(d-1)$ of them must be reduced to unity while all other factors retain their full value.

(9) In Class III, the $2C/3V$ class, we can form 10 subclasses, according to the positions occupied by the 2 consonants—*CCVVV*, *CVCVV*, *CVVCV*, *CVVVC*, *VCCVV*, *VCVCV*, *VCVVC*, *VVCCV*, *VVCVC*, *VVVCC*. It is obvious that each of these subclasses will yield the same number of words. Applying the general formula derived above to any of the foregoing ten subclasses, say the subclass *CCVVV*, we have $20\times1\times6\times6\times6=4320$ words for this subclass. Since there are 10 subclasses to each of which the same formula is applicable, we obtain a grand total of $10\times4{,}320=43{,}200$ words of the $2C/3V$ class.

(10) Proceeding with Class IV as with Class III, we find that there are again 10 subclasses, corresponding to the 10 possible positions of the consonants. Taking any subclass, for example, *CCCVV*, and applying the general formula

we have $20\times20\times1\times6\times6=14{,}400$ words. For 10 subclasses we have $10\times14{,}400=144{,}000$ words.

(11) Similarly Class V ($4C/1V$) will yield $20\times20\times20\times1\times6=48{,}000$ words for each position of the single vowel, or a total of $5\times48{,}000=240{,}000$ words for the entire Class.

(12) Class VI ($5C$) will give $20\times20\times20\times20\times1=160{,}000$ words.

2. We are ready now to study Table II with a view to finding an answer to the question posited: What is the theoretical maximum number of 5-letter code words that can be constructed with a minimum 2-letter difference, each code word containing a minimum of two vowels?

3. Three hypotheses may be considered with respect to how we can obtain from Table II the maximum number of code words conforming to the foregoing specifications. They are:

1st hypothesis: The maximum number may be obtained by taking *all* the words of any one class.

2nd hypothesis: The maximum number may be obtained by taking *all* the words of two or more classes without introducing any "conflicts," i.e., without violating the 2-letter differential in the case of even a single pair of words.

3rd hypothesis: The maximum number may be obtained by taking words from two or more classes, the words being selected in such a way that no conflicts will be introduced.

4. Classes V and VI can immediately be eliminated from consideration, for they do not conform to the requirement that each word contain at least two vowels. There are left for consideration, therefore, only Classes I to IV, inclusive.

5. It is obvious that if two or more of the four classes remaining for consideration can be found to conform to the requirements of the second hypothesis, we shall obtain more words than can be obtained by an adherence to the first hypothesis. Let us begin, therefore, with the class which by itself gives the greatest total number of words, viz., Class IV. Is it possible to combine Class IV words with *all* the words of any of the other three classes, without introducing any conflicts? Let us try to combine Class IV with the class showing the next greatest total number of words, viz., Class III. Let us take subclass 23 of Class IV, of the form *VCCVC* (see Table I); can we combine words of this subclass with words of subclass 9 of Class III, of the form *VCVVC*, without conflict? To be even more specific, can words of subclass 23, as exemplified in the word ABBAB be used without conflict with words of subclass 9, as exemplified in the word ABAAB? The answer must be in the negative, because as these two words now stand they show only a 1-letter difference. No matter what the specific constitution of any *VCCVC* word, it is bound to conflict with some *VCVVC* word, if *all* the words of both subclasses are taken. This is true with respect to all subclasses of Classes III and IV; hence these two classes, each taken in its entirety, cannot be used together without conflict.

6. Can words of Classes II and IV be used together without conflict? Fol-

lowing the same reasoning as in Paragraph 5, it will become apparent that they can. For, let us take any subclass of Class IV, subclass 23 for example, *VCCVC*, and combine it with any subclass of Class II, subclass 5 for example, *VCVVV*; no conflicts can arise because the third and the fifth positions in these subclasses will always be occupied in the one case by two consonants, in the other case, by two vowels. Take another example: combine subclass 3, *VVVCV*, with subclass 25, *VCVCC*; again no conflicts can arise because the second and fifth positions in these two subclasses will always be occupied in the one case by two vowels, in the other, by two consonants. This holds for all the subclasses of Classes II and IV when combined. Hence, the two classes can be used together without conflict, yielding a maximum total of $6{,}480+144{,}000=150{,}480$ words.[1]

7. It is obvious that words of Class I cannot be included with words of Classes IV and II without producing conflicts, since words of Classes I and II will show conflicts between themselves.

8. The only other classes that might be employed together in their entireties are Classes I and III, but they will yield a total of only 44,496 words. If, then, the maximum number of words is to be obtained by the association of complete classes, it will be by employing Classes II and IV in complete series, yielding a total of 150,480 words.

9. We are, however, not justified in calling this number the *maximum* number possible unless and until we can dispose of the third hypothesis set forth in paragraph 3: can the maximum number be obtained by taking *some* words from one class and adding *some* words from the other classes, the words being selected in stch a way that no words will conflict with one another? Let us see.

10. We have already associated Classes II and IV. We cannot add to the total number of words by taking any words from Class I. All that any Class I words can do is to *replace* words of Class II, and that will not add to the total. As a matter of fact, it will decrease it. With Class III, however, the case is different. In forming words of Class II we use, or need use, only six different consonants. Just as the words of Class I are formed by associations of one or another of six different vowels in each of the five positions, so the words of Class II use the same six vowels in four of the five positions, and six different consonants in the remaining position. Let us assume that the six different consonants used are BCDFGH.

11. In the two positions in the words of Class III where consonants are used, *all* the consonants are employed. As a consequence we can add a certain number of words from Class III without conflicting with any of the words of Class II. Thus, we shall have in Class II, say, the word AEIOB. We can now take AEIJK, AEIKL, AEILM, AEIMN, etc., from Class III, and still preserve the 2-letter differential between the words of Class II and those of Class III.

12. There are fourteen consonants not employed in words of Class II. These

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[1] Credit for the association of Class IV with Class II to produce the number 150,480 is due to Dr. L. H. Canfield, of the College of the City of New York.

14 consonants will form $6\times6\times6\times14=3{,}024$ combinations, which when multiplied by the ten possible positions of the two consonants in each word will yield 30,240 Class III words. All these words can be used with the complete series of words of Class II without interfering with the 2-letter differential.

13. We have, however, in addition to the complete set of words of Class II, used, in obtaining our 150,480 words, a complete set of words of Class IV. Before concluding that the 30,240 words just obtained can be added to our total, we must see whether they or any of them conflict with the Class IV words.

14. Both Class III and Class IV, at those places in their words where they employ consonants, use, for the complete set of words, the full number of twenty consonants. The only difference between a word of Class III and one of Class IV is that at one point where a word of Class III has a vowel a word of Class IV will have a consonant. It is therefore impossible to add words of Class III to a complete set of words of Class IV. We can *substitute* a word of Class III for one of Class IV, but this is an even exchange and will not increase our word total. If, then, we increase our word total by adding, to the words of Class II, non-conflicting words from Class III, we lose one word from Class IV for each word so added, and at the end we are exactly where we were before.

15. One possibility remains: it might be possible to omit *some* of the words of Class II or Class IV and substitute for each word so omitted more than one word from Class I or Class III. But if we replace any word of Class II by a word of Class I, we lose five words in Class II for each one that is taken for such replacement in Class I. And if we replace any word of Class II by a word of Class III, we shall gain ten words in Class III for every five so replaced in Class II—but we shall lose ten words of Class IV for every ten words taken from Class III. That is, for every ten words gained, 15 are lost as a result of the substitution.

16. It has been shown that we can obtain 150,480 words by using all the words of Class II and all those of Class IV. This total cannot be increased by adding words of Class I or Class III. Neither can it be increased by dropping some of the words of Class II or Class IV and adding a greater number of Class I or Class III. That is, it cannot be increased at all, and is the maximum number obtainable.

Query II

The 2-letter difference code words of modern 5-letter codes are usually compiled by reference to tables known under various designations, such as "Permutation Table," "Mutilation Chart," "Error Detector Chart," etc. An example of a typical table is shown in Table III below. We shall refer to such a table as a code-word construction table.

The code words afforded by the foregoing table are formed from the table by combining three elements: (1) a pair of letters from section 1, (2) a single letter from section 2, and (3) a pair of letters from section 3; and the combination must be made according to the rule that the initial pair and the middle letter must lie in the same vertical line (extended), the middle letter and the final pair must lie in the same horizontal line (extended).

Section 1

aa	ab	ac	ad	ae	af	ag	ah	ai	aj	ak	al	am	an	ao	ap	aq	ar	as	at	au	av	aw	ax	ay	az
bb	bc	bd	be	bf	bg	bh	bi	bj	bk	bl	bm	ba	bo	bp	bq	br	bs	bt	bu	bv	bw	bx	by	bz	bn
cc	cd	ce	cf	cg	ch	ci	cj	ck	cl	cm	ca	cb	cp	cq	cr	cs	ct	cu	cv	cw	cx	cy	cz	cn	co
dd	de	df	dg	dh	di	dj	dk	dl	dm	da	db	dc	dq	dr	ds	dt	du	dv	dw	dx	dy	dz	dn	do	dp
ee	ef	eg	eh	ei	ej	ek	el	em	ea	eb	ec	ed	er	es	et	eu	ev	ew	ex	ey	ez	en	eo	ep	eq
ff	fg	fh	fi	fj	fk	fl	fm	fa	fb	fc	fd	fe	fs	ft	fu	fv	fw	fx	fy	fz	fn	fo	fp	fq	fr
gg	gh	gi	gj	gk	gl	gm	ga	gb	gc	gd	ge	gf	gt	gu	gv	gw	gx	gy	gz	gn	go	gp	gq	gr	gs
hh	hi	hj	hk	hl	hm	ha	hb	hc	hd	he	hf	hg	hu	hv	hw	hx	hy	hz	hn	ho	hp	hq	hr	hs	ht
ii	ij	ik	il	im	ia	ib	ic	id	ie	if	ig	ih	iv	iw	ix	iy	iz	in	io	ip	iq	ir	is	it	iu
jj	jk	jl	jm	ja	jb	jc	jd	je	jf	jg	jh	ji	jw	jx	jy	jz	jn	jo	jp	jq	jr	js	jt	ju	jv
kk	kl	km	ka	kb	kc	kd	ke	kf	kg	kh	ki	kj	kx	ky	kz	kn	ko	kp	kq	kr	ks	kt	ku	kv	kw
ll	lm	la	lb	lc	ld	le	lf	lg	lh	li	lj	lk	ly	lz	ln	lo	lp	lq	lr	ls	lt	lu	lv	lw	lx
mm	ma	mb	mc	md	me	mf	mg	mh	mi	mj	mk	ml	mz	mn	mo	mp	mq	mr	ms	mt	mu	mv	mw	mx	my
no	np	nq	nr	ns	nt	nu	nv	nw	nx	ny	nz	nn	nb	nc	nd	ne	nf	ng	ni	nh	nj	nk	nl	nm	na
op	oq	or	os	ot	ou	ov	ow	ox	oy	oz	on	oo	oc	od	oe	of	og	oh	oj	oi	ok	ol	om	oa	ob
pq	pr	ps	pt	pu	pv	pw	px	py	pz	pn	po	pp	pd	pe	pf	pg	ph	pi	pk	pj	pl	pm	pa	pb	pc
qr	qs	qt	qu	qv	qw	qx	qy	qz	qn	qo	qp	qq	qe	qf	qg	qh	qi	qj	ql	qk	qm	qa	qb	qc	qd
rs	rt	ru	rv	rw	rx	ry	rz	rn	ro	rp	rq	rr	rf	rg	rh	ri	rj	rk	rm	rl	ra	rb	rc	rd	re
st	su	sv	sw	sx	sy	sz	sn	so	sp	sq	sr	ss	sg	sh	si	sj	sk	sl	sa	sm	sb	sc	sd	se	sf
tu	tv	tw	tx	ty	tz	tn	to	tp	tq	tr	ts	tt	th	ti	tj	tk	tl	tm	tb	ta	tc	td	te	tf	tg
uv	uw	ux	uy	uz	un	uo	up	uq	ur	us	ut	uu	ui	uj	uk	ul	um	ua	uc	ub	ud	ue	uf	ug	uh
vw	vx	vy	vz	vn	vo	vp	vq	vr	vs	vt	vu	vv	vj	vk	vl	vm	va	vb	vd	vc	ve	vf	vg	vh	vi
wx	wy	wz	wn	wo	wp	wq	wr	ws	wt	wu	wv	ww	wk	wl	wm	wa	wb	wc	we	wd	wf	wg	wh	wi	wj
xy	xz	xn	xo	xp	xq	xr	xs	xt	xu	xv	xw	xx	xl	xm	xa	xb	xc	xd	xf	xe	xg	xh	xi	xj	xk
yz	yn	yo	yp	yq	yr	ys	yt	yu	yv	yw	yx	yy	ym	ya	yb	yc	yd	ye	yg	yf	yh	yi	yj	yk	yl
zn	zo	zp	zq	zr	zs	zt	zu	zv	zw	zx	zy	zz	za	zb	zc	zd	ze	zf	zh	zg	zi	zj	zk	zl	zm

b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a
c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b
d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c
e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d
f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e
g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f
h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g
i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h
j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i
k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j
l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k
m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l
n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m
o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n
p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q
s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r
t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s
u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t
v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u
w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v
x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w
y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x
z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z

Section 2

TABLE III

An Example of a Code Word Construction Table.

Reproduced by permission of the Code Compiling Co., Inc., New York, publishers of the Universal Trade Code, 1921. All the cells in Section 1 and Section 3, many of which are vacant in the original table, have been filled in with the proper pairs of letters by the present authors, to meet the special requirements of this topic.

aa	bb	cc	dd	ee	ff	gg	hh	ii	jj	kk	ll	mm	nz	on	po	qp	rq	sr	ts	ut	vu	wv	xw	yx	zy
am	ba	cb	dc	ed	fe	gf	hg	ih	ji	kj	lk	ml	ny	oz	pn	qo	rp	sq	tr	us	vt	wu	xv	yw	zx
al	bm	ca	db	ec	fd	ge	hf	ig	jh	ki	lj	mk	nx	oy	pz	qn	ro	sp	tq	ur	vs	wt	xu	yv	zw
ak	bl	cm	da	eb	fc	gd	he	if	jg	kh	li	mj	nw	ox	py	qz	rn	so	tp	uq	vr	ws	xt	yu	zv
aj	bk	cl	dm	ea	fb	gc	hd	ie	jf	kg	lh	mi	nv	ow	px	qy	rz	sn	to	up	vq	wr	xs	yt	zu
ai	bj	ck	dl	em	fa	gb	hc	id	je	kf	lg	mh	nu	ov	pw	qx	ry	sz	tn	uo	vp	wq	xr	ys	zt
ah	bi	cj	dk	el	fm	ga	hb	ic	jd	ke	lf	mg	nt	ou	pv	qw	rx	sy	tz	un	vo	wp	xq	yr	zs
ag	bh	ci	dj	ek	fl	gm	ha	ib	jc	kd	le	mf	ns	ot	pu	qv	rw	sx	ty	uz	vn	wo	xp	yq	zr
af	bg	ch	di	ej	fk	gl	hm	ia	jb	kc	ld	me	nr	os	pt	qu	rv	sw	tx	uy	vz	wn	xo	yp	zq
ae	bf	cg	dh	ei	fj	gk	hl	im	ja	kb	lc	md	nq	or	ps	qt	ru	sv	tw	ux	vy	wz	xn	yo	zp
ad	be	cf	dg	eh	fi	gj	hk	il	jm	ka	lb	mc	np	oq	pr	qs	rt	su	tv	uw	vx	wy	xz	yn	zo
ac	bd	ce	df	eg	fh	gi	hj	ik	jl	km	la	mb	no	op	pq	qr	rs	st	tu	uv	vw	wx	xy	yz	zn
ab	bc	cd	de	ef	fg	gh	hi	ij	jk	kl	lm	ma	nn	oo	pp	qq	rr	ss	tt	uu	vv	ww	xx	yy	zz
az	bn	co	dp	eq	fr	gs	ht	iu	jv	kw	lx	my	na	ob	pc	qd	re	sf	tg	uh	vi	wj	xk	yl	zm
ay	bz	cn	do	ep	fq	gr	hs	it	ju	kv	lw	mx	nm	oa	pb	qc	rd	se	tf	ug	vh	wi	xj	yk	zl
ax	by	cz	dn	eo	fp	gq	hr	is	jt	ku	lv	mw	nl	om	pa	qb	rc	sd	te	uf	vg	wh	xi	yj	zk
aw	bx	cy	dz	en	fo	gp	hq	ir	js	kt	lu	mv	nk	ol	pm	qa	rb	sc	td	ue	vf	wg	xh	yi	zj
av	bw	cx	dy	ez	fn	go	hp	iq	jr	ks	lt	mu	nj	ok	pl	qm	ra	sb	tc	ud	ve	wf	xg	yh	zi
au	bv	cw	dx	ey	fz	gn	ho	ip	jq	kr	ls	mt	nh	oi	pj	qk	rl	sm	ta	ub	vc	wd	xe	yf	zg
at	bu	cv	dw	ex	fy	gz	hn	io	jp	kq	lr	ms	ni	oj	pk	ql	rm	sa	tb	uc	vd	we	xf	yg	zh
as	bt	cu	dv	ew	fx	gy	hz	in	jo	kp	lq	mr	ng	oh	pi	qj	rk	sl	tm	ua	vb	wc	xd	ye	zf
ar	bs	ct	du	ev	fw	gx	hy	iz	jn	ko	lp	mq	nf	og	ph	qi	rj	sk	tl	um	va	wb	xc	yd	ze
aq	br	cs	dt	eu	fv	gw	hx	iy	jz	kn	lo	mp	ne	of	pg	qh	ri	sj	tk	ul	vm	wa	xb	yc	zd
ap	bq	cr	ds	et	fu	gv	hw	ix	jy	kz	ln	mo	nd	oe	pf	qg	rh	si	tj	uk	vl	wm	xa	yb	zc
ao	bp	cq	dr	es	ft	gu	hv	iw	jx	ky	lz	mn	nc	od	pe	qf	rg	sh	ti	uj	vk	wl	xm	ya	zb
an	bo	cp	dq	er	fs	gt	hu	iv	jw	kx	ly	mz	nb	oc	pd	qe	rf	sg	th	ui	vj	wk	xl	ym	za

Section 3

Given a specific, completely-filled code-word construction table for constructing 5-letter words with a 2-letter difference, this question now arises: How many words are there of each of the six classes discussed under Query I?

Solution

1. The initial step in the solution to this query consists in determining by actual count the number of words of the $5V$ class in the table under investigation. Let us assume that in a certain table this was found to be 236 words.

2. Words of the class $4V/1C$ comprise five subclasses, according to the position of the consonant, (see Table I). Each of these subclasses will yield 1,296 2-letter difference words. Any word of the class $VVVVV$ will eliminate one word from each of the foregoing subclasses of $4V/1C$ words. For example, AEIOU would eliminate AEIOC, AEICU, AECOU, ACIOU, and CEIOU. If then, we use 236 five-vowel words, we shall have remaining $1{,}296-236=1{,}060$ words in each of the five subclasses comprising the class $4V/1C$, making a total of $5\times 1{,}060=5{,}300$ words.

3. Words of the class $3V/2C$ comprise ten subclasses, according to the positions of the two consonants (see Table I). Each of these subclasses will yield 4,320 2-letter difference words. Certain subclasses of class $4V/1C$ will conflict with certain subclasses of class $3V/2C$. For example, words of subclass 2, such as AEIOR, will conflict with words of subclass 7, such as AEIQR, as well as with words of subclass 8, such as AEQOR, of subclass 9, such as AQIOR, and of subclass 10, such as QEIOR. Thus, each word of subclass 2 will eliminate *four* words from class $3V/2C$ words. Since there are 1,060 words of subclass 2 ($VVVVC$) and 4,320 words in each of the subclasses 7, 8, 9 and 10, (see Table I) we shall have remaining in each of these four last-named subclasses the difference between 4,320 and 1,060, which is 3,260 words. Applying the same reasoning to all the subclasses of class II, we draw up the following table:

TABLE IV

Subclass	2—*VVVVC* conflicts with	Subclass	3—*VVVCV* conflicts with
"	7—*VVVCC*, with	"	7—*VVVCC*, with
"	8—*VVCVC*, with	"	11—*VVCCV*, with
"	9—*VCVVC*, and with	"	12—*VCVCV*, and with
"	10—*CVVVC*	"	13—*CVVCV*
Sublcass	4—*VVCVV* conflicts with	Subclass	5—*VCVVV* conflicts with
"	8—*VVCVC*, with	"	9—*VCVVC*, with
"	11—*VVCCV*, with	"	12—*VCVCV*, with
"	14—*VCCVV*, and with	"	14—*VCCVV*, and with
"	15—*CVCVV*	"	16—*CCVVV*

Subclass	6—*CVVVV* conflicts with
"	10—*CVVVC*, with
"	13—*CVVCV*, with
"	15—*CVCVV*, and with
"	16—*CCVVV*

4. Each of the ten subclasses 7 to 16, inclusive, is here duplicated in conflicts; but we need, of course, to eliminate words only once, in order to dispose of the conflicts with retained words of subclasses 2 to 6, inclusive. Therefore, we shall have remaining in each of subclasses 7 to 16, inclusive, 4,320 − 1,060 or 3,260 words, and hence the total number remaining, after deduction for conflicts, for the whole of class III is 3,260 × 10 or 32,600 words.

5. If we continue this process with the remaining classes of words, the following final results are obtained, shown in condensed mathematical form:

TABLE V

Class	*Description*	*Calculation*		*No. of Words*
I	$5V$	(as found by actual count in a specific table)	=	236
II	$4V/1C=$	1,296 − 236 = 1,060; 1,060 × 5	=	5,300
III	$3V/2C=$	4,320 − 1,060 = 3,260; 3,260 × 10	=	32,600
IV	$2V/3C=$	14,400 − 3,260 = 11,140; 11,140 × 10	=	111,400
V	$1V/4C=$	48,000 − 11,140 = 36,860; 36,860 × 5	=	184,300
VI	$5C$	= 160,000 − 36,860 = 123,140		123,140
			Grand total	456,976

6. It will be noted that the grand total in the foregoing calculation checks, since $26^4 = 456{,}976$. In similar calculations applicable to any other code-word construction table, while the individual totals may vary (according to the number of $5V$ words formed by the table), the grand total for a completely filled construction table must obviously be the same as that obtained above, viz., 456,976 words.

Query III

Experience has shown that if a code contain two such code words as, for example, ABCDE and BACDE, confusion may arise from the accidental transposition (in writing or telegraphing) of the letters A and B. It has accordingly been found advisable to construct codes so that no two code words differing from each other merely in the transposition of two adjacent letters will be included in the same code. A code-word construction table affording code words which will show no transpositions of adjacent letters can, however, be made when the number of different letters, λ, used in its construction is odd. The English alphabet contains 26 letters. To drop one letter in order to make λ odd would reduce the total number of words available. We may, however, add an extra character to the alphabet, giving $(\lambda+1)$ characters, construct a table without transpositions, and then eliminate all words containing the extra character. This will leave only words containing the λ letters, and these will contain no transpositions. We shall, however, lose a certain number of words that can be made from λ letters. How many shall we lose?

Solution

1. We may experiment with the miniature $(\lambda+1)$ table below, where the extra character, added to the alphabet to make the λ letter alphabet a $(\lambda+1)$ letter alphabet, is represented by the asterisk:

TABLE VI

	Section I														
	AA	AB	AC	AD	AE	AF	A*								
	BB	BC	BD	BE	BF	B*	BA								
1st and	CC	CD	CE	CF	C*	CA	CB								
2nd	DD	DE	DF	D*	DA	DB	DC				($\lambda=6$)				
letters	EE	EF	E*	EA	EB	EC	ED								
	FF	F*	FA	FB	FC	FD	FE								
	**	*A	*B	*C	*D	*E	*F								
	A	B	C	D	E	F	*	AA	BB	CC	DD	EE	FF	**	
	B	C	D	E	F	*	A	BA	CB	DC	ED	FE	*F	A*	
3rd	C	D	E	F	*	A	B	CA	DB	EC	FD	*E	AF	B*	Section
letter	D	E	F	*	A	B	C	DA	EB	FC	*D	AE	BF	C*	III
	E	F	*	A	B	C	D	EA	FB	*C	AD	BE	CF	D*	
	F	*	A	B	C	D	E	FA	*B	AC	BD	CE	DF	E*	
	*	A	B	C	D	E	F	*A	AB	BC	CD	DE	EF	F*	
	Section II							4th and 5th letters							

2. Examination of this miniature table of $(\lambda+1)$ characters, where λ is 6, will show that it can yield words containing only the original λ letters[1] as follows:

(1) One set of words having $\lambda^2+\lambda(\lambda-1)(\lambda-1)$ words
(2) $(\lambda-1)$ sets of words each having $\lambda(\lambda-1)+(\lambda-1)(\lambda-1)(\lambda-1)$ words
(3) One set of words having $\lambda(\lambda-1)(\lambda-1)$ words

Adding all words we have:

$$\lambda^4-\lambda^3+\lambda^2-\lambda+1 \text{ or } \lambda^4-(\lambda^2+1)(\lambda-1)$$

3. Now λ letters arranged in a construction table will give λ^4 2-letter difference words. Hence, by adding the extra character to the table and eliminating words in which the extra character appears, we shall lose $(\lambda^2+1)(\lambda-1)$ words.

4. Accordingly, if $\lambda=26$, we shall lose from the complete table for 26 letters $(26^2+1)25=677\times 25=16{,}925$ words. This leaves $456{,}976-16{,}925=440{,}051$ words.

5. It may be interesting to know how many words will be eliminated by the process described, from the complete table based on $(\lambda+1)$ letters. This table will yield $(\lambda+1)^4$ words, which equals $\lambda^4+4\lambda^3+6\lambda^2+4\lambda+1$ words. As already shown, if we omit words containing the extra character, we shall have $\lambda^4-\lambda^3+\lambda^2-\lambda+1$ words. Hence

[1] One special case must be considered. The added letter will appear twice in each column of Section I and twice in each row of Section III, except that there will be one column and one row respectively where it will appear only once. Should the arrangement of the code-word construction table as a whole be such that this one column and one row are not associated in forming words, one additional word will be lost. The deduction of the words from the total for λ letters will then be $(\lambda^2+1)(\lambda-1)+1$ and the deduction from the total for $\lambda+1$ letters will be $5\lambda^3+5\lambda^2+5\lambda+1$.

$$(\lambda + 1)^4 - (\lambda^4 - \lambda^3 + \lambda^2 - \lambda + 1) = 5\lambda^3 + 5\lambda^2 + 5\lambda$$

words will be eliminated.

For 27 characters the net total would therefore be:

$$27^4 - [5(27^3) + 5(27^2) + 5(27)] = 531,441 - (87,880 + 3380 + 130)$$
$$= 531,441 - 91,390 = 440,051.$$

6. It may further be interesting to see what is gained by this process over the simpler method of constructing a table with $(\lambda-1)$ letters. This table would give $(\lambda-1)^4=\lambda^4-4\lambda^3+6\lambda^2-4\lambda+1$ words. The table containing $\lambda+1$ letters gives, as we have seen, when words containing the extra letters are rejected, $\lambda^4-(\lambda^2+1)(\lambda-1)$ words. Subtracting $\lambda^4-4\lambda^3+6\lambda^2-4\lambda+1$ from the latter quantity, we have $3\lambda^3-5\lambda^2+3\lambda$ as the difference between the respective numbers of words yielded by the two tables. If $\lambda=26$, a table of $(\lambda-1)$ or 25 letters will give 390,625 words, while a table of $(\lambda+1)$ or 27 letters, omitting words containing the extra letter, will give 440,051, as has been shown.

Note

So far as words of Category B are concerned, since no limitations are placed upon their composition by the present regulations, the total number of code words with a 2-letter difference available for code compilers and code users is 26^4 or 456,976 words. If nontransposability of adjacent letters referred to in the preceding section is taken into consideration in the elaboration of the construction table, this total becomes reduced to either 440,051 words or 390,625 words, depending upon the method selected.

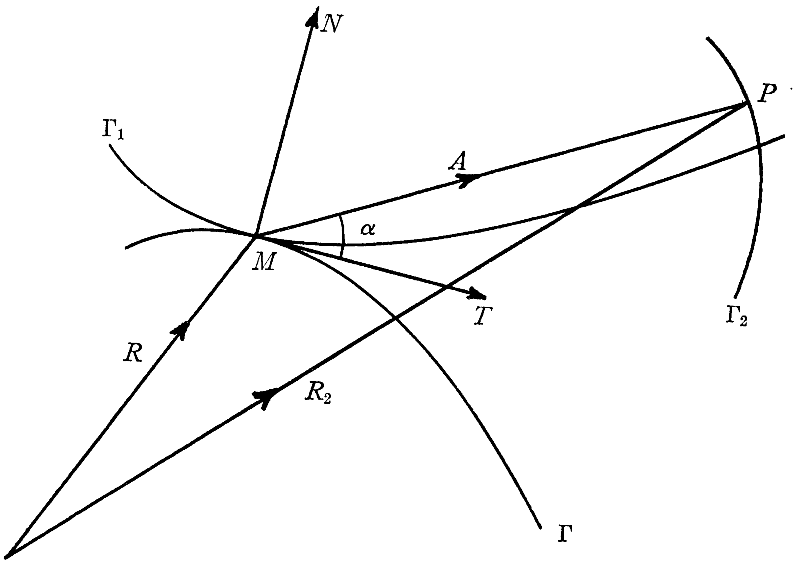
CYCLOIDAL CURVES

By SOLOMON BILINSKY, Washington University

When one plane curve rolls upon another, every point fixed relative to the rolling curve and in its plane describes a new curve. In the particular instance where the fixed curve is a straight line and the rolling curve a circle every point on the circle describes an ordinary cycloid. Other particular instances are quite as well known. It is the purpose of this discussion, however, to treat the general case, and to demonstrate the simplicity and straightforwardness with which vector analysis provides a means for the study of such curves, which will in the general case be designated as cycloidal curves.[1]

We suppose that the curve Γ_1 rolls upon the fixed curve Γ and that the point P, fixed relative to Γ_1, describes a curve Γ_2. As Γ_1 rolls along Γ the tangents and

[1] A different treatment of this subject may be found in A. Mannheim, *Géométrie Descriptive*, 2nd edition (1886), pp. 175–180.



normals to each curve are coincident and equal lengths of arc are measured off on each. We will agree that the unit normal N is to be drawn from the point of contact M so that κ_1, the curvature of Γ_1 is always positive, in which case the sign of κ, the curvature of Γ, will be positive or negative according as the concavity of Γ is in the same or opposite direction to that of Γ_1. The unit tangent T is to be drawn in the direction of increasing arc-length. Let A be a unit vector making an angle α with T and drawn so that aA is the vector MP.

It will first be necessary to find explicit expression[1] for a and the first derivatives of both a and A. Since this does not entirely involve the problem of rolling curves we will assume for the moment that Γ_1 is fixed and that the extremity of the vector R, drawn from a chosen origin, runs along Γ_1. Let the point P be fixed by the vector R_P, drawn from the same origin as R. We have

$$R_P = R + aA, \tag{1}$$

$$A = T\cos\alpha + N\sin\alpha, \tag{2}$$

and, in addition, a unit vector B, perpendicular to A:

$$B = N\cos\alpha - T\sin\alpha. \tag{3}$$

By scalar multiplication with T, we have from (2) and (3)

$$T\cdot B = -\sin\alpha \quad T\cdot A = \cos\alpha. \tag{3.1}$$

[1] The values of a and its derivatives obviously depend only upon the form of the curve Γ_1. A is determined by its orientation with respect to T and N which are coincident at M. The derivatives of A, however, depend upon changes in T and N, and these latter depend only upon the curvature of Γ.

Denoting differentiation with respect to s_1, the arc length of Γ_1, by an accent and subscript 1, we have from (2) and (3)

$$A_1' = \nu_1 B, \quad \nu_1 = \alpha_1' + \kappa_1;$$

from (1)

$$0 = T + a_1' A + a(\nu_1 B).$$

By scalar multiplication of the last equation with A and B in turn, we have

$$\sin\alpha = a\nu_1 \quad \cos\alpha = -a_1', \tag{3.2}$$

in view of relations (3.1).

Curve Γ_1 is now allowed to roll along Γ and the terminus of R kept on the point of contact. Denoting differentiation with respect to s, the arc length of Γ, which is the same as s_1, by an accent, we have from (2) and (3), as before

$$A' = \nu B, \quad \nu = \alpha' + \kappa. \tag{3.3}$$

Let the locus of P be determined by the vector R_2, drawn from the origin of R. Then

$$R_2 = R + aA.$$

Differentiating with respect to s:

$$T_2 s_2' = T + a'A + a\nu B. \tag{4}$$

The scalar product of the last equation with A yields

$$T_2 \cdot A s_2' = \cos\alpha - \cos\alpha = 0,$$

by (3.1) and (3.2), with the observation that $a_1' = a'$.

It follows that T_2 is perpendicular to A; a fact that is otherwise evident from the consideration that M is the instantaneous center of rotation. We shall write $T_2 \equiv -B$, and since N_2 is perpendicular to T_2, $N_2 \equiv -A$. The choice of sign is arbitrary and only serves to fix the directions along which s_2 and κ_2 are measured.

We have now

$$T_2' = -B',$$

or

$$\kappa_2 s_2' N_2 = \nu A,$$

the last member being obtained from (3). Since $N_2 \equiv -A$,

$$\kappa_2 = -\frac{\nu}{s_2'} = \frac{\nu}{a\nu - \sin\alpha},$$

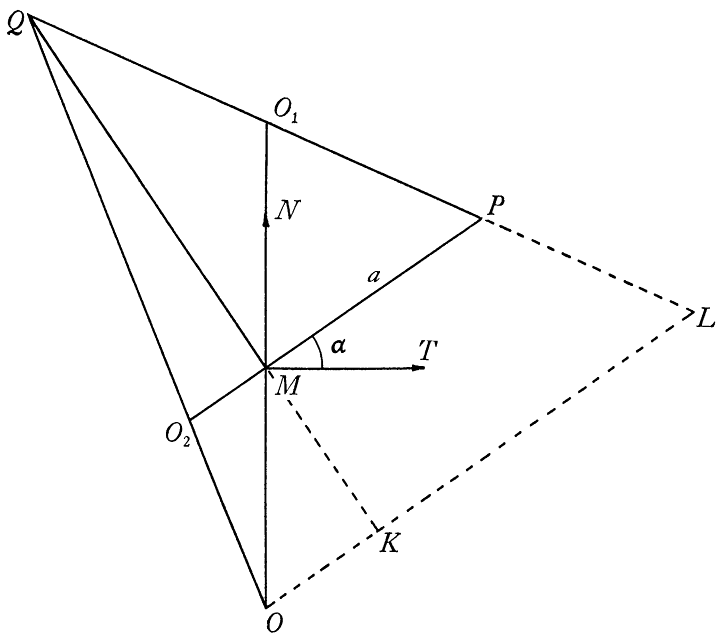
for by scalar multiplication of (4) with B, it is found that $s_2' = \sin\alpha - a\nu$. Now replacing ν and $\sin\alpha$ from the expression in (3.3) and (3.2),

$$\kappa_2 = \frac{a(\kappa - \kappa_1) + \sin\alpha}{a^2(\kappa - \kappa_1)}. \tag{5}$$

In terms of the radius of curvature ρ, (5) becomes

$$\rho_2 = \frac{a^2(\rho_1 - \rho)}{a(\rho_1 - \rho) + \rho\rho_1 \sin \alpha}. \tag{6}$$

The center of curvature of Γ_2 at P can easily be found by geometrical construction.[1] Though the method is general we shall consider for simplicity the case in which the center of curvature O_1 of Γ_1 and O of Γ are on opposite sides of the point of contact M.



Draw the lines PO_1, and MQ perpendicular to MP at M. Draw the straight line joining the point of intersection Q of these two lines with O. Then QO and PM intersect in O_2, the desired center of curvature of Γ_2 at P.

PROOF: Draw OL parallel to MP, cutting O_1P in L and QM in K. Then, by inspection of the figure, remembering that we have chosen the case for which ρ is negative, we have

$$OK = -\rho \sin \alpha, \quad \frac{OL}{a} = \frac{\rho_1 - \rho}{\rho_1}, \quad \frac{\rho_2}{a} = \frac{OL}{OL - OK};$$

and we are easily led to equation (6).

[1] Mannheim, p. 175. The construction is here attributed to Euler and Savary.

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

A REMARK ON DEDEKIND CUTS

By THEODORE WEAVER and THEODORE SUCKAU, Ohio State University

There are several ways of defining real numbers. One of the most widely used is that due to Dedekind, namely the Dedekind cut in the domain of rational numbers. The question arises as to whether or not there are cuts which do not define a rational number. The commonest example of such a cut is the following:

$$r \text{ is in the class } L \text{ if } r > 0 \text{ and } r^2 < 2, \text{ or if } r \leqq 0;$$

$$r \text{ is in the class } U \text{ if } r > 0 \text{ and } r^2 > 2.$$

To show that this cut does not define a rational number we must demonstrate that the class L has no largest element (number) and that the class U has no smallest element (number). The usual way to show this is to prove that given $r>0$ in L we can find h such that $0<h$ and such that $(r+h)^2<2$; also for $R>0$ in U we can find h such that $(R-h)^2>2$, and $R-h>0$.

While working on this problem in an introductory course in analysis conducted by Professor Tibor Rado at the Ohio State University we found a way of proving this which was both interesting to the class and to the instructor. We thought that it might be interesting to others because of its connection with more advanced mathematics.

We found that by properly choosing h we can obtain an expression which could be used to show not only that there is no smallest element in the upper class U, but also that there is no largest element in the lower class L. This expression is

$$\bar{r} = \frac{4 + 3r}{3 + 2r},$$

where r is any positive rational number. We find

$$\bar{r}^2 = 2 + \frac{r^2 - 2}{(3 + 2r)^2},$$

$$\bar{r}^2 - r^2 = \frac{4(2 + 3r + r^2)(2 - r^2)}{(3 + 2r)^2};$$

and hence, if r is in U, $\bar{r}$ is also in U and is $<r$; and if $r>0$ is in L, then $\bar{r}$ is also in L and is $>r$.

The reader who is familiar with the theory of linear transformations will recognise the above expression for $\bar{r}$ as a linear hyperbolic transformation with an attractive fixed point at $+\sqrt{2}$.

It is easily found that the most general such transformation with rational coefficients may be written in the form

$$\bar{r} = \frac{br + 2a}{ar + b},$$

where a and b are positive integers such that

$$\begin{vmatrix} b & 2a \\ a & b \end{vmatrix} > 0.$$

Notes on a Diophantine Equation

By H. Grossman, De Witt Clinton High School, New York, N. Y.

Given the equation $ax^3+bx^2+cx+d=y^2$, where a, b, c, d are rational, $a\neq 0$, $d=\square\neq 0$, the following is an interesting device for constructing a chain of rational solutions.

If (x_i, y_i) is any solution of the original equation other than the trivial ones $(0, \pm\sqrt{d})$, it may be shown that

$$x_{i+1} = (cx_i + 2d \pm 2\sqrt{d}\, y_i)/(ax_i^2) \tag{1}$$

gives a solution (x_{i+1}, y_{i+1}).

From (1) it follows that $\sqrt{d}+cx_i/(2\sqrt{d})-ax_{i+1}x_i^2/(2\sqrt{d})=\mp y_i$. If we square both sides, subtract from the original equation (with $x=x_i$, $y=y_i$), and divide by $x_i^2\neq 0$, we obtain

$$\left(\frac{a^2x_{i+1}^2}{4d}\right)x_i^2 - \left(\frac{acx_{i+1}}{2d} + a\right)x_i + \left(\frac{c^2}{4d} - ax_{i+1} - b\right) = 0. \tag{2}$$

The x_i-discriminant of this equation,

$$\frac{a^2c^2x_{i+1}^2}{4d^2} + \frac{a^2cx_{i+1}}{d} + a^2 - \frac{a^2x_{i+1}^2}{d}\left(\frac{c^2}{4d} - ax_{i+1} - b\right),$$

must be a rational square which we may call $a^2y_{i+1}^2/d$. Therefore

$$a^3x_{i+1}^3/d + a^2bx_{i+1}^2/d + a^2cx_{i+1}/d + a^2 = a^2y_{i+1}^2/d,$$

or multiplying by d/a^2, $ax_{i+1}^3+bx_{i+1}^2+cx_{i+1}+d=y_{i+1}^2$; i.e., x_{i+1} is also a solution of the original equation. Since x_i and x_{i+1} enter symmetrically into equation (2), it follows that if x_{i+1}, like x_i, $\neq 0$, the relation between them is reciprocal and

$$x_i = (cx_{i+1} + 2d \pm 2\sqrt{d}\, y_{i+1})/(ax_{i+1}^2).$$

Let us extend the definition (1) of x_{i+1} to the case $x_i=0$. We must take the negative determination of the double sign (if we take y_i positive) so that the numerator remains zero and the fraction does not become infinite. By double differentiation of both numerator and denominator and substitution of the limiting values of x and y, we find the limit of the fraction as x approaches 0 to be $(c^2-4bd)/(4ad)$. Since 0 is a solution of the original equation, this by continuity must also be a solution. The corresponding y is $(c^3-4bcd+8ad^2)/(8ad^{3/2})$. From this solution, by (1), we may derive the solution whose x is $8d(c^3-4bcd+8ad^2)/(c^2-4bd)^2$, etc. In general, we may construct a chain of solutions by successive applications of (1), making at each step that choice of the double sign which gives an x_{j+2} distinct from x_j (since for a definite x_j and x_{j+1} only one choice of the double sign in $x_j=(cx_{j+1}+2d\pm 2\sqrt{d}\,y_{j+1})/(ax_{j+1}^2)$ may be made), so that $x_{j+2}=x_j\pm(4\sqrt{d}\,y_{j+1})/(ax_{j+1}^2)$ (where again only one choice of the double sign may be used). The ambiguous sign makes it desirable to take only positive values of y.

The case $d\neq\square$, if we know one of its solutions (h, k) (and no trivial solution is available), may be reduced, as is well known, to the previous case by the transformation $x=X+h$. To the X of each solution of the new equation we may add h in order to obtain the x of the corresponding solution of the original equation. We may treat similarly the case $d=0$ (and $d=\square$). We can not however take $h=0$. If x_1 is a solution of $ax^3+bx^2+cx=y^2$, c/ax_1 (suggested by (1)) may be shown to be a solution also.

The equation $ax^3+bx^2y+cxy^2+dy^3=z^2$ may be solved by setting $y=d$ or any other positive or negative odd power of d (or similarly setting $x=a$ or any other positive or negative odd power of a), thus reducing to the first case.

For the equation $x^3+x^2+x+1=y^2$ (Dickson's *History of the Theory of Numbers*, Vol. I, p. 56), the method of this paper gives the following two chains of solutions:

$$(0,1),\ (-3/4,\ 5/8),\ (40/9,\ 287/27),\ (561/400,\ 21359/8000),\ \cdots$$

$$(-1,0),\ (1,2),\ (7,20),\ (-31/49,\ 246/343),\ (-161/961,\ 27560/29791),\ \cdots$$

If $f(x)=x^3+x^2+x+1=(1+x)\ (1+x^2)=\square$, then $f\{(1-x)/(1+x)\}=4(1+x^2)/(1+x)^3=\square$, so that if x is a solution, $(1-x)/(1+x)$ is also a solution, the two being in reciprocal relation. This relation exists between each solution in the first chain and a corresponding solution in the second chain.

This method, though generally giving fractional solutions, furnishes for $x^3+x^2+1=y^2$ the integral solution (4, 9) and for $x^3+x+1=y^2$ the integral solution (72, 611).

Real Roots of a Class of Reciprocal Equations

By L. S. JOHNSTON, University of Detroit

Consider the reciprocal equations $P(x, s, n)=0$ $(s=1, 2, \cdots; n=1, 2, \cdots)$, separated into two classes,

$$\text{(I)} \qquad P(x, s, 2p) = x^{2p} + {}_sH_1x^{2p-1} + {}_sH_2x^{2p-2} + \cdots + {}_sH_{p-1}x^{p+1} + {}_sH_px^p + {}_sH_{p-1}x^{p-1} + \cdots + {}_sH_2x^2 + {}_sH_1 + 1,$$

$$\text{(II)} \quad P(x, s, 2p-1) = x^{2p-1} + {}_sH_1x^{2p-2} + {}_sH_2x^{2p-3} + \cdots + {}_sH_{p-1}x^p + {}_sH_{p-1}x^{p-1} + \cdots + {}_sH_2x^2 + {}_sH_1x + 1,$$

where, using the notation of Chrystal,

$$ {}_sH_r = \frac{s(s+1)(s+2)\cdots(s+r-1)}{r!}, \quad {}_sH_0 = 1.$$

We shall need also to use ${}_sC_r$ with its usual meaning in the theory of combinations, and the identities

$$\text{(A)} \qquad {}_sH_r = {}_sH_{r-1} + {}_{s-1}H_r,$$

$$\text{(B)} \qquad {}_sH_r = {}_{s-1}H_r + {}_{s-1}H_{r-1} + {}_{s-1}H_{r-2} + \cdots + {}_{s-1}H_2 + {}_{s-1}H_1 + 1,$$

$$\text{(C)} \qquad {}_sC_r = {}_{s-1}C_r + {}_{s-1}C_{r-1}.$$

We shall prove the following

THEOREM. The equation $P(x, s, n) = 0$ has just k real roots, counting a root of multiplicity j as equivalent to j roots, where $n \equiv k \pmod 4$, and $0 \leqq k < 4$; except that

(a) $P(x, 1, 2p) = 0$ has no real roots,

(b) $P(x, s, 2p-1) = 0$ has just one real root, negative unity, for s less than 3.

PROOF:

(a) $P(x, 1, 2p) = (x^{2p+1}-1)/(x-1)$, and $x^{2p+1}-1=0$ has no negative roots.

(b) $P(x, 1, 2p-1) = (x^{2p}-1)/(x-1)$, and $x^{2p}-1=0$ has negative unity as its only negative root.

$P(x, 2, 2p-1) = [P(x, 1, p)]\ [P(x, 1, p-1)]$; from the last two statements above it follows that only one of these factors has a real zero, the real zero being at negative unity.

Before attacking the general theorem we remark that a reciprocal equation of odd degree with all signs positive has negative unity as a real root; hence $P(x, s, 2p-1) = 0$ has this root for every s and every p. Furthermore, if a reciprocal equation of even degree has negative unity as a root, this must be a root of even multiplicity. This last statement will be used to show that $P(x, 2, 4r-2) = 0$ has negative unity as a double root for every r.

To prove the general theorem, we shall prove that except for the special cases already proved

(1) $P(x, s, n) = 0$ has at least k real roots, and

(2) $P(x, s, n) = 0$ has no more than k real roots.

For convenience we shall refer to these as Theorem (1) and Theorem (2); we shall also separate $P(x, s, n) = 0$ into four subclasses

(Ia) $P(x, s, 4r) = 0$ (IIa) $P(x, s, 4r-3) = 0$

(Ib) $P(x, s, 4r-2) = 0$ (IIb) $P(x, s, 4r-1) = 0$.

Theorem (1). The theorem is immediately seen to be true for (Ia) and (IIa). For (Ib) we see that $P(0, s, 4r-2) = 1$, and

$$\begin{aligned} P(-1, s, 4r-2) &= -{}_sH_{2r-1} + 2{}_sH_{2r-2} - 2{}_sH_{2r-3} + \cdots + 2{}_sH_2 - 2{}_sH_1 + 2 \\ &= {}_sH_{2r-1} - 2({}_sH_{2r-1} - {}_sH_{2r-2} + {}_sH_{2r-3} \cdots - {}_sH_2 + {}_sH_1 - 1). \end{aligned}$$

Using identities (A) and (B) above, this last quantity can be shown to be $-({}_{s-2}H_{2r-1} + {}_{s-2}H_{2r-3} + {}_{s-2}H_{2r-5} + \cdots + {}_{s-2}H_3 + {}_{s-2}H_1)$, which is zero for $s=2$ and negative for s greater than 2. Hence negative unity is a double root of $P(x, 2, 4r-2) = 0$, and, for s greater than 2, there exist two distinct negative roots of $P(x, s, 4r-2) = 0$.

For (IIb), we remove the factor $(x+1)$ from $P(x, s, 4r-1) = 0$, deriving a new equation

$$\begin{aligned} \overline{P}(x, s, 4r-2) = x^{4r-2} &+ ({}_sH_1 - 1)x^{4r-3} + ({}_sH_2 - {}_sH_1 + 1)x^{4r-4} + \cdots \\ &+ ({}_sH_{2r-2} - {}_sH_{2r-3} + \cdots + {}_sH_2 - {}_sH_1 + 1)x^{2r} \\ &+ ({}_sH_{2r-1} - {}_sH_{2r-2} + \cdots - {}_sH_2 + {}_sH_1 - 1)x^{2r-1} \\ &+ ({}_sH_{2r-2} - {}_sH_{2r-3} + \cdots + {}_sH_2 - {}_sH_1 + 1)x^{2r-2} \\ &+ \cdots \\ &+ ({}_sH_2 - {}_sH_1 + 1)x^2 + ({}_sH_1 - 1)x + 1 = 0. \end{aligned}$$

Now $\overline{P}(0, s, 4r-2) = 1$, and $\overline{P}(-1, s, 4r-2)$ can be shown, by using identities (A) and (B) above, to have the value

$$-({}_{s-3}H_{2r-1} + 2{}_{s-3}H_{2r-3} + 3{}_{s-3}H_{2r-5} + \cdots + r{}_{s-3}H_1),$$

which is zero for $s=3$ and negative for s greater than 3. Hence negative unity is a triple root of $P(x, 3, 4r-1) = 0$, and there exist two distinct negative roots, in addition to negative unity, of $P(x, s, 4r-1) = 0$ for s greater than 3. Theorem (1) is thus completely proved.

To prove Theorem (2) we construct an auxiliary set of equations $Q(x, s, n) = (x-1)^s P(x, s, n) = 0$, and show that $Q(-x, s, n)$ has exactly k variations in sign.

It can be shown that

$$Q(x, s, 2p) = x^{s+2p} - Ax^{s+p-1} + Bx^{s+p-2} - \cdots + (-1)^{s-2}Bx^{p+2} + (-1)^{s-1}Ax^{p+1} + (-1)^s.$$

The terms in $x^{s+2p-1}, x^{s+2p-2}, \cdots, x^{s+p}$, and their corresponding terms in $x, x^2, \cdots, x^p$ vanish, since the coefficient of x^{s+2p-m}, for $m \leqq p$, is

$${}_sH_m - {}_sH_{m-1}{}_sC_1 + {}_sH_{m-2}{}_sC_2 - \cdots + (-1)^{m-1}{}_sH_1{}_sC_{m-1} + (-1)^m{}_sC_m,$$

and this coefficient can be shown, by using identities (A) and (C) in succession, to be zero for every s and every $m \leqq p$. There are, therefore, $s+2p+1-2p=s+1$ terms in $Q(x, s, 2p)$, with s variations in sign; hence the signs alternate throughout the latter polynomial. It is easily shown that $Q(-x, s, 4r)$ has no variations in sign and that $Q(-x, s, 4r-2)$ has just two variations in sign. Hence the general theorem is completely proved for $P(x, s, 2p)=0$.

To prove Theorem (2) for (IIa) we can easily show that

$$Q(x, s, 4r-3) = x^{s+4r-3} - Ax^{s+2r-2} + Bx^{s+2r-3} - \cdots + (-1)^{s-2}Bx^{2r} + (-1)^{s-1}Ax^{2r-1} + (-1)^{s}.$$

The missing terms are explained by the same argument which explains the terms missing from $Q(x, s, 4r-2)$. When s is even, it is easily seen that there is a repeated sign, that of $(-1)^{s/2}$, in the center; when s is odd, the middle term of $Q(x, s, 4r-3)$ vanishes also. $Q(x, s, 4r-3)$ has therefore $s+2$ terms when s is even and $s+1$ terms when s is odd, with s variations in sign in both cases. It can be shown without difficulty that $Q(-x, s, 4r-3)$ has just one variation in sign, which proves the general theorem for (IIa). The proof of Theorem (2) for (IIb) follows exactly the same lines as that for (IIa); we find that $Q(-x, s, 4r-1)$ has exactly three variations in sign. The general theorem is thus completely proved.

It has been shown that negative unity is a triple root of $P(x, 3, 4r-1)=0$. It is also clear that this is the only one of the equations of the general type under consideration which has a triple real root at all, and that none of the other equations of odd degree has even a double real root. It is easily shown that $P(x, 3, 4r-1)=(x+1)^3P(x^2, s, 2r-2)$.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

The Taylor Series. An Introduction to the Theory of Functions of a Complex Variable. By P. Dienes. Oxford University Press, 1931. 552 pages. \$7.50.

Professor Dienes has here performed a notable service to the mathematical world in assembling and ordering in a thoroughgoing manner the modern theories relating to the Taylor series. From its title and subtitle one might suppose the book to be another elementary text book on complex function theory from the Weierstrass point of view. The title however is too modest. The book is a pioneer in its field and makes no inconsiderable demands on the maturity of its readers.

The first half of the book, after a foundation consisting of the theory of aggregates and fundamental ideas concerning functions of a real variable, presents most of the material of a first course in complex variable theory with the ideas of Cauchy and Weierstrass uppermost. The seven chapter headings of this part are I, Real Variables; II, Complex Algebra; III, Infinite Series; IV, Elementary Functions; V, Complex Differentiation; VI, Geometrical Language; VII, Complex Integration. The choice of material in these chapters is guided largely by the needs of the second part of the book. The treatment of specific functions (such as the doubly periodic functions) is off the main track of the author's purpose and is therefore dealt with briefly or not at all. On the other hand the geometrical ideas associated with Jordan's theorem, culminating in a treatment of Stieltjes integrals, are carried further than the requirements of a first course. Incidentally the cult of the analyst is shown in the author's use of 'mathematics' and 'geometry' as mutually exclusive terms.

The second half, which is the *raison d'être* of the book, has the following seven chapter headings: VIII, Biuniform Mapping, Picard's Theorem; IX, Representation of Analytic Functions; X, Singularities of Analytic Functions; XI, Overconvergence and Gap Theorems; XII, Divergent Series; XIII, The Taylor Series on its Circle of Convergence; XIV, Divergence and Singularities. The central problem as stated in the author's preface is that of detecting the properties of a function from the sequence of coefficients of its Taylor series and from the formal properties of these series. The pursuance of this central problem leads sometimes to theorems not always associated with Taylor series. Thus the inversion of Taylor series leads up to the classical theorems of Picard (on integral functions and essential singularities) and Riemann (on biuniform mapping). The detection and location of singularities by means of the Taylor coefficients and their relation to the character of the divergence occupies a large share of this half of the book. For this purpose a chapter is devoted to the theory of divergence and summability. This in the reviewer's opinion is not a treatment to which a beginner should come for his first ideas but rather a systematic exposition of the subject in a logical rather than a chronological order.

In regard to the manner of presentation, an effort is made to place due emphasis on important results so that the reader will not lose sight of the forest on account of the trees. This is very necessary in a book with so many laborious proofs which depend on obtaining finer and finer inequalities. Errata, consisting of misprints and slips (only a few of which are in the list of errata at the beginning) are sufficiently numerous to give zest to the reading of the book but will cause no real trouble to the reader. Among these we pause only to mention the curious number of slips in the discussion of arc cos x on page 117 and the example 6 on page 405 in which $g_{k+1}(a)$ should be $\Gamma(kt+t+1)/\Gamma(k+2)$ which necessitates a modification in this proof. In place of footnotes, which would otherwise need to be numerous, a bibliography is placed at the end of the book, to which references throughout the text are made by name and date.

The author makes commendable use of suggestive terms such as *starlike*,

star domain, *analytic landscape* and *picture* (in place of *image*). On the other hand few of us will condone the omission of the word *singular* from the term *essential point*; incidentally the statement on page 232 that "$f(z)$ is completely indeterminate in the neighborhood of an essential point" is misleading. Again *cyclometric* seems a ponderous term for the inverse circular functions and *complex differentiation and integration* for *differentiation and integration of functions of a complex variable* seems to border on slang. There is some confusion between paths and integrals over these paths (pages 247 and 250).

Turning from these minor points let us close by a further reference to the author's main purpose, that of giving a systematic and up to date account of those researches, mostly by European mathematicians, into the properties of functions of a complex variable in so far as they relate to the expression of these functions by power series. The author's own researches form an important contribution to the subject. The book is a difficult and useful piece of work well done and forms a significant addition to mathematical literature.

NORMAN MILLER

Catalogue of an Exhibition at Columbia University to Commemorate the One Hundredth Anniversary of the Birth of Lewis Carroll (Charles Lutwidge Dodgson) 1832–1898. Columbia University Press, New York, 1932. 153 pages $.35.

Lewis Carroll would have been the last man to refer to himself as a mathematician. He always insisted that Lewis Carroll was the author of the "Alice" books and that Charles Lutwidge Dodgson was the author of such mathematical works as *Condensation of Determinants*. Since we know that the familiar name is a variant of the unfamiliar one through "Lewis = Ludovicus = Lutwidge; Carroll = Carolus = Charles," we shall speak of him by either name. The world of his day learned with much astonishment of the identity of these two men. The world of the present moment will not give the same start of surprise over it.

No claim can be made for Dodgson as an original mathematician. He did little or no research work and was the originator of no epoch-making ideas in his field. His publications in the realm of mathematics relate rather to improvements and aids for the parts of the science which he presented in his university instruction. The Exhibit includes (1) the first edition of *Notes on the First Two Books of Euclid*, 1860, his first publication; (2) the first edition and "the author's own copy, with his initials 'C.L.D.' on front end paper, and corrections in pen" of *A Syllabus of Plane Algebraical Geometry*, 1860, his first book; (3) the first edition of *The Formulae of Plane Trigonometry*, 1861; (4) *Open Letter to Mathematical Teachers*, 1862; (5) *A Guide to the Mathematical Student*, 1864; (6) *Condensation of Determinants*, 1866; (7) *An Elementary Treatise on Determinants*, 1867; (8) *The Fifth Book of Euclid Treated Algebraically*, 1868; (9) *Euclid, Book V Proved Algebraically*, 1874; (10) *Examples in Arithmetic*, 1874; (11) *Euclid Books I, II*, 1875; (12) *Euclid and His Modern Rivals*, 1879; (13) *A Tangled Tale* (Answers to Knots I and II), 1880; (14) *A Tangled Tale* (Knots I to VII), 1880–1882, "*A Tangled Tale* is a series of mathematical problems written as sto-

her heard her say that she did not stand there alone but that the spirit of Lewis Carroll accompanied her and that she received the honor in his stead.

LAO G. SIMONS

Outline of the History of Mathematics. By Raymond Clare Archibald. The Society for the Promotion of Engineering Education. No. 18. February, 1932. 53 pages $.30.

This work appears as "Bulletin Number 18 of the Summer School for Engineering Teachers," an indication of the end in view in its preparation.

The "Outline" is far from corresponding to the usual meaning of that term. It is that, to be sure, but it is far more. It is an interesting account of the history of mathematics with emphasis on only the important persons and contributions to the field. Here and there less important items are introduced to enliven the lecture but these always flow naturally from the matter which they follow. We have, for example, the connection of Gauss with Dase: (p. 44) "Gauss was a notable instance of a youthful arithmetical prodigy who later achieved greatness in mathematics. The most extraordinary mental calculating prodigy who ever lived was Zacharias Dase of Hamburg who died in 1861, aged 37. He multiplied two 8-figure numbers in 54 seconds, two 20-figure numbers in 6 minutes, two 40 figure numbers in 40 minutes and two 100-figure numbers in $8\frac{3}{4}$ hours, and found the square root of a hundred figure number in 52 minutes. Gauss tried to turn his abilities to some useful purpose and as a result we have Dase's seven-place tables of the natural logarithms of numbers from 1 to 10,500 (1850) and his factor tables of all numbers in the sixth, seventh, and eighth million (1862–65)." One other of quite a different kind may be included: (p. 28) "Purely as a thrilling intellectual experience, without any imaginable practical application, Apollonius of Perga, and other Greeks, developed a marvelous body of knowledge with regard to conic sections. Then suddenly, 1800 years later, to a Kepler this knowledge had most illuminating practical applications."

There is a touch of genius in saying familiar things in a new way. An author who does this adds something to his subject at the same time that he is presenting its various phases. This author in his "Outline" makes a contribution in more ways than the name indicates.

The "Literature List" which closes the book is a fine set of references to books and magazine articles bearing on the contents of the work. These references are referred to by number as the various mathematicians and mathematical concepts are introduced. They present a wide field for further study in each direction. The list should be used by libraries to check up their resources bearing on the history of mathematics.

Table of Contents. Prefatory Note. Synopsis. History of Mathematics before the Seventeenth Century: A. Babylonian and Egyptian Mathematics 3500 B.C.–600 B.C., B. Greek Mathematics 600 B.C.–500 A.D., C. Hindu, Arabic, Persian Mathematics 500–1200, D. European Mathematics 1200–1600. History of

Mathematics after the Sixteenth Century: A. The Seventeenth Century, B. The Eighteenth Century, C. The Nineteenth Century and Later.

To an instructor in the history of mathematics this presentation will offer a variation in the handling of the course, and will be most welcome for that reason. An index would have increased its usefulness here. It will be welcome as well for the very readable story told so well and so authoritatively, which latter reason recommends it to any reader.

LAO G. SIMONS

College Algebra. By Louis J. Rouse. New York, John Wiley & Sons, 1931. x+345 pages. $2.25.

The preface to this book states that "This book, though primarily intended for use as a text-book in the freshman year of colleges and technical schools, is suitable for use in any school offering a course in advanced algebra." This statement, that the book may be used in schools of lower grade than the college, seems to the reviewer to be necessary, for otherwise the amount of elementary material included seems unnecessarily large for a college text. To be sure, it is claimed that this can serve as a basis of review for the college student, but it seems as if he should be able to do this reviewing by himself, with the aid of his previous text-books. The only advantage from the point of view of review is that the material is easily accessible for reference. However, the inclusion of a large amount of elementary review material is not peculiar to this text, but seems to be a feature of many of the present day college algebras.

In a few places the book is not so inclusive as it might be: for example, in the chapter on the theory of equations, no treatment is given of zero and infinite roots. The explanations and developments of theory are well presented in considerable detail; and we found that when the book was actually tried with a class, the students grasped these explanations more readily than they do the more concise proofs given in some other frequently used texts.

There is an abundance of well chosen problems, so that the student is afforded ample practice. Those on determinants, mathematical induction and the binomial theorem are especially good. In this connection, we observe that the authors state that "the many solved examples are intended not so much to serve as models for working similar exercises, as to illustrate the principles and new theories developed in the text. The rules given are usually to be considered as statements of processes which have been previously discussed rather than to be used mechanically in solving problems." This is a worthy aim in itself, but we fear that it will not be realised, and that the average student will do exactly what the author does not want him to do: he will use the rules and solved examples as short cuts to aid in solving other problems, without having to spend much time studying the underlying theory.

In general, we may say that the book contains the usual material, well-presented, and with ample problems. There is nothing extraordinary in either the choice or the presentation of material, nor any such originality of treatment

as in the case of the recent algebra by Smail. But the book, though not unusual, is a good one, and may be recommended for use.

Before concluding, we wish to call attention to a minor point of criticism: the development of a three-rowed determinant by an elaborate scheme of curved lines, as given on page 221, seems an unnecessary complication. It is difficult for the student to learn the exact positions of the lines, and they are easily forgotten. In view of the fact that the student will sooner or later have to learn the development by minors, it seems more logical to develop three-rowed determinants in this way from the start. The student "catches on" quickly, and in the reviewer's experience always remembers it.

MILDRED WATERS DEAN

Ordinary Differential Equations. By E. L. Ince. London, Longmans, Green and Co., 1927. vi+558 pages.

This volume, "the first to be launched into the world by a member of the staff of the newly-founded Egyptian University," seems to have received curiously little attention from reviewers. It is a very useful book for study or reference, treating as it does all phases of its subject.

Part I deals with the real domain. Introductory chapters on formal solution are followed by chapters on existence theorems, the Lie theory, the properties of linear equations in general and those with constant coefficients in particular, solutions by infinite series and by definite integrals, and the modern forms of the Sturm theories. Part II deals with the complex domain. After proving the usual existence theorems, the author proceeds first to non-linear equations; then discusses linear equations, including solutions by series and by contour integration, and the study of linear systems; the part closes with a chapter on oscillation theorems in the complex domain. Appendices deal with the history of formal integration, with numerical integration, and with bibliography.

The form of treatment is not quite uniform. The early chapters on formal integration are written entirely in the spirit of the usual elementary text-book, with the familiar vagueness as to what a *solution* of a differential equation really is, and with no critical examination of the accuracy of the steps used in "solving" the equation. The chapter on continuous groups exhibits the optimism regarding the behavior of the functions, which is expected of writers in this field. In practically all the rest of the book, theorems are precisely stated and proved with scrupulous attention to logical detail. It seems likely that the author has followed his sources rather closely, both as to matter and manner. For instance, the chapters on algebraic properties, Sturmian theorems, and boundary value problems are even in small detail like the original papers of Birkhoff and Bôcher.

It is very convenient to have so excellent a treatment of so many phases of the theory of ordinary differential equations gathered into one volume, and every worker in this branch of analysis must be grateful to the author for the valuable service rendered by his book.

The typography is excellent; errors are few and trivial.

W. A. HURWITZ

Note on a Greek Papyrus in Vienna

There has recently been published by the Nationalbibliothek at Vienna, in its *Mitteilungen aus der Papyrussammlung*, "Eine stereometrische Aufgabensammlung im Papyrus Graeccus Vindobonensis 19996," an event of considerable interest in the history of mathematics. The translation and extended description is the work of Dr. H. Gerstinger of Vienna, who read and transcribed the text; Dr. Oellacher of Salzburg; and Dr. K. Vogel, who prepared the principal part of the text,—"Kapital C. Mathematischer Inhalt des Papyrus." Dr. Vogel has already proved himself to be one of the coming historians of mathematics, and his present contribution confirms the impression which his previous articles have made.

The manuscript consists of a number of fragments of a papyrus roll containing thirty-eight problems relating to the metrical side of solid geometry, twenty-three being accompanied by drawings. Although the manuscript is not unique as to content, since it is quite like two others that have been described, one in the Field Museum in Chicago and the other in Berlin, it loses nothing of its value because of this fact.

The Greek text is given so far as it is decipherable, after which Dr. Vogel discusses at length the nature and significance of the work. He first gives a general survey of the manuscript, followed by a detailed statement of the methods employed by the writer and his Greek predecessors. In the discussion he pays special attention to the truncated pyramid which has recently been the subject of so much investigation in connection with the Moscow Papyrus (*c.*1850 B.C.), and upon which he has already taken a definite stand in his article in the *Journal of Egyptian Archaeology* (vol. 16, 1830, pp. 242–249). He then gives a list of mathematical terms used in the papyrus, together with abbreviations, symbols, and certain rather unusual number forms, both integral and fractional. The list will be particularly helpful to students of Greek mathematics as giving certain variants of the common numerals that were used by such writers as Diophantus and Nicomachus.

Dr. Vogel then gives a translation of the entire text, part of which is necessarily conjectural, and adds numerous explanatory notes. This part of the article (pp. 54–72) will naturally be considered the most important. It includes the mensuration of the rectangular solid, the pyramid, frustum of a pyramid, prism, cylinder, cone, and truncated cone, but not the sphere.

The manuscript seems to have been a schoolbook; that is, a copy made by a pupil. It was found in Dimêh, the ancient Soknopaiu Nesos, in the Fayum. The date is uncertain, but Dr. Vogel places it before the fourth century to which Professor R. C. Archibald had tentatively assigned it (Chace ed. of the Rhind Papyrus, vol. I, p. 124).

The manuscript was evidently written early in the Christian era and the date will doubtless be found with a satisfactory approach to accuracy after the work has been subjected to further study.

David Eugene Smith

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

A.

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research in the preparation of papers in the field of mathematical science to be presented at its regular meetings.

CHAPTER REPORTS
1931–1932

Alabama Alpha Chapter of Pi Mu Epsilon.

The officers for 1931–1932 were: Joseph S. Gelders, Director; Eric Rodgers, Vice Director; William F. Adams, Secretary; H. S. Thurston, Treasurer; Sara Ella Haughton, Librarian.

Alabama Alpha began the year with twenty-two active members. In October 27, 1931, thirteen new members were initiated. On March 29, 1932, four new members were initiated. This brings our total up to thirty-nine active members.

The meetings and programs were as follows:

September 29, 1931: "A problem in collineation groups" by Mr. Fred A. Lewis.
November 24, 1931: "Euclid's parallel postulate" by Miss Sara Ella Haughton.
February 2, 1932: "The use of mathematics in economics" by Mr. Paul Kringer.
February 23, 1932: "Extension of differentiation and integration" by Mr. G. N. Carmichael.
March 29, 1932: "A complete system of differential equations" by Mrs. Charles E. Watkins.
April 26, 1932: "Some geometry in connection with the bilinear transformation" by Mrs. J. W. Sledge; "A study of the second degree equation in isotropic coordinates" by Miss Margaret Gorrie; "Analytical proofs of some theorems on the triangle" by Miss Boyce Garrett; "A study of seismology" by Mr. Eric Rodgers.

Alabama Alpha has held two social meetings—a bridge party in the fall and a picnic this spring.

In addition to the above activities, we have contributed fifty dollars to the storm victims in this locality.

WILLIAM F. ADAMS, *Secretary*

Arkansas Alpha Chapter of Pi Mu Epsilon.

The officers for 1931–1932, elected May, 1931, by vote of the Chapter were: Guilford V. Smith, Director; Roberta Currie, Vice Director; Marion Brashiers, Secretary; Martha Bond, Treasurer; Lulu Mae Holland, Librarian; Mr. Paul Cramer, Faculty Advisor.

We now have twenty-eight active members. Seven new members were initiated at a banquet March 31, 1932. Our meetings were held monthly, and our programs were as follows:

October 21, 1931: "Mathematics as the tool of science" by Professor Roberds.
November 19, 1931: "History of the trisection of an angle" by Robert Vining; "Analysis situs" by Dr. V. W. Adkisson.

February 10, 1932: "Lewis Carroll" by Dr. H. M. Hosford; "The golden section" by Guilford Smith.

March 15, 1932: "Arithmetic of 100 years ago" by Miss Helen Graham.

March 31, 1932: "Contribution of science to 'Believe it or not' " by Frank Davis; "History of Pi Mu Epsilon" by A. B. Carson.

April 14, 1932: "Exponential form of complex numbers" by L. M. Martin; "Mathematics in correlation charts" by Dr. J. R. Guberich.

May 17, 1932: "Mathematicians of Michigan" by Jane Stelzner; "Mathematicians of California" by Louise Love.

A picnic was held on the campus for pledge services March 22, 1932, and the activities of the year ended with a picnic at Ghost Hollow on Saturday afternoon, April 30, 1932.

MARION J. BRASHIERS, *Secretary*

B.

LOCAL MATHEMATICS CLUBS

The Mathematics Klub of Adelphi College.

The officers for 1931–1932 were: Dr. Joseph Bowden, Honorary President; Theresa Cartereau, President; Sarah Gordon, Vice President; Dorothy Hill, Secretary; Dorothy Hagemann, Treasurer.

The officers are elected in May of each year by vote of the Klub. There were 32 active members this year. The purpose of the Klub is to stimulate further interest in mathematics outside of regular college courses. Membership is open to any student interested in mathematics. Regular meetings are held twice a month, On November 17, Dr. Dalman, professor of Chemistry at Adelphi discussed at the meeting the importance of Mathematics in the Development of Science. Our president, Miss Cartereau, read an article to the Klub "What is New in Math" on December 3. At the next meeting, December 5, one of the members, Miss Adele Shrage, spoke on "Oddities in Mathematics." January 5, our president gave a report on "Place-Value in the Number System." This was followed, February 20, by "Geometric Fallacies" reported on by Miss Madeline Sniffen. April 11, Miss Sarah Gordon, Vice-President of the Klub, discussed Newton, the Calculus and Leibnitz. During the year besides the regular meetings, a party and two supper meetings were held. The party was given to welcome new Freshmen on October twentieth. Bridge was played and refreshments served. The two supper meetings were December third and May third, the first at the college dining room, and the second at a restaurant in Hempstead, Long Island. Alumni were invited and several attended during the year. The regular meetings ended this year with the nominations of officers for 1932–1933.

DOROTHY HILL, *Secretary*

The Mathematics Club of the Cooper Union Institute of Technology.

The officers for 1931–1932 were: G. Kosolapoff, '32, President; W. W. Rigrod, '33, Vice President; C. H. Kropp, '33, Secretary; E. H. Ryan, '33, Treasurer; J. Watkins, '33, Assistant Secretary.

The club has a membership of 216.

The meetings and programs were as follows:

November 9, 1931. "Generalization in mathematics" by Prof. T. S. Fiske, Columbia University.

November 24, 1931. "Higher plane curves" by J. F. Skelly, '34.

December 15, 1931. "Theory and use of the slide rule" by E. A. Drago, '32.

January 12, 1932. "Non-Euclidean geometry" by A. Rubinsky, '34.

January 26, 1932. "The mathematics of lens surfaces" by A. Ginsberg, '32.

February 9, 1932. "Systems of coordinates" by K. Itkin, '34.

February 23, 1932. "Elementary vector analysis in physics" by D. Briansky, '32.

March 8, 1932. "Transformation of a theorem in geometry" by L. Green, '34.

March 22, 1932. "Determinants" by C. A. Wamser, '34.
April 5, 1932. "Matrices" by Mr. C. H. Lehmann, Instructor, Department of Mathematics; Election of Officers.

C. H. LEHMANN, *Faculty Adviser*

The Mathematics Club of the University of Buffalo.

The Mathematics Club of the University of Buffalo, organized in 1929, is open to all students of mathematics. At present there are about twenty members. The club holds monthly or bimonthly meetings, when students or outside speakers give the program. The officers for the year 1932 were: William Corse, President; Alice Link, Vice-president; Oakland Becker, Secretary-treasurer.

The Mathematics Club is very proud to have established this year the Wilfred H. Sherk Memorial Prize in Mathematics. This is an annual prize to be awarded for the best student paper in mathematics, and was established in memory of the late Professor Sherk who was for many years the head of the department of mathematics at the University of Buffalo. The recipient of the award for the year 1932 is John W. Wrench of the class of 1933 whose subject was "The impossibility of solving the quintic."

The programs of the year follow:

November: "Mathematics and psychology" by Benjamin B. Sharpe; "Number systems" by Robert R. Lyle.
February: "Changing methods in teaching secondary school mathematics" by Miss Mary Crofts of Fosdick-Masten Park High School.
April: April fool puzzle party.
May: The club was entertained at the home of Professor Gehman. The Sherk Prize was awarded and plans for the coming year discussed.

The following is the basis for determining the winner of the Wilfred H. Sherk Memorial Prize in Mathematics:

1. This prize of five dollars shall be awarded once a year to the student who has submitted the best paper on any branch of mathematics, pure or applied.
2. All undergraduate students of mathematics in the University of Buffalo are eligible to compete.
3. The paper need not be entirely original, but must be the product of independent research by the student. It must be accompanied by an adequate bibliography.
4. The paper must be submitted to the judges on or before April 15 of each year.
5. The name of the winner will be announced at the May meeting of the Mathematics Club.
6. The judges shall consist of two students, chosen by vote of the Mathematics Club, and three members of the mathematics department.
7. The award shall be given to one who receives the majority vote of the board of judges.
8. The judges shall base their decision on the following points: (a) Evidences of careful independent research; (b) Originality; (c) Literary style and form of presentation; (d) Mathematical background of the student. This should make it possible for freshmen and sophomore papers to be judged on a par with junior and senior papers.

OAKLAND BECKER, *Secretary-Treasurer*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing, and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3562. *Proposed by Vladimir F. Ivanoff, San Francisco, Calif.*

If $\phi(x, y, z)=0$, $\psi(x, y, z)=0$, prove that

$$\begin{vmatrix} d^2x & d^2y & d^2z \\ \phi_x & \phi_y & \phi_z \\ \psi_x & \psi_y & \psi_z \end{vmatrix} = 0,$$

under suitable conditions.

3563. *Proposed by Otto Dunkel, Washington University.*

To construct approximately an angle which is $1/n$th of a given angle, let the given angle be BAC, with C as any convenient point on the side AC. Take D on BA produced so that $AD=AC$, and draw DC. Lay off from A along BA produced lengths equal to $(n-1)DC/2$ and $(n-1)AC$ terminating, respectively, in M and N. Determine the point P which divides MN in the ratio $(8-n)$: $4(n-2)$; then the angle BPC is approximately $1/n$th of the angle BAC.

This construction is exact for $n=2$, 4 and for the trivial case $n=1$. For $n=3$ it is essentially the method given by Wedderburn in the second part of his problem 2972 [1922, 224]; and the error in this case was discussed in the solution [1925, 96–97]. For $n=3$ a more convenient but less accurate method may be obtained by taking P as the mid-point of MN. Determine the approximate error for given angles not greater than 45° for the cases $n=5$, 7, and for the modified construction when $n=3$.

3564. *Proposed by H. T. R. Aude, Colgate University.*

A determinant of order $n+1$ has for the elements of its first and second rows, respectively, the successive powers of α and x with exponents from 0 to n inclusive. The third row is the derivative of the second row, and the elements of any row after the second are given by the relation

$$\alpha_{i+1,j} = \frac{1}{i-1}\frac{d\alpha_{i,j}}{dx}, \quad i = 2, 3, \cdots, n.$$

Prove that this determinant has the value $(x-\alpha)^n$.

3565. *Proposed by Orrin Frink, Jr., Pennsylvania State College.*

Find the ellipse of least area circumscribing a given triangle.

SOLUTIONS

3500 [1931, 340]. *Proposed by Emma M. Gibson, Springfield High School, Springfield, Mo.*

Show that the primitive of the differential equation, $p^2(1-x^2)=(1-y^2)$ is $x^2+y^2-2Axy=1-A^2$ and derive this equation by taking the sine of the sum of two angles, both of which are arcsines.

Solution by J. D. Leith, University of North Dakota.

Extracting the square root of each side of the differential equation, separating the variables, and integrating, we obtain the primitive at once in the form arcsin $y=$arcsin $x+$arcsin k, where the plus sign alone has been used on the right. Taking the sine of each side, and setting $k^2=1-A^2$, we obtain the required result after isolating on one side the radical term containing x and then squaring each side. Since A may be positive or negative, the double sign has been taken into account in the result.

Also solved by H. M. Feldman, F. Underwood, and the proposer.

3503. (1931, 408) *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

The apex angles of two cones of revolution are α_1 and α_2; the cones intersect in an ellipse whose semi-axes are a and b; show that the angle β which the axes of the cones make with each other is given by the equation

$$\cos\beta = \left[\sin^2 \tfrac{1}{2}\alpha_1 - (b/a)^2 \cos^2 \tfrac{1}{2}\alpha_1\right]^{1/2}\left[\sin^2 \tfrac{1}{2}\alpha_2 - (b/a)^2 \cos^2 \tfrac{1}{2}\alpha_2\right]^{1/2}$$
$$- \left[1 - (b/a)^2\right] \cos \tfrac{1}{2}\alpha_1 \cos \tfrac{1}{2}\alpha_2.$$

Solution by F. Underwood, University College, England.

Let γ_1 and γ_2 be the angles which the axes of the cones make with the plane of the ellipse. (See the solution, immediately following, of problem 3504 for the relation $\cos\gamma_1=e\cos\frac{1}{2}\alpha_1$.) The angle β between the axes of the two cones may be $\gamma_1-\gamma_2$, $\gamma_1+\gamma_2$, or $\pi-\gamma_1-\gamma_2$ according to the senses in which the angles γ_1 and γ_2 are measured, and according as an acute or obtuse angle is taken as β. Now

$$\cos\gamma_1\cos\gamma_2 = e^2\cos\tfrac{1}{2}\alpha_1\cos\tfrac{1}{2}\alpha_2 = \left\{1-(b/a)^2\right\}\cos\tfrac{1}{2}\alpha_1\cos\tfrac{1}{2}\alpha_2 = p, \text{ say.}$$

If γ_2 is acute (as well as γ_1),

$$\begin{aligned}\sin\gamma_1\sin\gamma_2 &= (1-e^2\cos^2\tfrac{1}{2}\alpha_1)^{1/2}(1-e^2\cos^2\tfrac{1}{2}\alpha_2)^{1/2}\\ &= \left\{\sin^2\tfrac{1}{2}\alpha_1+(b/a)^2\cos^2\tfrac{1}{2}\alpha_1\right\}^{1/2}\left\{\sin^2\tfrac{1}{2}\alpha_2+(b/a)^2\cos^2\tfrac{1}{2}\alpha_2\right\}^{1/2}\\ &= q, \text{ say.}\end{aligned}$$

Then

(a) if $\beta = \gamma_1 - \gamma_2$, $\cos\beta = p + q$

(b) if $\beta = \gamma_1 + \gamma_2$, $\cos\beta = p - q$

(c) if $\beta = \pi - \gamma_1 - \gamma_2$, $\cos\beta = -p + q$

Also solved by W. B. Campbell, A. Pelletier, and Paul Wernicke.

3504 (1931, 408). *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Two cones of revolution whose apex angles are α_1 and α_2 intersect in an ellipse whose semi-major and semi-minor axes are a and b; show that

$$\cos\gamma_1/\cos\gamma_2 = \cos\tfrac{1}{2}\alpha_1/\cos\tfrac{1}{2}\alpha_2$$

in which γ_1 and γ_2 are the angles which the axes of the cones make with the plane of the ellipse.

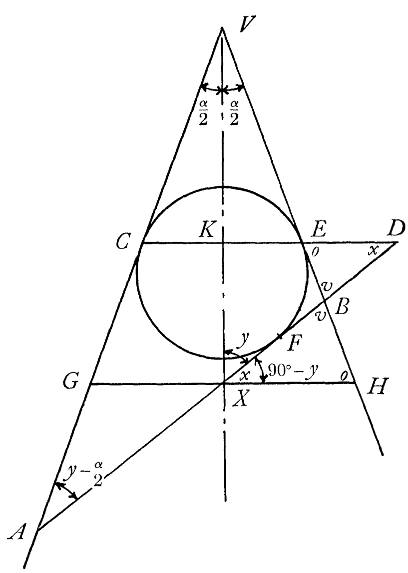
Solution by V. F. Murray, Hoboken, N.J.

Let a circular cone of vertical angle $AVB=\alpha$ be cut by a plane AXB making an angle $VXB=\gamma$ with the axis VX (see figure).

Inscribe the sphere FCE touching the plane AXB in the point F and the cone in the circle CKE. Let the planes AXB, CKE meet in the line D. Draw the line XH parallel to CE, to meet VE in H.

The point F is a focus of the ellipse and the line D is the corresponding directrix (see C. Smith, Geometrical Conics, 1894, p. 180).

The eccentricity e equals BF/BD or BE/BD, BF and BE being equal tangents.



In the triangle BXH,

$$\begin{aligned}\sin BXH/\sin BHX &= BH/BX\\ &= BE/BD, \text{ by similar triangles,}\\ &= e.\end{aligned}$$

Since VXH is a right angle, $\sin BXH=\cos\gamma$ and $\sin BHX=\cos\frac{1}{2}\alpha$. Thus $\cos\gamma/\cos\frac{1}{2}\alpha=e$.

The construction given is quite general so that when a circular cone of apex angle α is cut by a plane making an angle γ with the axis, the eccentricity of the resulting conic section is given by $\cos\gamma/\cos\frac{1}{2}\alpha$.

In the problem, the ellipses being given the same, the eccentricities obtained from the two cones are the same, therefore

$$\cos\gamma_1/\cos\tfrac{1}{2}\alpha_1 = \cos\gamma_2/\cos\tfrac{1}{2}\alpha_2$$

or

$$\cos\gamma_1/\cos\gamma_2 = \cos\tfrac{1}{2}\alpha_1/\cos\tfrac{1}{2}\alpha_2.$$

This is true for all parallel planes in each case. To determine VX corresponding to a major axis $2a$; in the triangle VAB,

$$VB = \frac{2a\sin(\gamma-\frac{1}{2}\alpha)}{\sin\alpha},$$

and in the triangle VBX,

$$VX = VB\,\frac{\sin(\gamma+\frac{1}{2}\alpha)}{\sin\gamma} = 2a\,\frac{\sin(\gamma+\frac{1}{2}\alpha)\sin(\gamma-\frac{1}{2}\alpha)}{\sin\alpha\sin\gamma}.$$

from which the values of VX in the two cones can be calculated.

Also solved by W. B. Campbell, J. B. Ennis, A. Pelletier, F. Underwood, and Paul Wernicke.

3506 [1931, 408]. *Proposed by E. B. Escott, Oak Park, Illinois.*

Solve these n simultaneous equations in n unknowns:

$$\begin{vmatrix} x_1x_2\cdots x_{n-1}\\ x_nx_1\cdots x_{n-2}\\ \cdots\cdots\\ x_3x_4\cdots x_1\end{vmatrix} = a_1; \quad \begin{vmatrix} x_2x_3\cdots x_n\\ x_1x_2\cdots x_{n-1}\\ \cdots\cdots\\ x_4x_5\cdots x_2\end{vmatrix} = a_2; \cdots$$

$$\begin{vmatrix} x_nx_1\cdots x_{n-2}\\ x_{n-1}x_n\cdots x_{n-3}\\ \cdots\cdots\\ x_2x_3\cdots x_n\end{vmatrix} = a_n.$$

Solution by T. L. Smith, Carnegie Institute of Technology

Let $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ be a set of n mutually orthogonal unit vectors along the corresponding coordinate axes in n-space. Then the system of equations given above may be more compactly written in the form

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \cdots \mathbf{e}_n \\ x_n & x_1 & x_2 \cdots x_{n-1} \\ x_{n-1} & x_n & x_1 \cdots x_{n-2} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ x_2 & x_3 & x_4 \cdots x_1 \end{vmatrix} = a_1\mathbf{e}_1 \pm a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \pm a_4\mathbf{e}_4 + \cdots,$$

where the double sign is $+$ or $-$ according as n is odd or even.

From this form, the vector with components $(a_1, \pm a_2, a_3, \cdots)$ is perpendicular to each of the $n-1$ vectors whose components are the elements in the rows of the determinant in the last equation. This fact gives rise to the $n-1$ equations

$$\begin{cases} a_1x_n \pm a_2x_1 + a_3x_2 \pm \cdots = 0, \\ a_1x_{n-1} \pm a_2x_n + a_3x_1 \pm \cdots = 0, \\ a_1x_{n-2} \pm a_2x_{n-1} + a_3x_n \pm \cdots = 0, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{cases},$$

which on rearrangement becomes

$$\begin{aligned} \pm a_2x_1 + a_3x_2 \pm a_4x_3 + \cdots &= 0, \\ a_3x_1 \pm a_4x_2 + a_5x_3 \pm \cdots &= 0, \\ \pm a_4x_1 + a_5x_2 \pm a_6x_3 + \cdots &= 0, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ . \end{aligned}$$

The solution of this system of $n-1$ homogeneous linear equations determines the ratio of the unknowns,

$$x_1 : x_2 : \cdots : x_n = \alpha_1 : \alpha_2 : \cdots : \alpha_n,$$

where

$$\alpha_1 = \begin{vmatrix} a_3 \pm a_4 \cdots \\ \pm a_4 \quad a_5 \cdots \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{vmatrix}, \alpha_2 = \begin{vmatrix} \pm a_4 \quad a_5 \cdots \\ a_5 \pm a_6 \cdots \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{vmatrix}, \cdots, a_n = \begin{vmatrix} \pm a_2 \quad a_3 \cdots \\ a_3 \pm a_4 \cdots \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{vmatrix}.$$

($+$ or $-$ according as n is odd or even).

Finally, we may solve for any of the unknowns, say x_r, by eliminating all the other unknowns from one of the original given equations, say the first, by means of the relations

$$x_i = \alpha_i x_r / \alpha_r.$$

On substituting these relations in the first of the given equations, multiplying both members by α_r^{n-1} for symmetry, and factoring out x_r^{n-1}, we obtain

$$x_r^{n-1}\begin{vmatrix} \alpha_1 & \alpha_2 \cdots \alpha_{n-1} \\ \alpha_n & \alpha_1 \cdots \alpha_{n-2} \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \alpha_3 & \alpha_4 \cdots \alpha_1 \end{vmatrix} = a_1\alpha_r^{n-1}.$$

Whence finally

$$x_r = \alpha_r a_1^{\frac{1}{n-1}}\begin{vmatrix} \alpha_1 & \alpha_2 \cdots \\ \alpha_n & \alpha_1 \cdots \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \alpha_3 & \alpha_4 \cdots \end{vmatrix}^{\frac{1}{1-n}}.$$

In this final result, the radical may be interpreted as any one of the $(n-1)$th roots of the quantity underneath it. The determinant under the radical is obtained from the determinant in the first of the given equations by replacing the x_i by the corresponding α_i.

Also solved by W. V. Parker and F. Underwood.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor R. A. Millikan, of the California Institute of Technology, has been awarded the Roosevelt medal of the Roosevelt Memorial Association.

The Franklin Institute has awarded a Cresson medal to Professor P. W. Bridgman, of Harvard University.

Juan de la Cierva has been awarded the Guggenheim gold medal for the promotion of aeronautics, for his work in the development of the theory and practice of the autogyro.

Mrs. Anna Johnson Pell-Wheeler of Princeton, N. J. received the honorary degree of Doctor of Science at the eleventh annual commencement exercises of New Jersey College of Women, June 4, 1932.

Professor Marston Morse, of Harvard University, has been elected a member of the National Academy of Sciences.

The following have been elected members of the American Academy of Arts and Sciences: C. R. Adams, A. A. Bennett, Einar Hille, and D. V. Widder.

Professor Werner Heisenberg, of the University of Leipzig, is taking part in the fifth annual symposium in theoretical physics at the University of Michigan, June 27–August 19, 1932.

Dr. Beulah M. Armstrong, of the University of Illinois, has been promoted to be associate in mathematics.

Dr. H. W. Bailey has been promoted to an assistant professorship at the University of Illinois.

Assistant Professor B. H. Brown has been promoted to a professorship of mathematics at Dartmouth College.

J. P. DenHartog, chief of the dynamic section of the Westinghouse Electric and Manufacturing Co., has been appointed assistant professor of applied mechanics at Harvard University.

Dr. J. L. Dorroh has been appointed research assistant at Princeton University.

Dr. O. J. Farrell has been appointed professor of mathematics at Union College.

Mr. S. E. Field, instructor at the University of Michigan, has been appointed head of the department of mathematics at Junior College, Ironwood, Michigan.

Dr. W. W. Flexner has been promoted to be associate in mathematics at Bryn Mawr College.

Dr. W. O. Menge has been promoted to an assistant professorship in mathematics at the University of Michigan.

Dr. J. E. Powell has been promoted to an assistant professorship of mathematics at Michigan State College.

Assistant Professor M. H. Stone, of Yale University, has been promoted to an associate professorship.

Dr. Oscar Zariski has been promoted to an associate professorship in mathematics at Johns Hopkins University.

Dr. Leo Zippin has been appointed research assistant at Princeton University.

The following appointments to instructorships are announced:

Brooklyn College of the City of New York, Dr. J. M. Feld, Dr. L. S. Kennison, Mrs. Jennie P. Kormes.

University of Minnesota, Mr. Carl H. Fischer.

Northwestern University, Dr. N. E. Rutt, Dr. W. J. Trjitzinsky.

Ohio State University, Dr. L. E. Bush

Pennsylvania State College, Dr. T. C. Benton, Dr. Beatrice L. Hagen.

Princeton University, Mr. M. M. Flood.

Stanford University, Mr. F. A. Butter.

University of Wisconsin, Mr. K. W. Wegner.

Yale University, Mr. W. R. Church.

Dr. L. A. Bauer, Director emeritus of the Department of Terrestial Magnetism of the Carnegie Institute of Washington, died April 12, 1932, at the age of sixty-seven.

Mr. J. D. Grant, of the department of mathematics at the University of Illinois, died July 9, 1932. He had been a member of the Association since 1928.

Professor C. G. Simpson, of the department of mathematics of the School of Engineering, Milwaukee, died February 5, 1932, at the age of sixty.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Sixteenth Summer Meeting of the Association, Los Angeles, Calif., Aug. 29-30, 1932.

Seventeenth Annual Meeting of the Association, Atlantic City, N.J., Dec. 27-30, 1932.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1932 and reported to the Secretary.

ILLINOIS, Urbana, May 6-7.
INDIANA, Indianapolis, May 6-7.
IOWA, Cedar Falls, April 29-30.
KANSAS, Topeka, Feb. 13.
KENTUCKY, Lexington, May.
LOUISIANA-MISSISSIPPI, Oxford, Miss., March 11-12.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7.
MICHIGAN, Ann Arbor, March 19.
MINNESOTA, River Falls, Wis., May 7.
MISSOURI.
NEBRASKA, Omaha, May, 6-7.
OHIO, Columbus, Ohio, April 7.
PHILADELPHIA, Philadelphia, Pa., Nov. 26.
ROCKY MOUNTAIN, Laramie, Wyo., April 15-16.
SOUTHEASTERN, Gainesville, Fla., Mar. 18-19.
SOUTHERN CALIFORNIA, San Diego, March 26.
TEXAS, Austin, Jan. 30.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXIX, 1932

NUMBER 8, OCTOBER

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND ITHACA, N.Y.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

THE EIGHTH ANNUAL MEETING OF THE LOUISIANA—MISSISSIPPI SECTION

The eighth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at the University of Mississippi, Oxford, Mississippi, on March 11 and 12, 1932, in connection with the annual meetings of the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics and the Mississippi Academy of Sciences. Professor A. C. Maddox, Chairman of the Section, presided at the session on Friday afternoon at which there were about thirty in attendance.

At the business meeting the following officers were chosen for next year: Chairman, C. D. Smith, Mississippi State College; vice-Chairmen, T. A. Bickerstaff, University of Mississippi, and H. L. Smith, Louisiana State University; Secretary, May Hickey, Delta State Teachers College. The 1933 meeting of the Section will be held at Ruston, Louisiana.

The Oxford Branch of the American Association of University Women served tea at the Graduate Building on Friday afternoon, and the University of Mississippi entertained at the joint banquet on Friday evening.

Professor Dunham Jackson, University of Minnesota, lectured on "Least Squares and Shortest Distances" at the joint meeting held on Friday evening.

The following papers were presented:

1. "An elementary problem in vector analysis" by Professor T. J. Lindsey, Mississippi State College, by invitation.
2. "Small oscillations of the neutral helium atom near the straight line positions" by Professor H. E. Buchanan, Tulane University.
3. "On Cayley's formulas for orthogonal determinants" by Professor D. S. Dearman, Mississippi State Teachers College.
4. "Trigonometric interpolation" by Professor Dunham Jackson, University of Minnesota.
5. "A study of certain problems from the field of investment" by Professor I. C. Nichols, Louisiana State University.
6. "Some theorems on symmetric determinants" by Professor W. V. Parker, Mississippi Woman's College.
7. "Least squares and shortest distances" by Professor Dunham Jackson, University of Minnesota.

Abstracts of some of these papers follow:

3. The definition of orthogonal determinants is given in terms of $\frac{1}{2}n(n+1)$ independent relations. By use of Cayley's formulas, the n^2 elements of an orthogonal determinant are given in terms of $\frac{1}{2}n(n-1)$ independent parameters. Cayley gives the development of these formulas in Crelle's Journal, Vol. 32. By means of further work done by Brioschi in the 1850's and by Siacci in the

1870's, the development of these formulas can be made more general. This latter development is given in Kowalewski's Determinantentheorie.

4. This paper is intended to supplement the one on Fourier series presented by the author at the New Orleans meeting of the Association. The purpose is to see how far the organization of material suggested there can be adapted to the related problem of trigonometric interpolation. While there are considerable differences of detail, it is found possible to supply proofs both of convergence and of uniform convergence comparable in simplicity and in generality with those given for Fourier series in the earlier paper.

6. The principal theorems of the paper may be stated as follows: Theorem 1. If $D=|a_{ij}|$ is a real symmetric determinant of order greater than four with $a_{ii}=0$, then if all fourth order principal minors of D are zero, D vanishes.

Theorem 2. If $D=|a_{ij}|$ is a real symmetric determinant of order n, $n>5$, with $a_{ii}=0$, and M is any principal minor of D of order $n-1$, then if all fourth order principal minors of M are zero, D vanishes.

Theorem 3. If $D=|a_{ij}|$ is a symmetric determinant of order n, $n>5$, with $a_{ii}=0$, and M is any principal minor of D of order $n-1$, then if all fourth order principal minors of D which are not minors of M are zero, D vanishes.

Certain results of these theorems were stated.

7. This paper was a detailed presentation of the elementary facts of analytic geometry which lead to the algebraic notion of orthogonality, and, when carried over into space of n dimensions, to the determination of a regression coefficient as a parameter marking the foot of the perpendicular from a point to a line.

NOLA LEE ANDERSON, *Secretary*

THE TENTH ANNUAL MEETING OF THE SOUTHEASTERN SECTION

The Southeastern Section of the Mathematical Association of America met in its tenth annual session at the University of Florida, Gainesville, March 18 and 19, 1932. In the absence of the chairman, C. D. Killebrew, Professor A. B. Morton presided. Sessions were held in the afternoon and evening on the 18th, and on the morning of the 19th. Professor Dunham Jackson, of the University of Minnesota, was the guest speaker.

Forty-eight persons attended the meeting, including the following seventeen members of the Association: D. F. Barrow, C. S. Cox, Forrest Cumming, B. F. Dostal, Floyd Field, L. D. Hampton, D. C. Harkin, P. R. Hill, Dunham Jackson, F. W. Kokomoor, F. A. Lewis, A. B. Morton, W. P. Ott, Z. M. Pirenian, A. J. Robinson, T. M. Simpson and E. F. Weinberg.

The following officers were elected for the next year: Chairman T. M. Simpson, University of Florida; vice-Chairman, D. F. Barrow, University of Georgia; Secretary, W. W. Rankin, Duke University. A committee, composed of one

member from each of the colleges represented, was appointed to study the teaching of mathematics in the high schools and colleges of the southeast, and to make a comprehensive report of their findings.

The next annual meeting will be held at the University of Georgia.

On the evening of the 18th, a dinner in Primrose Grill was held in honor of the visiting speaker, Professor Dunham Jackson. At this time vice-Chairman T. M. Simpson presided, and President J. J. Tigert of the University of Florida gave the address of welcome, while Dean Floyd Field of Georgia School of Technology responded.

Papers were presented as follows:

1. "How can a high school teacher advance professionally?" by Professor F. W. Kokomoor, University of Florida.

2. "The value of college mathematics to the high school teacher" by Professor Dunham Jackson, University of Minnesota.

3. "Problems in the teaching of high school mathematics" by D. M. Bernard, Robt. E. Lee High School, Jacksonville, Fla., by invitation.

4. "Problems in the teaching of freshman college mathematics" by Professor T. M. Simpson, University of Florida.

5. "Some observations on the correlation of these two problems" by U. P. Davis, University of Florida.

6. "The study of mathematics" by Professor Jackson.

7. "A conference method" by Professor E. F. Weinberg, Rollins College.

8. "Solution of a certain transcendental equation" by Professor D. F. Barrow, University of Georgia.

9. "Frequency functions" by Professor Jackson.

10. "Graphical determination of moments of inertia" by H. H. Germond, University of Florida, by invitation.

11. "Remarks on non-Aristotelian mathematics" by B. F. Dostal, University of Florida.

12. "Calculations with approximate data" by Professor C. G. Phipps, University of Florida, by invitation.

Abstracts of some of these papers follow:

1. Professor Kokomoor emphasized five points, as follows: (1) Strenouus application to the work until one is born into the spirit of teaching when one's interest becomes contagious and students are infected. (2) The development of an outlook from which pupils can be shown how almost every field of human endeavor is shot through and through with mathematics. (3) The study of order, so that there may be adequacy in the selection of material, in the manner of presenting it, in one's own method of study and of directing students in their advance. (4) The acquisition of more of the content of mathematics so as to keep apace in the march of knowledge. (5) The spirit of cooperation and the exercise of initiative in seeing and satisfying the needs of the department.

3. In discussing Algebra Mr. Bernard's remarks centered around the formation and solution of the equation, and the symbols were presented as forming a powerful system of shorthand. The use of charts in geometry was discussed at length and a method of teaching pupils to prove theorems was presented. Indirect proofs, loci, the Pythagorean theorem and similar figures were discussed as important subjects which present difficulty and suggestions were offered for treating these topics. In solid geometry it was recommended that theorems of minor importance and those with long and difficult proofs be treated as assumptions if necessary to give ample time for a thorough study of numerical exercises.

4. Professor Simpson discussed the following questions: How can we stimulate scholarship among freshman? What shall we teach in a beginning course? Is there any best method of teaching mathematics? What are some of the deficiencies in the preparation of freshmen?

5. One of the principal causes of the unsatisfactory preparation in mathematics for college entrance is the lack of a definite objective on the part of the high school teacher. Modern conditions demand that college preparation should not be this objective. This need should be met by enriching the course of study for the better students only, in such a way that, for prospective college students, the foundations in mathematics will be strengthened. Every accredited high school should issue two different grades of diplomas, one of which does not admit the holder to college.

8. In this paper, Professor Barrow sets up an infinite exponential which satisfies the equation $x \log y = y \log x$ and whose graph consists of a segment of a straight line joined at right angles by a curve which approaches an asymptote parallel to the x-axis and one unit above it.

10. If an area enclosed by a curve in the x, y plane be remapped by the transformation $u=x$, $v=y^2/2$, the area enclosed by the new curve (in the u, v plane) will numerically equal the moment of the original area about the x-axis. The transformation $u=x$, $v=y^3/3$, results in an area in the u, v plane equal to the moment of inertia about the x-axis of the original area in the x, y plane. The area obtained by remapping in a u, v plane an area in the x, y plane by the transformation $u=x$, $v=\pi y^4/2$, is equal to the moment of inertia about the axis of rotation of the volume formed by rotating the original area about the x-axis. Remapping a curve from the r, θ plane to the plane $\rho=r^2/\sqrt{2}$, $\phi=\theta$, the area enclosed by the new curve equals the polar moment of inertia of the area enclosed by the original curve.

The use of suitably prepared coordinate papers facilitates the application of this method of determining the moment of inertia.

11. Professor Dostal discussed the bearing of the work of the Polish School of Intuitionistic Formalists upon the questions raised by Eddington in his address of January 4, 1932, (reported in London Nature, Feb. 13). According to Eddington, we have already reached the stage, in Theoretical Physics, where it is considered desirable to abandon Determinism, at least for the time being.

However, even Eddington admits the desirability, for certain practical purposes, of using Deterministic means as a sort of "symbolic operational" method for obtaining results quickly and conveniently. For astronomical purposes, for example, we could continue to use what may be called, perhaps, "Astronomical Determinism," to calculate results, whereas for Electron Physics we could make use of the "Indeterministic" or "Secondary" Laws (as Eddington terms them), that is, the Probability Methods of the New Quantum Mechanics. Between these two extreme cases, however, there would be a set of intermediate cases And this would lead us into logical difficulties, for the notion of Probability has no recognized place in classical Deductive Logic. Furthermore, Determinism enters the scientific arena mainly through the door of Deductive Logic, and if we are to modify Determinism we will have to modify our whole system of thinking. The necessity for such a fundamental revision of our methods of reasoning is strongly suggested in Eddington's statement that while naive realism, materialism, and the Mechanistic Hypothesis were simple, yet it was only by closing our eyes to the essential nature of experience that they could be made to seem credible. Suppose now, it were possible to construct a multiple-valued Logical System (instead of the simple Two-valued System of classical Logic, as exemplified, for instance, in Whitehead and Russell's Principia Mathematica) of such a sort that it could constitute a valid and satisfactory basis for a whole set of distinct Theories of Sets; then the way would be open for the introduction of the notion of Probability into Deductive Logic, and the logical difficulties mentioned above might be smoothed out. Now, as a result of the work of Chwistek (Mathematische Zeitschrift, Vols. 26 and 30), and of Lesniewski, Tarski, Lukasiewicz, Greniewski, and others, together with the fundamental work of Korzybski (a forthcoming book by him will explain the work of the Polish School), a start in this direction has already been made, and a multi-valued system has actually been constructed, and looks to be very promising for the purposes mentioned.

12. This paper discusses the process of "rounding off" as applied to numbers, the absolute and relative errors in sums, differences, products and quotients. Mention is made of the accuracy in connection with logarithms and trigonometric functions. The reverse problem is taken up, namely, the allowable error in the data if there is a certain desired accuracy in the result.

W. W. RANKIN, *Secretary*

THE NINTH ANNUAL MEETING OF THE INDIANA SECTION

The ninth annual meeting of the Indiana Section of the Mathematical Association of America was held on Friday and Saturday, May 6, and 7, 1932, at Butler University, Indianapolis, Indiana.

There were sixty present at the meeting including the following twenty-six members of the Association:

Gladys L. Banes, G. A. Bliss, Stanley Bolks, G. E. Carscallen, H. T. Davis, J. E. Dotterer, W. E. Edington, P. D. Edwards, E. D. Grant, G. H. Graves, H. E. H. Greenleaf, F. H. Hodge, H. K. Hughes, E. N. Johnson, E. L. Klinger, Juna M. Lutz, T. E. Mason, H. A. Meyer, T. W. Moore, Eunice C. Orr, Saul Pollock, J. A. Reising, C. K. Robbins, L. S. Shively, R. O. Virts, K. P. Williams.

On Friday afternoon at 5:30 a reception was given to the visiting members and their guests which was followed by a banquet attended by fifty-five persons. Dean J. W. Putnam of Butler University officiated as toast master and made an address of welcome to the members of the Section and their guests. Responses were made by Professor Carscallen of Wabash College, chairman of the Section, Professor Edington of De Pauw University and Professor Cora B. Hennel of Indiana University.

At 8:15 a public lecture was given by Professor G. A. Bliss of the University of Chicago on the subject, "The Structures of Pure and Applied Mathematical Sciences." In this address Professor Bliss characterized pure mathematical science as consisting of postulates, definitions, and theorems. The theory of the real number system based upon four simple postulates for positive integers was cited as an example. The structure of an applied mathematical science is similar except that the usefulness of the theory depends upon the accuracy with which the postulates correspond to simple observed data, and the logical conclusions of the theory to the results of more complicated observations. It is not true that there is a unique mathematical theory for the correlation of a particular set of natural phenomena. On account of the looseness of the fit between theory and observation, which is always present, a multiplicity of theories for the correlation of the same set of observed data is always possible. Euclidean and non-Euclidean theories of plane geometry, the Ptolemaic and Copernican theories in astronomy, and the various quantum theories were described briefly as illustrations.

Following the address of Professor Bliss, Professor Saul Pollock, of the Indiana State Teachers' College at Terre Haute, gave a public exhibition of "Skew Curve Projection." This exhibition consisted in the creation of space curves by throwing light upon string models of various types of surfaces. By means of the device of photographing curves of high order and using these lantern slides in turn to generate new space curves, it was possible to obtain curves of remarkably high orders.

At the session on Saturday morning, presided over by Professor G. E. Carscallen, Wabash College, chairman, the following officers were elected: Professor K. P. Williams, Indiana University, Chairman; Professor Juna M. Lutz, Butler University, vice-Chairman; Professor H. T. Davis, Indiana University, Secretary-Treasurer.

A chairman's address was made by Professor G. E. Carscallen on the subject, "The Pathology of Mathematics." In this address Professor Carscallen called attention to the fact that "educational research" during the past two or

three decades has encroached by leaps and bounds upon other departments in college and university curricula. This encroachment presents to mathematics in particular an especial menace, since too many people, ignorant of its aims and unappreciative of its importance, have attempted foolish modifications of the mathematics courses both in intermediate and college teaching. Present low standards of attainment by graduates are attributable to these causes. The speaker cited the lowered standards of the North Central Association with regard to mathematics as an evidence of this dangerous trend. Professor Carscallen particularly urged the members of the Indiana Section to take a more vigorous part in the framing of curricula and in other activities where the cause of mathematics could be more effectively defended.

By special invitation Dr. Cornelius Lanczos of Frankfort University, Germany, and Purdue University, made an hour's address on the subject, "An Elementary Development of Riemannian Geometry with Application to Relativity." In this address Dr. Lanczos considered the geometry of the line element, $ds^2=g_{11}dx_1^2+2g_{12}dx_1dx_2+g_{22}dx_2^2$, and showed how interpretations in Euclidean geometry could be generalized in the Riemannian and Lobachevskian cases. Making use of the variation principle applied to the line integral $I=\int ds$, the speaker showed how the mechanics of the Keplerian orbits could be obtained as an interpretation of the geometrical picture. Proceeding from this elegant discussion, the speaker carried the generalization into four dimensions and showed how the mechanics of the Einstein physics came as a natural consequence of the geometrical considerations.

The remainder of the program consisted of the following papers. Due to illness Professor Heath was unable to give his paper, but was represented by George Manning of Franklin College.

1. "A study in Keplerian elliptic motion" by M. Wiles Keller, Indiana University, by invitation.
2. "Functions analogous to Hermite polynomials in the problem of curve fitting" by Professor H. E. H. Greenleaf, De Pauw University.
3. "Language, logic, and mathematics" by Dr. A. F. Bentley, Paoli, Indiana, by invitation.
4. "Maxima and minima of radii of curves" by Professor F. H. Hodge, Purdue University.
5. "Some composite polyhedrons" by Professor D. H. Heath, Franklin College.
6. "A new technique in the analysis of trend lines" by Professor H. T. Davis, Indiana University.
7. "Higher geometry in the college curriculum" by Professor J. E. Dotterer, Manchester College.
8. "Some elementary geometrical applications of group theory" by Professor W. E. Edington, De Pauw University.

Abstracts of the papers follow, the numbers corresponding to the list of titles.

1. In this paper the expression for the angular velocity about a point between the foci in Keplerian elliptic motion was determined and the cubic equation derived, the roots of which give the values for which the angular velocity is a maximum and a minimum. Some of the properties of this cubic, the coefficients of which depend upon two parameters, were given and their relationship to the angular velocity noted.

2. Professor Greenleaf discussed the least-square approximation to data in which the variates are equally spaced and the observed frequencies are given the binomial coefficients as weighting factors. By a method parallel to the Gram-Charlier development of continuous variates in the type-A curve, a set of functions analogous to the Hermite polynomials were found for discrete variates. These functions have a generating function and a recurrence formula; each satisfies a given difference equation of second order, and has an orthogonality property similar to that of the Hermite polynomials. The coefficients to be used with these functions are computed and tabulated, permitting the determination of the least-square equation with a minimum amount of computation.

At the close of the paper Professor E. H. Hildebrandt of De Pauw University discussed these functions showing how they fit into the general system of polynomials connected with the Charlier expansion, using his paper published in the Annals of Mathematical Statistics, November, 1931, as the basis of his discussion.

3. Dr. Bentley undertook to show the manner in which language, logic and mathematics may be inspected as differentiated aspects of that historical field of human behavior, indicated by the word "knowledge" and by various associated terms. For the investigation of this common field a procedure to be called "Semantic Analysis" was suggested as wider in scope and more powerful than logic. Semantic Analysis in this sense is closely akin to Korzybski's Non-Aristotelian Semantics, but basically different from Chwistek's Semantik. The recent trend towards the reconstruction of logic in the work of Russell, Hilbert, Brouwer, Chwistek, Lukasiewicz, Tarski, Lesniewski and Korzybski was briefly sketched.

4. In testing for the maximum or minimum values of the radius of curvature of the parabola, $y^2=4ax$, differentiation with respect to x fails to indicate the origin as a critical point while differentiation with respect to y does indicate this point. In the case of the ellipse in standard form, differentiation with respect to x indicates one pair of points while differentiation with respect to y indicates another pair. Similar results are found for the hyperbola. The aim of this paper is to point out the apparent exceptions to ordinary rules and to indicate the reasons for these conditions.

5. In this paper models were presented showing families of solids formed by the combinations of the regular convex polyhedrons. Professor Heath presented to the members of the Association pamphlets of patterns for the construction of twelve of these solids, as, for example, the formation of a solid composed of equal octohedrons mounted on each face of an icosahedron.

6. In the analysis of series of economic items, as, for example, the Dow Jones stock market averages, pig iron production, etc., it is desirable to have a technique for fitting trend lines of higher degree than the straight line and also to be able to compute the correlations of the deviations of these series from their trend lines. By means of simple formulas involving quadratic forms of the moments of the series it is possible to solve this problem in the sense of least squares. Explicit formulas for the coefficients of the forms in terms of the number of items and tables of their values are given for polynomials from the first to the seventh degrees inclusive. The speaker indicated the application of these formulas in an elaborate computation of economic constants which is being made by the Cowles Commission for Research in Economics of Colorado Springs, an affiliate of the Econometric Society.

7. In this paper Professor Dotterer presented the case of higher geometry in the college curriculum. He indicated the scope of this subject which seems desirable for presentation to undergraduates and pointed out the many contacts which this study makes with other mathematical disciplines.

8. In this paper Professor Edington discussed the results obtained by permuting the coefficients of such equations as $y=ax^2+bx+c$, $x^2+y^2+ax+by+c=0$, $ax+by+c=0$, and indicated the generalizations and some of the geometrical relations associated with group concepts. It was also pointed out that Veronese in 1881 had discovered that the permutations of the homogeneous coordinates of a point in a plane gave six points which are the vertices of a Pascal hexagon, and Professor Edington showed that the plane is divided up into regions such that if a point is taken from a given region, the kind of conic determined by the six points is always of the same kind. The extension to space was also indicated.

Resolutions were adopted by the members of the section expressing their appreciation and thanks to the authorities of Butler University for their hospitality, to Professor Bliss, Professor Pollock, and Professor Lanczos for their contributions to the program, and to the Mathematics Section of the State Teachers' Association and the Extension Division of Indiana University for their efforts in stimulating mathematical study through the recently inaugurated state wide contest in mathematics for high school study.

H. T. Davis, *Secretary-Treasurer*

THE THIRTEENTH ANNUAL MEETING OF THE ILLINOIS SECTION

The thirteenth annual meeting of the Illinois Section of the Mathematical Association of America was held at the University of Illinois, on Friday and Saturday, May 6 and 7, 1932.

The attendance was about ninety, including the following forty-five members of the Association: Beulah M. Armstrong, Edith I. Atkin, H. W. Bailey,

G. A. Baker, R. W. Barnard, Walter Bartky, H. R. Beveridge, Julia W. Bower, O. K. Bower, R. D. Carmichael, A. B. Coble, C. E. Comstock, A. R. Crathorne, H. B. Curry, D. R. Curtiss, Arnold Emch, Elinor B. Flagg, H. L. Garabedian, A. E. Gault, R. M. Ginnings, H. W. Haggard, Mildred Hunt, E. C. Kiefer, J. M. Kinney, W. C. Krathwohl, Luise Lange, Harry Levy, Mrs. Mayme I. Logsdon, E. B. Lytle, W. D. MacMillan, E. J. McShane, H. J. Miles, G. A. Miller, C. N. Mills, G. E. Moore, E. J. Moulton, J. W. Peters, W. T. Reid, C. K. Robbins, H. A. Simmons, H. E. Slaught, C. J. Stowell, E. H. Taylor, V. B. Teach, E. J. Townsend.

The Section chairman, Professor R. W. Barnard, presided at both the Friday afternoon and Saturday morning sessions.

At the business session officers for 1932–1933 were elected as follows: Chairman, W. C. Krathwohl, Armour Institute of Technology; vice-Chairman, E. B. Miller, Illinois College; Secretary-Treasurer, C. N. Mills, Normal University; Assistant Secretary-Treasurer, Edith I. Atkin, Normal University.

The program consisted of eight papers, as follows:

1. "An elementary development of Riemann's geometry with applications to relativity" by Professor Cornelius Lanczos, Purdue University, by invitation.
2. "A method of Prüfer for differential systems of the second order" by Doctor W. T. Reid, University of Chicago.
3. "The Hamilton-Jacobi theory for problems in parametric form" by Professor V. B. Teach, Armour Institute.
4. "On mathematical models and graphs," accompanied by lantern slides and exhibit of models, by Professor Arnold Emch, University of Illinois.
5. "Recent discoveries in the history of mathematics" by Professor G. A. Miller, University of Illinois.
6. "Permanent configurations in the four-body problem" by Professor Walter Bartky, University of Chicago.
7. "A problem of Klein concerning the real roots of an algebraic equation" by Doctor I. J. Schoenberg, University of Chicago, by invitation.
8. "The transformations of non-normal frequency distributions into normal distributions" by Professor G. A. Baker, Shurtleff College.

Abstracts of some of these papers follow:

1. The author wishes to sketch how far we can come in understanding the general principles of the Riemannian geometry, and its applications to physical problems, if we utilize only elementary knowledge, and do not introduce the general tensor-calculus. After bringing in the fundamental line-element of Riemann, and defining the straight line as the shortest line, he demonstrates as a first application the parallel development of plane, spherical, and hyperbolic geometry based upon the corresponding line elements, and there follows the deduction of the centrifugal and Coriolis force, with the help of the line-element as an example for "apparent-forces" in mechanics. The equivalence

hypothesis of Einstein and its first attempt to solve the problem of gravitation can also be included in this elementary development, obtaining finally a good foundation in order to study the higher chapters of Riemannian geometry and of relativity.

2. In this paper the method introduced by Prüfer (Mathematische Annalen, Vol. 95 (1926), pp. 499–518) is applied to a system of two ordinary linear homogeneous differential equations of the first order; and properties of the solutions of such a system, including comparison and oscillation theorems, are discussed.

3. A classical formulation of the Hamilton principle is given and the Lagrange differential equations defining the motion of a dynamical system derived and expressed in canonical form. Next a brief discussion is given on the theory relating to the partial differential equation of Hamilton and Jacobi, and on the Jacobi theorem on the determination of solutions of the Lagrange equations from a complete integral of this partial differential equation. Professor Teach then shows how the Hamilton-Jacobi theory may be extended, and how analogous results may be obtained when parametric representation is used.

6. A permanent configuration is a configuration of n-particles moving under their mutual gravitational attraction in such a way that ratios of mutual distances are not changed. The only permanent configurations in the problem of three bodies are the straight line and equilateral triangle solutions of Lagrange. A generalization of the straight line configuration to n-bodies is given in F. R. Moulton's Periodic Orbits. In this paper all remaining permanent configurations in the problem of four bodies are found.

7. In a paper entitled *Geometrisches zur Abzählung der reellen Wurzeln algebraischer Gleichungen*, Collected Works, vol. 2, pp. 198–208, Felix Klein initiated the problem to compare the efficiency of the various methods (Budan-Fourier, Descartes-Jacobi, Newton-Sylvester) which give upper limits for the number of real roots of an algebraic equation within a given interval. The method used by Klein is geometrical and gives definite results in case the discussion involves the consideration of spaces of two or three dimensions only. The method of the present paper is analytical and is based on some general results given by the author in his paper *Über variationsvermindernde lineare Transformationen*, Mathematische Zeitschrift, vol. 32 (1930), pp. 321–328. Klein has shown in particular that for equations of the second degree the method of Descartes-Jacobi is always more efficient than the method of Budan-Fourier (See H. Weber, *Lehrbuch der Algebra*, vol. 1, second edition, 111, pp. 354–357). H. Weber (loc. cit. p. 357) asks if this is true for any degree. In the present paper it is first proved that for equations of any degree the method of Descartes-Jacobi is always more efficient than a method given by Laguerre (Oeuvres, vol. 1, Paris 1898, p. 10). The comparison of the former method with the method of Budan-Fourier is reduced for a given degree n to the discussion of the signs of the minors of a particular matrix of $n+1$ rows and $2n+2$ columns whose elements are binomial coefficients. For a given degree this discussion will answer Weber's question by a finite number of trials.

8. By assuming that the transforming function necessary to transform a given non-normal distribution into a normal distribution can be represented by a Maclaurin's expansion, Professor Baker shows how to determine the coefficients of the transforming function in terms of the coefficients of the expansion of the given distribution. The transformation necessary to transform a given non-normal distribution into a normal distribution serves to specify the distributions in random samples of the estimates of the parameters of the non-normal distribution in terms of the distributions in random samples of the mean and standard deviation of the resulting normal distribution. Thus it is possible to determine the distributions in random samples of the parameters of the non-normal population.

C. N. MILLS, *Secretary*

CONFORMAL REPRESENTATION, WITH APPLICATIONS TO PROBLEMS OF APPLIED MATHEMATICS[1]

By WARREN WEAVER, University of Wisconsin

I. INTRODUCTION TO CONFORMAL REPRESENTATION

The history of the development of mathematics is intimately connected with the history of the development of the number concept. One of the most significant and fruitful of the various generalizations which the number concept has experienced is that which led to "imaginary" and "complex" numbers. Since the product of any real number, positive or negative, by itself is positive, it is clear that there is no ordinary or real number whose square is negative; and hence it was natural to call "imaginary" the square root of a negative number. Since, moreover, the customary laws of algebra permit one to write, for any number a,

$$\sqrt{-a^2} = \sqrt{(-1)a^2} = \sqrt{-1}\,a,$$

it is clear that the pure imaginary unit $\sqrt{-1} \equiv i$ can, so to speak, be made to bear the entire brunt of "imaginariness," and the square root of any negative number can be written as the product of the imaginary unit and a real number.

It is often convenient, when multiplying one real number a by a second positive real number k to give, on geometrical grounds, an operational interpretation to the product ka; and say that k is an operator which stretches a in the ratio of k to 1. If the stretching operator be negative, and say equal to $-k$, one naturally says that $-k=(-1)k$ carries out two operations, the first of which is a stretch in the ratio of k to 1, while the second is a reversal of direction. This reversal of direction may itself conveniently be viewed as a rotation through 180°, so that -1 is an operator which leaves lengths unchanged but which rotates through $+180°$. (See Figure 1.)

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[1] A lecture delivered by invitation at the S.P.E.E. Summer Session for Teachers of Mathematics to Engineering Students at Minneapolis, September, 1931.

If one seeks, similarly, to give an operational interpretation to the process of multiplying a real number b by $i\equiv\sqrt{-1}$, he receives a useful suggestion from the fact that a double application of the operation of multiplying by i is equivalent to the operation of multiplying by -1. It clearly is consistent with this fact to agree to treat $i\equiv\sqrt{-1}$ as an operator which rotates through half of 180° or 90°.

A mixed number $a+ib$ which consists of the sum of a real portion a and a pure imaginary portion ib is called a "complex" number, and the operational interpretation just suggested indicates the manner in which one may give a convenient geometrical interpretation to such a complex number. (See Figure 1.) When complex numbers are represented in this familiar way, the figure is called a Gauss-Argand diagram. It is clear, from such a diagram, that complex numbers add vectorially; that two complex numbers are equal if and only if their real and imaginary parts are separately equal. It is also clear that any

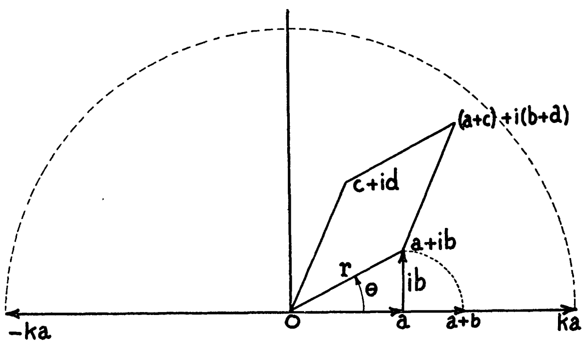

FIGURE 1

complex number $a+ib$ can be written in the equivalent form $r(\cos\theta+i\sin\theta)\equiv re^{i\theta}$. This latter expression is called the polar form of the complex number, r and θ being known as, respectively, the modulus and the amplitude of $a+ib$. The modulus, or "absolute value" of a complex number z is often represented by $|z|$.

It is readily argued, using the polar form, that the product (or quotient) of two complex numbers is a complex number whose modulus is the product (or quotient) of the moduli of the factors and whose amplitude is the sum (or difference) of the amplitudes of the factors.

A complex quantity whose real and pure imaginary parts are variable is called a complex variable, and may be denoted by $z=x+iy$ or by $w=u+iv$. A functional relationship $w=f(z)$ may exist between two such complex variables, so that to each chosen value of z there corresponds a value of w. Each of the complex quantities w and z can be represented on an Argand diagram, so that one speaks of the w-plane and the z-plane. The functional relation sets up a

correspondence between the points of the z-plane and points of the w-plane. Thus when the variable z wanders, in its plane, over a curve C the corresponding point w describes a curve C' in the w-plane. One of these curves is said to be the "map" of the other; and in this way a functional relationship between w and z maps the z-plane onto the w-plane.

If one seeks to calculate, for this situation, a quantity analogous to the derivative of a function of a real variable, significant new considerations enter. One naturally chooses, for the definition of the derivative,

$$\frac{dw}{dz} = \lim_{\Delta z \doteq 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \doteq 0} \frac{f(z + \Delta z) - f(z)}{\Delta z},$$

and the new feature arises from the fact that $\Delta z = \Delta x + i\Delta y$ has, geometrically speaking, two degrees of freedom and can approach zero along any path such as that shown in Figure 2, while Δx, in the simple previous case, was restricted to linear variation. In order that the derivative of w with respect to z have a

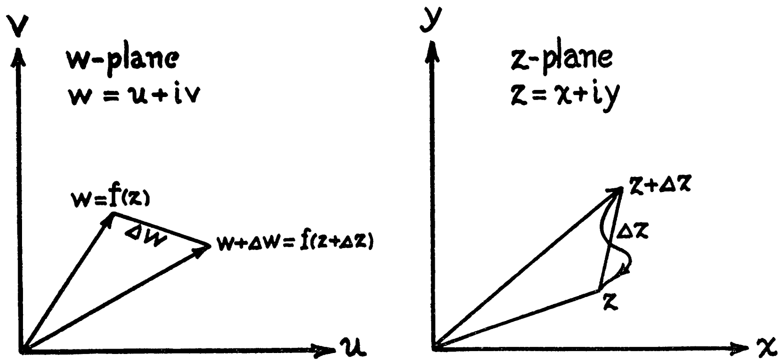

FIGURE 2

unique value for a given argument, it is necessary to demand that the limit of the above difference quotient be independent of the manner in which Δz approach zero. If one assume, for the moment, that $w = u + iv = f(z)$ have a unique derivative with respect to $z = x + iy$ one has

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{dw}{dz}\frac{\partial z}{\partial x} = \frac{dw}{dz},$$

while

$$\frac{\partial w}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = \frac{dw}{dz}\frac{\partial z}{\partial y} = i\frac{dw}{dz}\cdot$$

Therefore

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right),$$

from which, equating reals and pure imaginaries,

$$\left\{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned}\right. \tag{1}$$

These are known as the Cauchy-Riemann equations, and it can easily be shown that a necessary and sufficient condition that a complex function $w=u+iv$ possess a derivative with respect to $z=x+iy$ is that the partial derivatives of the functions u and v exist, are continuous, and satisfy equations (1). The above calculation proves the necessity of the condition just stated. A function of a complex variable which possesses a derivative at every point of a region is said to be "analytic," "regular," or "holomorphic" over that region.

Suppose now that $w=u+iv$ is, over a certain region, an analytic function of $z=x+iy$. The Cauchy-Riemann equations (1) are then satisfied. If one differentiates the first of these equations partially with respect to x, the second partially with respect to y, and then adds the two resulting equations he obtains

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{2}$$

This partial differential equation is known as the Laplace equation, and it is of very widespread occurrence in problems of pure and applied mathematics. This is the fundamental equation, for example, in problems of electrostatics, of magnetostatics, of steady state current flow, of the irrotational motion of a perfect non-compressible fluid, of elasticity, of sound, and of a wide variety of thermal and optical problems. By interchanging the variables when differentiating equations (1), one obtains

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, \tag{2'}$$

so that *every analytic function automatically furnishes one with a pair of real functions of two real variables each of which is a solution of Laplace's equation.* The function $v(x, y)$ is called the "conjugate" of the function $u(x, y)$, and vice versa; and the process of obtaining solutions of the Laplace equation in this way from analytic functions of a complex variable is known as the "method of conjugate functions."

The actual process of obtaining the pair of conjugate functions from a given analytic function is best indicated by means of an example. Thus, if

$$w = u + iv = \sin z = \sin (x + iy),$$

then

$$u + iv = \sin x \cos iy + \cos x \sin iy,$$
$$= \sin x \cosh y + i \cos x \sinh y;$$

so that

$$u(x, y) = \sin x \cosh y,$$
$$v(x, y) = \cos x \sinh y.$$

In general, the relations

$$(3) \qquad u = u(x, y), \quad v = v(x, y),$$

obtained by breaking up into real and pure imaginary parts an analytic function $w=u+iv=f(z)=f(x+iy)$, can be solved for x and y to obtain the equations

$$(3') \qquad x = x(u, v), \quad y = y(u, v).$$

These new functions x and y are also solutions of the Laplace equation, considering u and v as the independent variables. Thus, strictly speaking, each analytic function furnishes one with *four* solutions of Laplace's equation, but

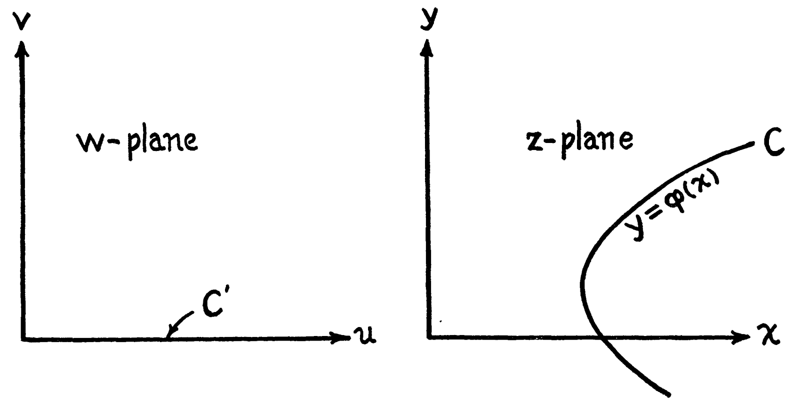

FIGURE 3

very seldom are all four sufficiently simple to be useful. The solution of (3) to produce (3′) is always theoretically possible, the Cauchy-Riemann conditions for analyticity assuring that the requirements for the possibility of such solution are fulfilled.

Specific examples of the method of conjugate functions will be given later, but it may be well to indicate here two general sorts of problems. Suppose that an analytic function $w=u+iv=f(z)=f(x+iy)$ maps a curve C of the z-plane

(see Figure 3), whose equation is $y=\phi(x)$ onto the entire real axis $v=0$ of the w-plane. This will obviously occur if and only if

$$v(x, \phi(x)) \equiv 0.$$

Then the function

$$\Phi(x, y) \equiv v(x, y),$$

clearly is a solution of Laplace's equation which reduces to zero on the curve C. In an important class of problems of applied mathematics one requires a solution of Laplace's equation which reduces to zero, or some other constant, on some given curve. Thus, one may, so to speak, go at such problems backward; and by plotting in the z-plane the curves $u(x, y)=$constant and $v(x, y)=$constant, he finds for what curves C a given analytic function solves the above problem. Similarly, one may interchange the roles of u, v and x, y and may plot in the w-plane the curves $x(u, v)=$constant and $y(u, v)=$constant. Thus a properly drawn picture of the plane transformation indicates to the eye what problems, of this sort, are solved by a given analytic function. It must be emphasized that the picture must be "properly drawn"; that is, one requires, in one plane, the two families of curves obtained by setting equal to various constants the coordinate variables of the other plane.

In a second and more general sort of problem it is necessary to obtain a solution $\Phi(x, y)$ of Laplace's equation which, on a given curve C whose equation is $y=\phi(x)$, reduces to some given function $\Phi^*(x, y)$. The previous problem is clearly a very special case of this second problem. Suppose now that an analytic function $w=f(z)$ map the curve C of the z-plane onto the axis of reals $v=0$, of the w-plane. Since the curve C maps onto $v=0$ in the w-plane, $v(x, \phi(x))\equiv 0$, and the values of Φ^* at points on C are equal to the values of

$$\Phi^*(x(u, 0), y(u, 0)) \equiv \Phi_*(u)$$

at the corresponding points on the transformed curve $v=0$. Suppose now that the function $\Psi(u, v)$ be a solution of Laplace's equation (viewing u and v as independent variables), such that

$$\Psi(u, 0) \equiv \Phi_*(u).$$

It is easily checked that

$$\Phi(x, y) = \Psi(u(x, y), v(x, y))$$

is a solution of Laplace's equation, viewing x and y as independent variables. Moreover, on the curve C one has

$$\Phi(x, \phi(x)) = \Psi(u, 0) = \Phi_*(u) = \Phi^*(x, y),$$

so that Φ is the solution sought.

The chief service, in this case, of the method of conjugate functions, is that the form of the boundary condition is much simplified. Rather than seeking a

function which takes on prescribed values on some *curve* C, one has rather to find a function which takes on prescribed values on a *straight line*, namely the axis of abscissas. This latter problem is so much simpler than the former that it can, indeed, be solved in general form for a very general function Φ_*. This solution will be referred to later, in III (C).

The foregoing discussion has briefly indicated the way in which an analytic function $w=f(z)$ maps the z-plane onto the w-plane. A characteristic and important feature of this mapping has not been mentioned as yet, and is this: If one consider two curves C_1 and C_2, in the z-plane, which intersect at a point P (See Figure 4), then it may easily be shown that, if the mapping function $w=f(z)$ be analytic, the angle of intersection τ of these two curves is equal to the angle τ' at which the transformed curves C_1' and C_2' intersect, in the w-plane, at the transformed point P'. This fact can conveniently be referred to by saying that the map is "angle-true." Mathematicians, however, refer to this

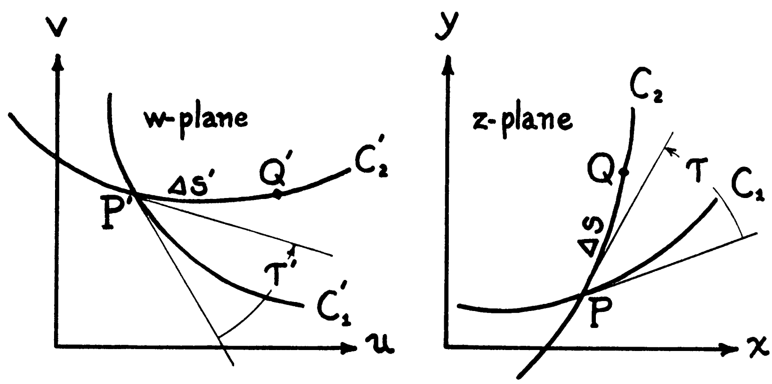

FIGURE 4

characteristic feature by saying that the map is "conformal," and one speaks of the conformal representation of one plane on the other.

As an immediate and important application of the fact that angles are preserved in the conformal map furnished by an analytic function, one may note that the curves $u=$constant of the z-plane are everywhere orthogonal to the curves $v=$constant, merely because the curves $u=$constant and $v=$constant are, in the w-plane, straight lines which intersect orthogonally. This orthogonality property is often very important in the application of conformal representation to problems of applied mathematics. If, for example, $u(x, y)$ represent the potential function in an electrical problem, then the orthogonal family of curves $v(x, y)=$constant gives the lines of force (and vice versa). Other examples of this will be mentioned later.

If one consider an increment of arc length $\Delta s=\overline{PQ}$ (Figure 4) along one of the curves in the z-plane, this transforms into $\Delta s'=\overline{P'Q'}$ in the w-plane; and it

follows at once from the uniqueness of the derivative that, regardless of the direction of the curve in question,

$$\lim \frac{\Delta s'}{\Delta s} = \left| \frac{dw}{dz} \right|.$$

Thus the region immediately around the point P of the z-plane experiences, when it is mapped on the w-plane, a uniform radial stretch whose magnitude (either greater or less than unity) is the absolute magnitude of the derivative, at that point, of w with respect to z. For this reason the quantity $|dw/dz|$ is called the "modulus of the transformation."

The above remarks have suggested the practical importance of being able to construct or determine an analytic function which will conformally map a given region of the z-plane onto some standard simple region (such as the interior of the unit circle or such as the upper half plane) of the w-plane. The fundamental theorem of conformal representation, first treated by Riemann in his doctoral dissertation, asserts that there always exist analytic functions which will accomplish this result, there being, in the simple forms of the theorem, mild restrictions on the nature of the original region of the z-plane. This fundamental theorem has been widely generalized, and has received much recent study.

II. EXAMPLES OF CONFORMAL MAPS

As a preparation for the consideration of applications, this section will present six specific instances of the conformal mapping of one plane on another. The examples chosen are not precisely those which one would select if, building up from the simplest cases, one were to study the mathematical theory in detail. The examples are chosen for their characteristic features, and because of their important and direct applications. The first case is:

(A) The transformation $w = z^m$, m a positive integer.

If one write both z and w in polar form, so that,

$$z = re^{i\phi},$$
$$w = Re^{i\Phi},$$

then

$$w \equiv Re^{i\Phi} = z^m = r^m e^{im\phi},$$

and

$$R = r^m,$$
$$\Phi = m\phi.$$

Thus the curve $r = \text{constant}$ in the z-plane (that is to say, a circle about the origin) transforms into a curve $R = \text{constant}$ in the w-plane (also a circle about the origin), the radius of the circle in the w-plane being equal to the mth power of the radius of the circle in the z-plane. Also, a radial line $\phi = \text{constant}$ in the z-

plane transforms into a new radial line $\Phi = \text{constant}$, the amplitude angle for the transformed radial line being m times the amplitude angle of the original radial line. Thus a sector of the z-plane of central angle $2\pi/m$ is "fanned out" to cover the entire w-plane, this sector also being stretched radially. (See Figure 5, drawn for $m=3$.) One notes the characteristic feature that a set of orthogonal curves in one plane transform into a set of orthogonal curves in the other plane.

This example suggests several interesting questions which cannot be discussed here. The "angle-true" property clearly does not hold at the origin, which indicates that this point deserves special study. Further, it is clear that only a portion of the z-plane maps onto the entire w-plane. In the case for which the figure is drawn it would require *three* w-planes, so to speak, if the entire z-plane were to be unambiguously mapped. This consideration leads to the use of

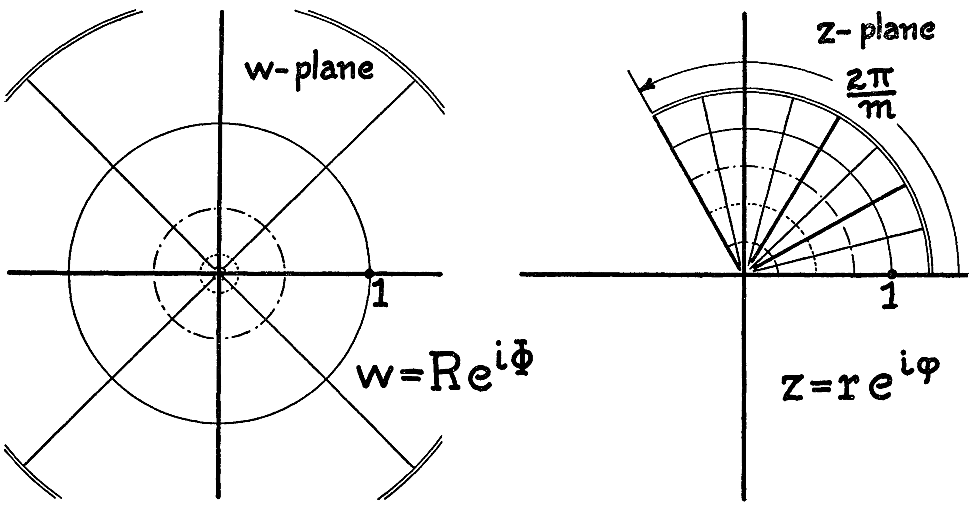

FIGURE 5
The Transformation $w=z^m (m=3)$

many-sheeted surfaces, called Riemann surfaces. Such questions and apparent difficulties correctly indicate that a thorough knowledge of the mathematical theory of analytic functions is essential to a proper and complete understanding of even simple instances of conformal representation.[1]

To get a clear idea of the way in which the z-plane maps onto the w-plane one may choose various convenient families of curves in one plane, and determine the corresponding curves in the other plane. The resulting picture, as was mentioned earlier, does not give any indication of the immediate physical applications of the transformation in question unless one of the sets of curves, in one plane or the other, consists of the straight lines parallel to the coordinate

[1] Bieberbach, L., *Einführung in die Konforme Abbildung*, Berlin, 1927: Lewent, L., *Conformal Representation*, London, 1925: Osgood, W. F., *Lehrbuch der Funktionentheorie*, vol. 1, ch. 14.

axes. It should thus be clear that Figure 5 does not give a direct indication of the type of problem immediately solvable by the transformation $w=z^3$. The curves in the w-plane obtained by setting $x=$constant and $y=$constant are, in fact, cubic curves; and no simple physical problem is directly solved by this transformation. This transformation may, however, be used to solve various physical problems for a wedge-shaped region, since the bounding curve C of such a wedge (say the line $\phi=0$ and the line $\phi=\pi/3$) is transformed into a curve C' of the w-plane which consists of the entire real axis. Thus the transformation can be used, in the way indicated in (I), to solve problems in which one desires a solution of the Laplace equation which reduces to a given function (or a constant) on the boundary of a wedge.

(*B*) *The transformation* $w=(z+1/z)/2$.

This again is a transformation which does not have immediate applicability. It has, however, interesting features, and subsequent discussion will indicate how it may be made to serve a practical purpose.

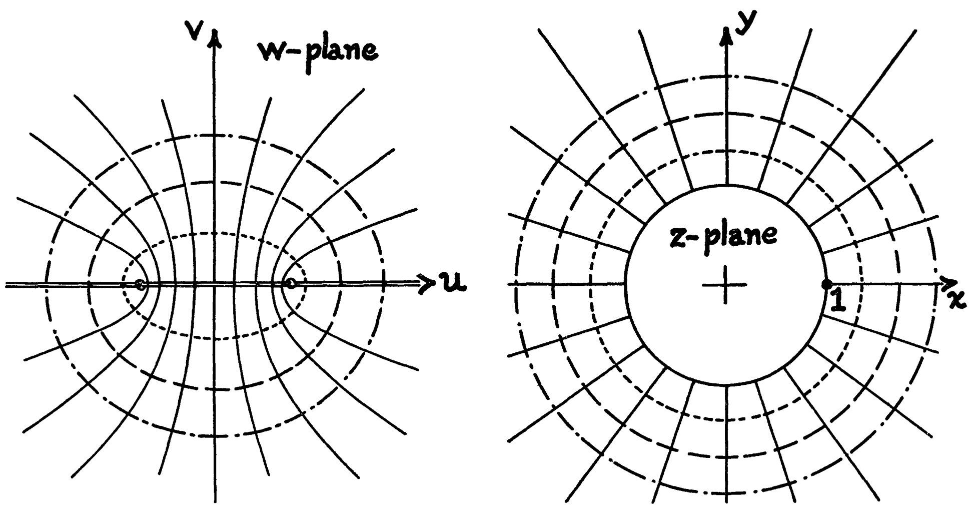

FIGURE 6
The Transformation $w=(z+1/z)/2$

If, as before, one write z in the polar form $re^{i\Phi}$, then

$$w = u + iv = (z + 1/z)/2 = re^{i\phi}/2 + e^{-i\phi}/2r,$$
$$= \tfrac{1}{2}(r + 1/r)\cos\phi + i\tfrac{1}{2}(r - 1/r)\sin\phi,$$

so that

$$u = \tfrac{1}{2}(r + 1/r)\cos\phi,$$
$$v = \tfrac{1}{2}(r - 1/r)\sin\phi.$$

Thus, eliminating ϕ and r in turn,

$$\frac{u^2}{(r+1/r)^2}+\frac{v^2}{(r-1/r)^2}=\tfrac{1}{4},$$

$$\frac{u^2}{\cos^2\phi}-\frac{v^2}{\sin^2\phi}=1.$$

From these equations it follows by inspection that the circles $r=$constant of the z-plane transform into a family of ellipses of the w-plane (See Figure 6), the ellipses being confocal since

$$(r+1/r)^2-(r-1/r)^2=4=\text{constant}.$$

It is also clear that two circles of reciprocal radii transform into the same ellipse. Similarly, the radial lines $\phi=$constant of the z-plane transform into a family of hyperbolas which, again, are confocal since

$$\cos^2\phi+\sin^2\phi=1=\text{constant}.$$

Thus the exterior of the unit circle of the z-plane transforms into the entire w-plane. The unit circle itself "flattens out" to form the segment from -1 to $+1$ of the real axis of the w-plane. All larger circles are less strenuously "flattened out" and form ellipses, while the radial lines of the z-plane form the associated confocal hyperbolas of the w-plane. A similar statement can be made for the inside of the unit circle.

(*C*) *The transformation* $w=e^z$.

If one set $w\equiv Re^{i\Phi}$ and $z=x+iy$, then

$$Re^{i\Phi}=e^{x+iy}=e^x\cdot e^{iy},$$

so that

$$R=e^x,$$

$$\Phi=y.$$

It is thus clear that vertical lines of the z-plane map into circles of the w-plane, the radius being greater or less than one, according as x is positive or negative. Horizontal lines of the z-plane, on the other hand, map into the radial lines of the w-plane, and it is clear that any horizontal strip of the z-plane of height 2π will cover the entire w-plane once. (See Figure 7.)

The curves in the w-plane of Figure 7 are drawn by setting equal to a constant one or the other of the coordinates of the z-plane. Thus these curves give direct indication of physical problems to which this analytic function may be applied. For example, one could obtain the electrostatic field due to a charged right circular cylinder, the lines of flow from a single line source of current or liquid, the circulation of a liquid around a cylindrical obstacle, etc.

By considering this example in conjunction with the preceding example, one gives new significance to Figure 6. In fact, if one start with the z-plane of Figure

7, and then use the w-plane of Figure 7 as the z-plane of Figure 6, it is clear that the curves drawn in the w-plane of Figure 6 then are obtainable by setting equal to various constants the coordinates of the z-plane of Figure 7. That is to

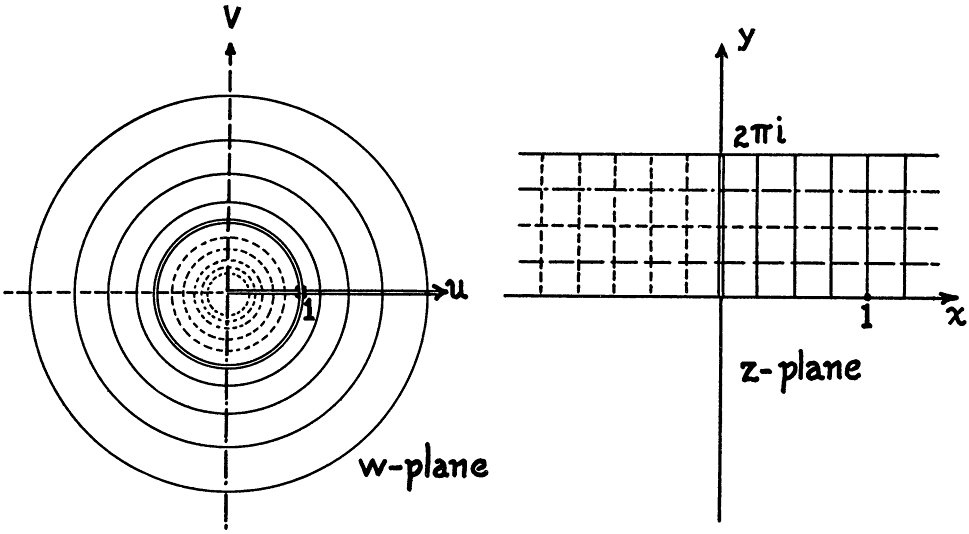

FIGURE 7
The Transformation $w = e^z$

say, the w-plane curves of Figure 6 give direct evidence of physical problems which can be solved by the pair of transformations

$$z_1 = e^z$$
$$w = (z_1 + 1/z_1)/2.$$

(*D*) *The transformation* $w = \cosh z$.

If, in the two preceding equations, one eliminate the intermediate variable z_1 (so he may pass directly from the z-plane of Figure 7 to the w-plane of Figure 6) the result is

$$w = (e^z + e^{-z})/2 = \cosh z.$$

Thus

$$u + iv = \cosh(x + iy) = \cosh x \cosh iy + \sinh x \sinh iy,$$
$$= \cosh x \cos y + i \sinh x \sin y,$$

so that

$$u = \cosh x \cos y,$$
$$v = \sinh x \sin y,$$

or

$$\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 x} = 1,$$

$$\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1.$$

This transformation is shown in Figure 8, and it may be used to obtain the electrostatic field due to an elliptic cylinder, the electrostatic field due to a

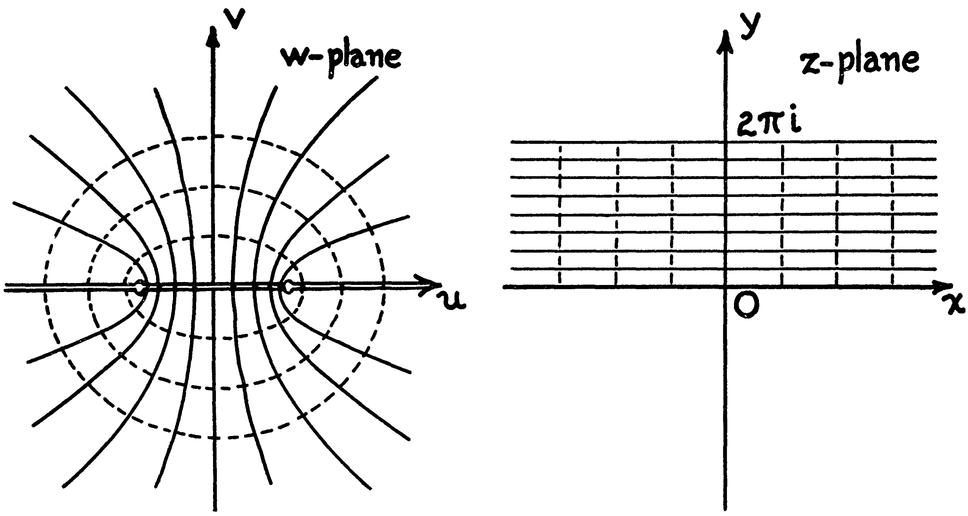

FIGURE 8
The Transformation $w = \cosh z$

charged plane from which a strip has been removed, the circulation of liquid around an elliptical cylinder, the flow of liquid through a slit in a plane, etc.

The transformation from the z-plane to the w-plane may be described geometrically as follows: Consider the horizontal strip of the z-plane between the lines $y=0$ and $y=\pi$; and think of these lines as being broken and pivoted at the points where $x=0$. Rotate the strip 90° counterclockwise, and at the same time fold each of the broken lines $y=0$ and $y=\pi$ back on itself, the strip thus being doubly "fanned out" so as to cover the entire w-plane.

(E) *The transformation* $w=z+e^z$.

One has

$$u + iv = x + iy + e^{x+iy},$$
$$= x + iy + e^x(\cos y + i \sin y),$$

so that

$$u = x + e^x \cos y,$$
$$v = y + e^x \sin y.$$

This transformation is shown in Figure 9. If one considers the portion of the z-plane between the lines $y=\pm\pi$, then the portion of the strip to the right of $x=-1$ is to be "fanned out" by rotating the portion of $y=+1$ (to the right of $x=-1$) counterclockwise and the portion of $y=-1$ (to the right of $x=-1$)

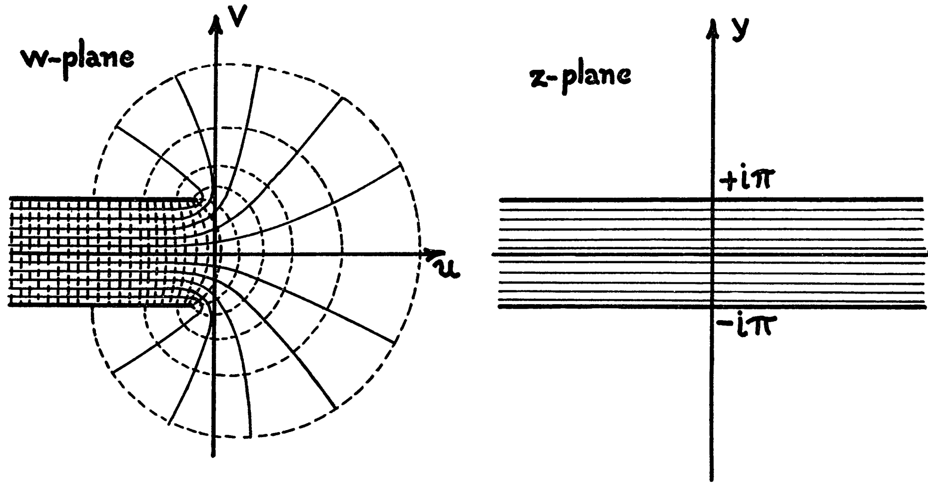

FIGURE 9
The Transformation $w=z+e^z$

clockwise until each line is folded back on itself. This transformation gives the electrostatic field at the edge of a parallel plate condenser, the flow of liquid out of a channel into an open sea, etc.

(*F*) *The Schwartz transformation.*

The transformations just considered are simple examples, and are necessarily very special in character. This list of illustrations will be concluded by a more general transformation. Suppose one have (See Figure 10) a rectilinear polygon, in the w-plane, whose sides change direction by an angle $\alpha_i\pi$ when one passes the ith vertex, going around the boundary of the polygon so that the interior lies to the left. The interior of this polygon can be mapped onto the upper half z-plane by the transformation

$$w = A\int \frac{dz}{(z-z_1)^{\alpha_1}(z-z_2)^{\alpha_2}\cdots(z-z_n)^{\alpha_n}} + B,$$

where $z_1, z_2, \cdots, z_n$ are the (real) points, on the x axis of the z-plane, onto which map the first, second, $\cdots$, nth vertex of the polygon; and where A and B are constants which are to be determined to fit the scale and location of the polygon. Three of the points z_i may be chosen at will, and the values of the remaining ones may be calculated.

This theorem may be used to find, for example, the analytic transformation

which solves the problem of determining the electrostatic field around a charged cylindrical conductor of any polygonal cross-section. It should be noted, however, that one requires for this purpose the function $y(u, v)$ whereas the theorem gives one w as a function of z. It is often exceedingly difficult and laborious to solve this relation for z as a function of w, so that one may obtain the function

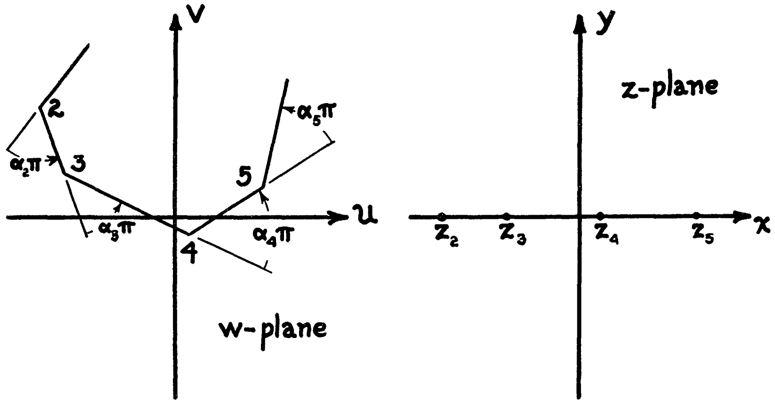

FIGURE 10

y. It should further be remarked that this theorem may be applied to polygons some of whose vertices are not located in the finite plane; and that the theorem is of wide applicability and importance in connections less direct and simple than the one just mentioned.

III. APPLICATIONS OF CONFORMAL REPRESENTATION

(*A*) *Applications to cartography.*

It is natural that a mathematical theory which discusses the "mapping" of one plane on another should have application to the problems connected with the drawing of geographic maps. Since the surface of a sphere cannot be made plane without distortion of some sort, one has to decide, when mapping a portion of the sphere on a plane, what type of distortion to choose and what to avoid. For some purposes it is essential that areas be represented properly; for other purposes it is most important that the angles on the map faithfully represent the actual angles on the sphere.

The first problem, in conveniently mapping a sphere on a plane, is to map the sphere on the plane in some fashion or other, and then, if this fashion be unsatisfactory, to remap this plane onto a second plane. The first problem can be done in a wide variety of ways[1] which includes as important examples, stereographic projection and Mercator's projection. Both of these examples are *conformal* projections, in that they preserve the true values of all angles. Having once mapped the sphere on the plane (or on a portion of the plane) one may now

[1] The Encyclopedia Britannica article on Maps lists and discusses nearly thirty such projections actually used in map making.

remap onto a second plane, and it is here that the theory of conformal representation finds its application; for one can determine the analytic function which will conformally remap the original map onto a new region of any desired shape and size. Not only are all angles preserved in this process of conformal remapping, but the distortion in the neighborhood of a point is always a pure magnification. Thus the shapes of all small objects or regions are preserved. Such maps do not give a true representation of areas, and for this reason many maps are based on compromises between conformal transformations and area-preserving transformations.

(*B*) *Applications to hydrodynamics.*

When the velocities of all particles of a moving liquid lie in planes parallel to one plane which we may conveniently choose as the x, y plane, and when all particles having the same x and y have equal velocities; then the motion is said to be two-dimensional. Such cases clearly arise if a very thin sheet of liquid is flowing in some manner over a plane; or if a thick layer of liquid circulates over a plane, there being no motion and no variation of motion normal to the plane. Let the x and y components of velocity at any point (x, y) be u and v respectively. The motion is said to be *irrotational* if the curl of the velocity vector vanishes. Analytically this demands that

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x},$$

whereas physically it states that the angular velocity of an infinitesimal portion of the liquid is zero. The equation just written assures that

$$-(udx + vdy)$$

is the perfect differential of some function, say Φ. This function is known as the velocity potential, since by a comparison of the two equations

$$d\Phi = -u\,dx - v\,dy,$$

$$d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy,$$

it follows that

$$u = -\frac{\partial\Phi}{\partial x}, \quad v = -\frac{\partial\Phi}{\partial y}.$$

Now if the liquid be incompressible the amount of it which flows into any volume in a given time must equal the amount which flows out. This demand imposes on the components of velocity the restriction that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

this being known as the "equation of continuity." From the last two equations it follows that

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \equiv \nabla^2 \Phi = 0.$$

That is, the velocity potential satisfies the Laplace equation.

Just as the vanishing of the curl of the velocity demands that $u\,dx+v\,dy$ be an exact differential, so the equation of continuity demands that $v\,dx-u\,dy$ be an exact differential of some function, say ψ. That is,

$$d\Psi = v\,dx - u\,dy,$$

$$d\Psi = \frac{\partial \Psi}{\partial x}\,dx + \frac{\partial \Psi}{\partial y}\,dy,$$

so that

(5) $$v = \frac{\partial \Psi}{\partial x}, \quad u = -\frac{\partial \Psi}{\partial y}.$$

From (4) and (5) it follows at once that

$$\frac{\partial \Phi}{\partial x}\frac{\partial \Psi}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial \Psi}{\partial y} = 0,$$

which expresses the geometrical fact that the curves Φ = constant and ψ = constant intersect everywhere orthogonally. It is clear from (4) that there is no component of velocity in the direction of the curves on which Φ is a constant, so that the velocity of the liquid is everywhere orthogonal to the equi-potential curves Φ = constant. That is, the curves ψ = constant depict everywhere the direction of flow. For this reason ψ is called the stream function and the curves ψ = constant are called the stream lines. From (5) and the vanishing of the curl of the velocity it follows that the stream function ψ is also a solution of the Laplace equation.

Thus the velocity potential Φ and the stream function Ψ in the case of the irrotational flow of a perfect incompressible liquid both satisfy the Laplace equation; while the curves Φ = constant and ψ = constant form two orthogonal families. Every analytic function therefore furnishes the solution to four such problems, the four solutions resulting from the fact that one may choose the pair x, y or the pair u, v as independent variables, and that one may interchange the roles of the potential function and the stream function. Figure 8, for example, indicates two of the four problems solved by the analytic transformation $w = \cosh z$. If one treat u and v as the independent variables, and identifies the (solid) curves $y(u, v)$ = constant in the w-plane with the curves Φ = constant, then the dotted curves $x(u, v) = \psi$ = constant give the stream lines, and one has

solved the problem of the circulation of liquid around an elliptic cylinder. If, however, one set $y(u, v) = \psi$ and $x(u, v) = \Phi$, then the solid curves of the w-plane are the stream lines, and one has solved the problem of the flow of liquid through a slit. The other two problems solved by this same function are to be obtained by drawing, in the z-plane, the curves $u(x, y) = \text{constant}$ and $v(x, y) = \text{constant}$, and identifying ψ and Φ with u and v (and vice versa). The z-plane curves $u = \text{constant}$ and $v = \text{constant}$ are very complicated and do not correspond to any simple or important physical problem, and hence they are not drawn on the figure. In fact, it is usually the case that only two of the possible four problems are sufficiently simple to be of any practical use.

It should be emphasized that it is never sufficient, in obtaining the analytical solution of a definite physical problem, merely to know that certain functions satisfy the Laplace equation. One must also have certain boundary conditions. The graphs shown above disclose to the eye what physical problem has been solved precisely because they show what sort of boundary conditions are satisfied. For example, if the dotted curves of Fig. 8 are stream lines, then the problem solved is the circulation around an elliptical obstacle just because these dotted stream lines satisfy the boundary condition for such a problem; namely, because the flow at any point on the boundary of the obstacle is parallel to the boundary of the obstacle.

It is interesting to note that this same transformation $w = \cosh z$ (or, slightly more generally, $w = a \cosh z$) can be used to solve a hydrodynamic problem of a different sort. When liquid seeps through a porous soil, it is found that the component in any direction of the velocity of the liquid is proportional to the negative pressure gradient in that same direction. Thus, in a problem of two dimensional flow,

$$u = -k\frac{\partial p}{\partial x}, \qquad v = -k\frac{\partial p}{\partial y}\cdot$$

If these values be inserted in the equation of continuity, namely, in the equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

the result is

$$\nabla^2 p \equiv \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0.$$

Suppose, then, one consider the problem of the seepage flow under a gravity dam which rests on material which permits such seepage. One seeks (see Figure 11) a function p which satisfies the Laplace equation, and which satisfies certain boundary conditions on the surface of the ground. Namely, the pressure must be uniform on the surface of the ground upstream from the heel of the dam, and must be zero on the surface of the ground downstream from the toe of the dam.

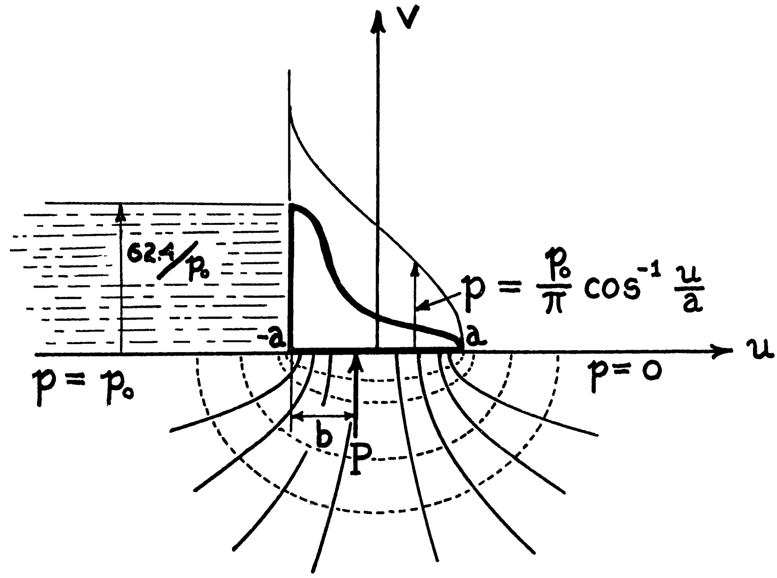

FIGURE 11
Seepage under a dam

If we choose a system of Cartesian coordinates u, v with origin at the mid-point of the base of the dam (Figure 11) and u-axis on the surface of the ground, then it is easily checked that $p(u, v) = p_0 y(u, v)/\pi$, where

$$w = u + iv = a \cosh (x + iy),$$

satisfies the demands of the problem. In fact, it was seen in II(D) where the transformation $w = \cosh z$ was studied, that the line $y = \pi$ of the z-plane folds up to produce the portion to the left of $u = -1$ of the u-axis in the w-plane; while the line $y = 0$ of the z-plane folds up to produce the portion to the right of $u = +1$ of the u-axis. The introduction of the factor a in the transformation merely makes the width of the base of the dam $2a$ rather than 2. These remarks show that $p(u, v)$ reduces to the constant π on the surface of the ground upstream from the heel of the dam. If the head above the dam is such as to produce a hydrostatic pressure p_0, one merely has to set

$$p(u, v) = p_0 y(u, v)/\pi.$$

One may now easily find the distribution of uplift pressure across the base of the dam. In fact, the base of the dam is the representation, in the u, v plane, of the line $x = 0$, $0 \leqq y \leqq \pi$ of the x, y plane. Hence on the base of the dam the equations

$$u = a \cosh x \cos y,$$
$$v = a \sinh x \sin y,$$

reduce to

$$u = a \cos y,$$
$$v = 0,$$

so that

$$p(u, 0) = \frac{p_0}{\pi} \cos^{-1} \frac{u}{a} \cdot$$

This curve is drawn in the figure. The total uplift pressure (per foot of dam)

$$P = \frac{p_0}{\pi} \int_{-a}^{+a} \cos^{-1} \frac{u}{a} \, du = p_0 a$$

which is what the uplift pressure would be if the entire base of the dam were subjected to a head just one-half the head above the dam; or if the pressure decreased uniformly (linearly) from the static head p_0 at the heel to the value zero at the toe. The point of application of the resultant uplift is easily calculated to be at a distance $b = 3a/4$ from the heel of the dam.

(*C*) *Applications to elasticity.*

If opposing couples be applied to the ends of a right cylinder or prism of homogeneous material the cylinder twists and shearing stresses are developed. Choose the axis of the prism for the z-axis of a rectangular system of coordinates. The angle of twist per unit length, say τ, and the shearing stresses, due to an applied couple T, can both be calculated if one can determine a function $\Phi(x, y)$ satisfying the Laplace equation and reducing, on the boundary of a section of the prism, to the function $\Phi^* = (x^2 + y^2)/2$. In fact[1]

$$\tau = T/C,$$

where

$$C = 2G \iint (\Phi - \Phi^*) dx \, dy,$$

in which G is the modulus of rigidity of the material; whereas the shearing stresses are given by

$$X_z = G\tau\left(\frac{\partial \Phi}{\partial y} - y\right),$$

$$Y_z = -G\tau\left(\frac{\partial \Phi}{\partial x} - x\right).$$

Exact analytical solutions of this important technical problem have been obtained for several simple sections, notably circular, elliptical, rectangular, and

[1] Love, A. E. H., *Theory of Elasticity*, 3d edition, 1920, pp. 315–333.

triangular.[1] Only very recently[2] the problem was solved for an infinite T section (See Figure 12). From the general discussion given in Section (I) of this paper it is clear that, to solve this latter problem, one requires first an analytic function which will map the boundary of this T section onto the entire real axis of

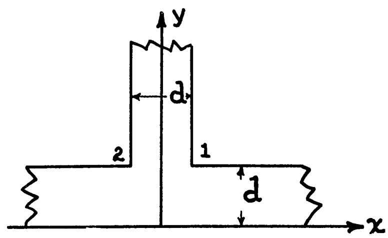

FIGURE 12
Infinite T section

the new w-plane. This section, moreover, is a rectilinear polygon, so that one can use the Schwarz transformation theory to produce the desired analytic relation. One finds that the desired mapping is carried out by the function

$$z = A \int \frac{(w^2 - 1)^{1/2}}{(w^2 - a^2)} dw + B,$$

$$= \frac{di}{\pi} \log [w + \sqrt{w^2 - 1} + \frac{2di}{\pi} \tan^{-1} \frac{\sqrt{w^2 - 1}}{2w} + \frac{d}{2} + id,$$

where the first line is furnished directly by the Schwartz Theorem, and where, in the second line, the constants a, A, B have been evaluated so as to fit the dimensions and location of the T section.

It is next necessary to break z up into its real and imaginary parts so as to obtain x and y as functions of u and v. These values, when substituted into

$$\Phi^* = (x^2 + y^2)/2$$

give, since $v = 0$ on the boundary of the section, the function

$$\Phi^*[x(u, 0), y(u, 0)] \equiv \Phi_*(u).$$

The remaining essential step is to obtain a function $\Psi(u, v)$ satisfying the La-

[1] See *The torsion of members having sections common in aircraft construction*, by W. Trayer and H. W. March, Bureau of Aeronautics Navy Department, Separate Report No. 334; also contained in the 15th Annual Report of the National Advisory Committee for Aeronautics, 1929, pp. 675–719.

[2] *On a solution of Laplace's equation with an application to the torsion problem for a polygon with reentrant angles*, I. S. Sokolnikoff, Transactions of the American Mathematical Society, vol. 33, pp. 719–732.

place equation and reducing, on the axis of reals $v=0$, to the function $\Phi_*(u)$. Such a function is[1]

$$\Psi = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\Phi_*(\xi)\rho \sin\theta}{\rho^2 - 2\xi\rho\cos\theta + \xi^2}\, d\xi,$$

where

$$w = u + iv = \rho e^{i\theta}.$$

The solution to the original problem is then given, as was earlier indicated in Section I, by $\Phi=\Psi$. It is, obviously, a difficult and laborious job to carry out these calculations, but formulas have been obtained, in the paper referred to, from which practical calculations can be made.

(*D*) *Applications to electrostatics.*

The methods of complex variable theory are peculiarly applicable to two-dimensional electrical problems. In order that the problems be two-dimensional we will understand that the conductors under consideration are exceedingly long cylinders whose axes are normal to the $z=x+iy$ plane. Under these circumstances the various electrical quantities do not change appreciably in the direction normal to the z-plane, and one has to determine these quantities as functions of x and y only. In certain problems one or more of the conductors present will have very small cross-sections, and will be given a charge of, say, e' per unit length. Such a conductor will be called a "line charge of strength e'."

The electrostatic problem, for such conductors, is solved when one has obtained a function $\Phi(x, y)$, known as the electrostatic potential, satisfying the following conditions:[2]

(6)

- (*a*) $\nabla^2\Phi=\partial^2\Phi/\partial x^2+\partial^2\Phi/\partial y^2=0$ at all points in free space.
- (*b*) Φ reduces, on the surface of the kth conductor, to a constant Φ_k.
- (*c*) In the neighborhood of a line charge of strength e', Φ becomes infinite as $(-e' \log r)/2\pi$, where r measures distance to the line.
- (*d*) Φ behaves at infinity as $(-\log R\Sigma e')/2\pi$, where $\Sigma e'$ is the total charge per unit length of all conductors present, and where R measures distance from some reference point in the finite plane. In case $\Sigma e'=0$, Φ approaches zero as $1/R$.

It is readily shown, by standard methods, that the solution of such a problem is unique. This remark is of great practical importance, since it assures one that a function Φ satisfying these conditions is, however it may have been obtained, the correct solution of the physical problem.

[1] Sokolnikoff, l.c. This formula is the general solution, spoken of in Section I, of the Laplace equation subject to specified boundary values on the entire axis of abscissas.

[2] See Mason and Weaver, *The Electromagnetic Field*, University of Chicago Press, 1929, pp. 134, 146, and Riemann-Weber, *Die Differentialgleichungen der Physik*, Vieweg & Sohn, 1927, Vol. II, p. 290. The units used in the above discussion are the rational units used in Mason and Weaver, l.c.

Physically one wishes to know the distribution of charge and the electrostatic force at any point. These data may be obtained from the function Φ in the following manner: the component E_s in any direction s of the electrostatic force per unit charge is given in terms of Φ by the relation

$$E_s = -\frac{\partial \Phi}{\partial s};$$

while the surface density of charge η on any conductor is given by

$$\eta = -\frac{\partial \Phi}{\partial n},$$

where n measures distance along the external normal to the conductor in question.

Now if

$$w = u + iv = f(z) = f(x + iy),$$

and if the function

$$\Phi(x, y) = u(x, y)$$

satisfies condition (6), then

$$E_x = -\frac{\partial \Phi}{\partial x} = -\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y},$$

the last step following from the Cauchy-Riemann equations (1) of Section I. Similarly

$$E_y = -\frac{\partial \Phi}{\partial y} = -\frac{\partial u}{\partial y} = +\frac{\partial v}{\partial x}\cdot$$

Thus

$$\begin{aligned} E_x - iE_y &= -\frac{\partial v}{\partial y} + i\frac{\partial u}{\partial y} = i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right), \\ &= i\frac{\partial w}{\partial x} = i\frac{dw}{dz}, \end{aligned}$$

the last step resulting from the fundamental fact that the value of the derivative of an analytic function w is independent of the mode in which z approaches zero. Now the complex number $a-ib$ is called the "conjugate" of the complex number $a+ib$ and one often denotes a conjugate by a bar, thus:

$$a - ib = \overline{a + bi}.$$

Using this standard notation, the "complex electric force" $\mathcal{E} \equiv E_x + iE_y$ is given by

$$\mathcal{E} \equiv E_x + iE_y = i\overline{\frac{dw}{dz}} = -\overline{i\frac{dw}{dz}}, \tag{7}$$

and the magnitude $\sqrt{E_x^2+E_y^2}$ of the electrostatic force at any point is given by

$$\sqrt{E_x^2 + E_y^2} = \left|\frac{dw}{dz}\right|. \tag{8}$$

If one chooses $\Phi \equiv v(x, y)$, then (7) and (8) are replaced by

$$\mathcal{E} = E_x + iE_y = \overline{\frac{dw}{dz}}, \tag{7'}$$

$$\sqrt{E_x^2 + E_y^2} = \left|\frac{dw}{dz}\right|. \tag{8'}$$

Three types of electrostatic problems will now be briefly considered. The first and simplest two dimensional electrostatic problem is that of a single long cylindrical conductor with a given charge. One then seeks a function which satisfies the equation of Laplace and which, in accordance with (6)b, reduces to a constant on the curve which bounds a section of the conductor. This is the analytical problem whose solution was indicated in Section I. One requires a function $w=u+iv=f(z)=f(x+iy)$ such that either a vertical straight line u $=$constant or a horizontal straight line $v=$constant of the w-plane maps into the bounding curve C of the conductor's section in the z-plane. Then $\Phi(x, y)$ $\equiv u(x, y)$ or $\Phi(x, y)\equiv v(x, y)$ solves the problem, and the physically important quantities are given by (7), (8) or by (7′), (8′) respectively.

Secondly, suppose that a single long cylindrical conductor is in the presence of a parallel line charge of strength e'. We suppose the line charge to be outside the conductor. Let C be the bounding curve in the z-plane of a section of the conductor, and let the line charge be located at $z=z_0$. We may conveniently suppose the cylindrical conductor to be grounded, so that we seek a solution of the Laplace equation which reduces to zero on C and which becomes infinite as indicated in (6c) at $z=z_0$. Let $\zeta=f(z)$ transform C onto the entire axis of reals and the exterior of C conformally upon the upper half ζ-plane. Then if

$$w = u + iv = \frac{e'}{2\pi}\log\frac{f(z) - \overline{f(z_0)}}{f(z) - f(z_0)},$$

the function $\Phi\equiv u(x, y)$ is the solution sought. In fact, for values of z sufficiently close to z_0, $f(z)-f(z_0)$ behaves, except for a constant factor, as $(z-z_0)$. Thus, if one write

$$z - z_0 = re'^{i\theta},$$

then for values of z very near to z_0,

$$w = u + iv = \frac{e'}{2\pi}\log\frac{1}{re^{i\theta}} + A = \frac{e'}{2\pi}\log\frac{1}{r} - i\frac{e'\theta}{2\pi} + A,$$

where A remains finite as $z \doteq z_0$. Therefore

$$u = \frac{e'}{2\pi}\log\frac{1}{r} + B,$$

where B remains finite as $z \doteq z_0$. Thus $u(x, y)$ has the proper type of infinity at $z = z_0$. Furthermore, for points z on C, $f(z)$ is on the axis of reals in the ζ-plane, so that the modulus of $f(z) - f(z_0)$ equals the modulus of $f(z) - \overline{f(z_0)}$. Hence the modulus of

$$\frac{f(z) - \overline{f(z_0)}}{f(z) - f(z_0)}$$

is unity. However, since

$$\log \rho e^{i\phi} = \log \rho + i\phi$$

it is clear that the real part of the logarithm of a complex quantity is the logarithm of the modulus of the complex quantity. Since the logarithm of unity is zero it is clear that u vanishes on C. As regards the behavior of $u(x, y)$ at infinity, one notes that u is the logarithm of the ratio of the (real) distances of $\zeta = f(z)$ to $\zeta_0 = \overline{f(z_0)}$ and to $\zeta_0 = f(z_0)$. As z becomes infinite this ratio differs from

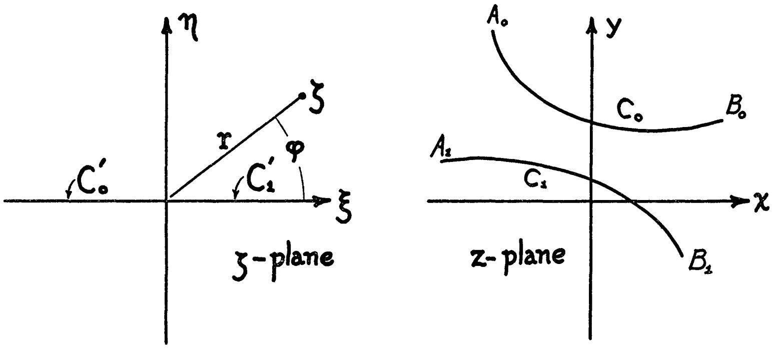

FIGURE 13

unity by an amount whose leading term is equal to or less than a constant times the reciprocal distance from $f(z)$ to one of the points $f(z_0)$ or $\overline{f(z_0)}$. Thus the leading term in the logarithm of this ratio is a constant times this reciprocal distance; and $\Phi = u$ behaves at ∞ in the required manner.

In the third type of problem there are two conductors present, one raised to the potential Φ_0 while the other is at a potential zero. Thus suppose that the cross-section of two long cylindrical conductors consists of two curves C_0 and C_1, such as those shown in Figure 13, which do not intersect at a finite point, but

which, if one takes account of the intersection of B_1 and B_0 and of A_0 and A_1 at $z=\infty$, divide the extended plane into two simply connected regions, one of which may be called the "interior" and the other the "exterior" of the closed curve C_0+C_1. Now suppose that $\zeta=f(z)$ map C_0 onto the entire negative axis of reals in the ζ-plane, with the infinitely distant point along B_0 mapped onto $\zeta=0$; and that $\zeta=f(z)$ also maps C_1 onto the entire positive axis of reals with the infinitely distant point along B_1, mapped onto $\zeta=0$; and that $\zeta=f(z)$ map the interior of C_0+C_1 conformally on the upper half ζ-plane. Then if

$$w = u + iv = \frac{\Phi_0}{\pi}\log f(z),$$

the function $\Phi=v(x, y)$ satisfies $\nabla^2\Phi=0$ at every point in the interior of C_0+C_1; reduces to zero on C_1; and reduces to Φ_0 on C_0. In fact, the imaginary part of the logarithm of a complex number is merely the amplitude of the complex number, and for points on $C_1, f(z)$ has an amplitude of π; while for points on $C_0, f(z)$ has an amplitude of zero. Then

$$\frac{dw}{dz} = \frac{\Phi_0}{\pi}\,\frac{ds/dz}{f(z)}$$

and the electrostatic force is given by (8) and (8′).

This third type of problem is of frequent and important practical occurrence. Many electrical engineering problems which have been solved by this method of conformal representation are referred to in an expository article, devoted largely to the Schwarz transformation, by E. Weber.[1] In an earlier article in the same journal,[2] for instance, the theory of conformal representation is applied to the problem of studying the leakage voltage and the breakdown potential between the high and low potential portions of oil-immersed transformers. The cases studied come under the third type of problem discussed above.

THE SUMMATION OF CERTAIN SERIES BY FOURIER EXPANSIONS

By JOSEPH B. REYNOLDS, Lehigh University

It is the purpose of this paper to obtain, in a simple direct manner, a method of evaluating the sum of the reciprocals of the even powers of the odd integers and the sum of the reciprocals, with alternating signs, of the odd powers of the odd integers.[3] The procedure is to develop a Fourier series, sub-

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[1] E. Weber, Archiv für Elektrotechnik, 18–18, 1926–27, p. 174.

[2] L. Dreyfus, Archiv für Elektrotechnik, 13, 1924, p. 123.

[3] These sums are given, for example, in Knopp's *Theory and Applications of Infinite Series*, English Translation, page 238.

stitute in it a particular value of the variable and thus determine a relation among the sums under discussion. For the even powers the Fourier series is made to fit the curve $y=x^p$ (p a positive odd integer) from $x=0$ to $x=\pi$ and $y=(\pi-x)^p$ from $x=\frac{1}{2}\pi$ to $x=\pi$. For the odd powers the series is made to fit $y=x^p$ from $x=0$ to $x=\pi$. In both cases the particular value of x used is $\frac{1}{2}\pi$.

From the Fourier series

$$f(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots + a_m \sin mx + \cdots, \tag{1}$$

we have for $x=\frac{1}{2}\pi$

$$f(\tfrac{1}{2}\pi) = a_1 - a_3 + a_5 - \cdots + (-1)^{1/2(m-1)} a_m + \cdots, \tag{2}$$

in which m is a positive odd integer.

If the series (1) is made to fit $y=x^p$ from $x=0$ to $x=\frac{1}{2}\pi$ and $y=(\pi-x)^p$ from $x=\frac{1}{2}\pi$ to $x=\pi$ and p is a positive odd integer

$$\begin{aligned}\tfrac{1}{2}\pi a_m &= \int_0^{1/2\pi} x^p \sin mx dx + \int_{1/2\pi}^{\pi} (\pi - x)^p \sin mx dx \\ &= (-1)^{m+1}\left[\frac{p\pi^{p-1}}{2^{p-2}m^2} - \frac{p(p-1)(p-2)\pi^{p-3}}{2^{p-4}m^4}\right. \\ &\qquad \left. + \frac{p(p-1)(p-2)(p-3)(p-4)\pi^{p-5}}{2^{p-6}m^6} - \cdots\right]\end{aligned}$$

Since m is a positive odd integer these values of a_m in (2) give

$$\begin{aligned}\frac{\pi^{p+1}}{2^{p+1}} &= \frac{p\pi^{p-1}}{2^{p-2}} S_2 - \frac{p(p-1)(p-2)\pi^{p-3}}{2^{p-4}} S_4 \\ &\quad + \frac{p(p-1)(p-2)(p-3)(p-4)\pi^{p-5}}{2^{p-6}} S_6 - \cdots \\ &\quad + (-1)^{1/2(n-2)} \frac{p!\pi^{p-n+1}}{(p-n+1)!2^{p-n}} S_n + \cdots\end{aligned}$$

in which

$$S_n = \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m^n}$$

where n is any positive even integer. From this relation, by setting $p=n-1$, we derive

$$\begin{aligned}S_n = (-1)^{1/2(n+2)}&\left[\frac{\pi^n}{2^{n+1}(n-1)!} - \frac{\pi^{n-2}}{2^{n-2}(n-2)!} S_2 + \frac{\pi^{n-4}}{2^{n-4}(n-4)!} S_4 - \cdots \right. \\ &\left. + \frac{(-1)^{1/2n}\pi^2}{2^2(2!)} S_{n-2}\right]\end{aligned}$$

For a chosen value of n, the series of terms in the bracket ends with the one having the subscript $n-2$. Thus, for $n=2$, we have

$$S_2 = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8};$$

for $n=4$ we obtain

$$S_4 = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = -\left[\frac{\pi^4}{32\cdot 6} - \frac{\pi^2}{4\cdot 2}\cdot\frac{\pi^2}{8}\right] = \frac{\pi^4}{96}$$

In this manner we may find

$$S_6 = \pi^6/960, \quad S_8 = 17\pi^8/32(7!), \quad S_{10} = 31\pi^{10}/8(9!), \quad \text{etc.}$$

It is interesting to note that from

$$S_{10} = \frac{31\pi^{10}}{8(9!)} = \frac{1}{1^{10}} + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \cdots,$$

we can determine the value of π correct to six figures by using one term only of the series and the value of log π correct to eight places of decimals by using two terms of the series.

For the other case we make series (1) fit $y=x^p$ from $x=0$ to $x=\pi$, in which p is a positive odd integer. In this case, for m a positive odd integer, in (2)

$$\begin{aligned}\tfrac{1}{2}\pi a_m &= \int_0^\pi x^p \sin mx\,dx \\ &= \frac{\pi^p}{m} - \frac{p(p-1)\pi^{p-2}}{m^3} + \frac{p(p-1)(p-2)(p-3)\pi^{p-4}}{m^5} - \cdots,\end{aligned}$$

and therefore,

$$\begin{aligned}\frac{\pi^{p+1}}{2^{p+1}} = \pi^p S_1 - \frac{p!\pi^{p-2}}{(p-2)!}S_3 + \frac{p!\pi^{p-4}}{(p-4)!}S_5 - \cdots \\ + \frac{(-1)^{(n-1)/2}p!\pi^{p-n+1}}{(p-n+1)!}S_n,\end{aligned}$$

in which

$$S_n = \sum_{m=1,3,\cdots}^{\infty} \frac{(-1)^{(m-1)/2}}{m^n},$$

where m is a positive odd integer. Replacing p by n we derive

$$\begin{aligned}S_n = (-1)^{1/2(n-1)}\Bigg[\frac{\pi^n}{2^{n+1}n!} - \frac{\pi^{n-1}}{n!}S_1 + \frac{\pi^{n-3}}{(n-2)!}S_3 - \frac{\pi^{n-5}}{(n-4)!}S_5 + \cdots \\ + \frac{(-1)^{(n-1)/2}\pi^2}{3!}S_{n-2}\Bigg]\end{aligned}$$

For $n=1$ we have the common series

$$S_1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4};$$

for $n=3$ we obtain

$$S_3 = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = -\left[\frac{\pi^3}{16(6)} - \frac{\pi^2}{6}\cdot\frac{\pi}{4}\right] = \frac{\pi^3}{32}\cdot$$

By this formula we readily find

$$S_5 = 5\pi^5/2^6(4!),\ \ S_7 = 61\pi^7/2^8(6!),\ \ S_9 = 12465\pi^9/2^{10}(9!),\ \text{etc.}$$

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE APPROXIMATE SOLUTION OF CERTAIN EQUATIONS

By RAYMOND GARVER, University of California at Los Angeles

In a recent issue of this MONTHLY[1] Mr. E. C. Kennedy discusses the approximate solution of equations of the type $x^x=c$. The present paper is in a sense a continuation of his discussion, since it supplies a theoretical justification for the method and gives applications to a number of different kinds of equations.

In the first place, Mr. Kennedy's method is really not a "new method," as the title of his note would indicate. It is simply the familiar Newton's method, applied, however, not to the given equation, but to a related equation. To show this we may as well consider a more general case. If a is an approximation to a real root of the equation $f(x)=c$, where c is a positive constant, then Newton's method gives $a+[c-f(a)]/f'(a)$ as a second approximation. In certain well-known cases this second value will be closer to the root than a. But in some instances we can do better by applying Newton's method, not to $f(x)=c$, but to $\log f(x)=\log c$. This gives as a second approximation

$$a + \frac{f(a)}{f'(a)}[\log c - \log f(a)], \tag{1}$$

which will be a better approximation than a provided the function $\log f(x)-\log c$ has suitable properties and a is properly chosen.[2] Formula (1) gives Mr. Kennedy's method when applied to $x^x=c$, as the reader can verify at once.

[1] Vol. 38, 1931, pages 449–50.

[2] We must choose a so that $f(a)>0$.

The two second approximations obtained above can be easily compared. Suppose we write $\log c - \log f(a)$ as $\log [1+\{c-f(a)\}/f(a)]$. This is known to be less than $[c-f(a)]/f(a)$. If then $f'(a)$ is positive, formula (1) gives a smaller value than $a+[c-f(a)]/f'(a)$, if $f'(a)$ is negative, a larger value.

It follows at once that in some cases formula (1) will give better results than Newton's method applied to the given equation. A typical case is that in which the functions $f(x)-c$ and $\log f(x)-\log c$ both have their first and second derivatives positive in the neighborhood of the root we are seeking. Since Newton's method takes us along the tangent to the curve, speaking roughly, the first approximation a should be taken larger than the root to guarantee that the second approximation is closer. We are sometimes justified, however, in taking a too small, if we can get a better first approximation by so doing and if the graph of the function is fairly steep where we are working. In such a situation the second approximation will be too large, but probably closer than the first. But no matter which sort of a first approximation we wish to use, formula (1) will give a better second approximation than Newton's method, subject to our assumptions on $f(x)-c$ and $\log f(x)-\log c$. This is easily seen from a graph.

Mr. Kennedy's numerical example, $x^x=32.46$ is of this type. His first approximation, 3, with formula one, gives a second approximation 3.087+, which is too large, though it becomes a little too small when only the first three decimal figures are retained. From the statements of the last paragraph we know that Newton's method, applied to $x^x=32.46$, would give a larger and poorer second approximation. Further, formula (1), simplified to $(a+\log c)/(1+\log a)$, is easier to use than Newton's method.

Another example satisfying the same hypotheses is $e^x+x=18.56$, where 3 is a good (too large) first approximation to the real root. Newton's method gives 2.785 for the next approximation. Formula (1), with only a little more labor, gives 2.761, which is only .001 too large. Of course, the general theory does not tell this, only that formula (1) will give a better approximation. We may point out, however, that the formula gives the exact value of the root as the second approximation when the equation is $e^{kx}=c$, and can be expected to be especially satisfactory when the equation is similar (graphically) to this.

If the constant in the last example is changed to a smaller number, say 7.71, the situation changes somewhat. The graph of the function $\log (e^x+x)-\log c$ is now concave downward near the real root, and if we take 2 as our first approximation, formula (1) give a second approximation a little too small. However, by the remark of the last paragraph, we will expect the approximation to be close. It is, in fact, 1.7795, wereas 1.7800 is correct to 4 decimal places. Newton's method, applied to $e^x+x=7.71$, gives 1.8 as second approximation.

It may also be interesting to consider a few polynomial equations. The equation $x^3-6x^2+12x-12=0$ satisfies all of our original hypotheses near its real root. If 4 is taken as a first approximation to the real root, Newton's method gives 3.67 as the next approximation, while formula (1) gives 3.616. The real root is actually $2+4^{1/3}$, or about 3.58740.

The equation $x^3+2x-20=0$ illustrates a little different situation. The graph of the function $x^3+2x-20$ is concave upward near the real root, while that of $\log(x^3+2x)-\log 20$ is concave downward.[1] The real root appears to be closer to 2 than 3, but using 2 as a first approximation, and Newton's method, we will have our second approximation too large. It is, in fact, 2.57. The poorer first approximation 3 gives a better value, 2.55. Formula (1) has this advantage; we can start with the better first approximation, 2, and be sure in this case that our second approximation has *not* passed beyond the root. We get in fact 2.44, the correction being 6/7 of the difference $\log 20-\log 12$, or $\log 5-\log 3$. The root is 2.46955, to 5 places.

This last type of equation arises many times. We can find a close approximation easily, but the next approximation, by Newton's method, lies on the other side of the root, and perhaps a considerable distance. In some cases formula (1) will keep us on the same side of the root. Again, as in Mr. Kennedy's example, it may take us beyond the root, but not so far as Newton's method.

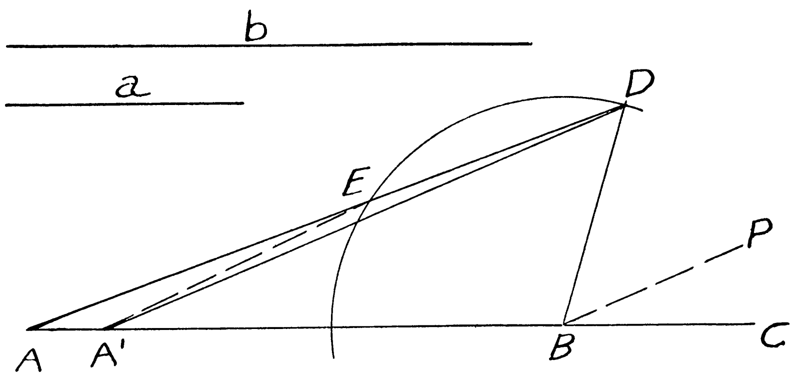
If we apply formula (1) to the equation $x^n=c$, we have the following interesting and perhaps useful

THEOREM. If a is smaller than the real nth root of c, then $a[1+(\log c)/n-\log a]$ is a better approximation to the root than a. Taking $c=10$, $n=3$, $a=2$, the theorem gives 2.149. If we call this 2.15, realizing it is too small, and take it as a, the formula gives 2.15443, which is correct to five decimal places.

ANGLE DIVISION

By E. C. KENNEDY, Univ. of Texas, College of Mines

Any given angle may be divided approximately by ruler and compass into any given ratio and to any desired degree of accuracy by the method described below. By way of illustration let us consider a given angle CBD and two given



[1] To find the second derivative of $\log f(x)$ conveniently, factor $f(x)$ if possible. If $f(x)$ is a polynomial with real, linear factors, the second derivative of $\log f(x)$ is always negative. This general theorem handles many cases, though we have purposely avoided it in our illustrations.

magnitudes a and b. Let it be required to find a point P so that $\angle CBP/\angle PBD = a/b$.

Method: Make $BD=a$, $BA=b$, $EA'=b-a$. Draw BP parallel to $A'D$. Then $\angle CBP/\angle PBD=a/b$, approximately. The accompanying figure is self-explanatory. For greater accuracy locate a point A'' from A' just as A' was located from A. That is, let $A'D$ cut circle at E'. Make $E'A''=EA'=(b-a)$, A'' being on AB. This may be repeated any number of times—the error rapidly approaching zero.

To trisect CBD make $BA=2BD$ and proceed as above. To obtain the angle $CBD/(2+\sqrt{2})$, let $BD=1$ and $BA=1+\sqrt{2}$, etc.

The following data give some idea concerning the accuracy of this method in the case of trisection. e, e', e'' represent the error.

CBD	A	A'	A''
$\pi/12$ rad.	$e=-0.0'\ 45.96''$	$e'=+0.47''$	$e''=-.0025''$
1 rad.	$e=-46'\ 16.40''$	$e'=+8'3.50''$	$e''=-1'19.87''$

An approximate measure of e' is given by

$$e' = 4\theta^5/3^7$$

which is quite accurate for small values of θ. Both e' and $\theta(=CBD)$ are measured in radians. This excellent approximation was suggested to the writer by Prof. Dunkel.

Thus it seems that any angle of about 60°, say, may be trisected with great accuracy by either (1) bisecting the given angle two or three times (thus dealing with a small angle) and locating A' or (2) by locating A''' or A^{IV} as described above.

The writer suggests that someone find a measure of $e^{(n)}$ for $CBD=15°$.

Additional Marginal Notes

By L. S. Johnston, University of Detroit

The title of this note is frankly and appropriately suggested by Professor Hildebrandt's "*Marginal Notes*" published in this Monthly, April, 1929. Like the material of that note, this also is culled from classroom experience. No claims for priority are made except where specifically so stated.

I. *Analytic Geometry.*

1. Several classic theorems of Projective Geometry which are either not provable at all by high school geometry or if provable are very difficult can be easily exhibited, and, by the more ambitious and able student, even actually

demonstrated, by use of the ordinary theory of the straight line and the circle in Analytic Geometry. Illustrative examples of such theorems are not at all difficult to set up, and I have found that by such exercises many students are attracted to the subject, and to mathematics as a whole, to a degree rarely found when only the more conventional exercises are used.

Some theorems which can be so approached are: Existence of the Euler Line for any triangle- that is, the collinearity of the centroid, the circumcenter, and the orthocenter; existence of the Nine Point Circle; Pascal's Hexagon Theorem and its dual, the Brianchon Theorem, using the degenerate two line conic and the circle rather than the general conic; Desargues' Theorem and the converse; Pascal's Pentagon Theorem and its use in drawing the tangent to any circle (or any conic, for that matter) at any point on it without using the center of the circle or any perpendicular lines.

2. *Constructions on the Ellipse and Hyperbola.* The following theorems on the ellipse are very easily proved, and suggest constructions useful in drawing the curves.

i) Let the equation of the ellipse be $x^2/a^2+y^2/b^2=1$, $c^2=a^2-b^2$, foci at F and F', X intercepts at A and A' as usual. The points $(0, a)$, $(c, b^2/a)$, and $(a, a-c)$ are collinear on the tangent to the ellipse at $(c, b^2/a)$.

ii) The points $(-a, 0)$, $(0, a-c)$, and $(c, b^2/a)$ are collinear, and, by (i) above, the line through $(c, b^2/a)$ and $(a, a-c)$ is tangent to the ellipse at $(c, b^2/a)$.

iii) If R and R' are respectively on the lines $x=a$ and $x=-a$, and FR is perpendicular to FR', the line RR' is tangent to the ellipse, as is well known. If FS be drawn so that the angles RFS and RFA are equal, the intersection of RR' and FS is the point of contact of RR' with the ellipse. This is proved in the writer's paper "*On Certain Sequences of Conics and Associated Sequences of Numbers*," this MONTHLY, April, 1929.

These three theorems are easily modified to apply to the hyperbola—the only modification, indeed, being that in the first theorem the point $(0, -a)$ is used instead of $(0, a)$.

I am not aware that any of these theorems have been mentioned in any elementary text.

3. *Simplification of Conics by Rotating Axes.* There is an increasing tendency to omit this topic from elementary courses, probably due to the difficulties inherent in the trigonometric and numerical calculations involved. Most of these difficulties can be eliminated if certain relations given in most elementary texts be used.

Let the translated form of the conic be $Ax^2+Bxy+Cy^2=F$, with the form $A'x^2+C'y^2=F'$ to be derived. The relations

$$A' + C' = A + C, \tag{1}$$

$$A' - C' = [(A - C)^2 + B^2]^{1/2}, \tag{2}$$

the sign of the square root to be taken to agree with the sign of B, and

$$\tan 2\theta = \frac{B}{A - C} \tag{3}$$

are well known, and are set as exercises in most texts. Tan θ is usually calculated from the identity $\tan^2 \theta = (1 - \cos 2\theta)/(1 + \cos 2\theta)$.

The entire simplification can be performed without using θ at all, by using relations (1) and (2) above. Tan θ is most easily calculated from the identity $\tan \theta = (\tan 2\theta)/(1 + \sec 2\theta)$, which, compared with (3) above, gives

$$\tan \theta = \frac{B}{A - C + A' - C'} \,. \tag{4}$$

II. *Calculus.*

1. Let $\rho = f(\theta)$ be the equation of a plane curve in polar coordinates; let ψ be the angle between the radius vector to any point (ρ_1, θ_1) and the tangent to the curve at this point, measured in positive direction from the radius vector; let τ be the angle between the tangent line to the curve and the polar axis, measured in positive direction from the polar axis. The usual method of deriving the formula for ψ in terms of ρ and θ seems unnecessarily laborious. I have found the following approach more easily handled in the classroom: From the relations $\rho^2 = x^2 + y^2$, $\tan \theta = y/x$, $\tan \tau = dy/dx$, we have

$$\begin{aligned} \tan \psi &= \frac{\tan \tau - \tan \theta}{1 + \tan \tau \tan \theta} = \frac{xdy - ydx}{xdx + ydy} \\ &= \sqrt{x^2 + y^2} \left(\frac{\dfrac{xdy - ydx}{x^2}}{1 + \dfrac{y^2}{x^2}} \right) \Bigg/ \left(\frac{xdy + ydx}{\sqrt{x^2 + y^2}} \right) = \frac{\rho d\left(\arc \tan \dfrac{y}{x} \right)}{d\rho} \\ &= \rho \frac{d\theta}{d\rho} \,. \end{aligned}$$

2. Let the parametric equations of motion in a plane be $x = x(t)$, $y = y(t)$, and let v_x, v_y, $v(=v_t)$, α_x, α_y, α, have their usual meanings; let

α_t = tangential acceleration
α_n = normal acceleration
v_R = radial velocity (velocity component along the radius vector)
α_R = radial acceleration
v_T = transverse velocity (velocity component normal to radius vector)
α_T = transverse acceleration.

The following simple relations are easily derived and applied:

$$\alpha_t = \frac{v_x\alpha_x + v_y\alpha_y}{v}; \quad \alpha_n = \frac{\begin{vmatrix} v_x & v_y \\ \alpha_x & \alpha_y \end{vmatrix}}{v}$$

$$v_R = \frac{xv_x + yv_y}{\sqrt{x^2+y^2}}; \quad v_T = \frac{\begin{vmatrix} x & y \\ v_x & v_y \end{vmatrix}}{\sqrt{x^2+y^2}}$$

$$\alpha_R = \frac{x\alpha_x + y\alpha_y}{\sqrt{x^2+y^2}}; \quad \alpha_T = \frac{\begin{vmatrix} x & y \\ \alpha_x & \alpha_y \end{vmatrix}}{\sqrt{x^2+y^2}}.$$

III. *Differential Equations.*

Consider the equation

$$\sum_{j=0}^{n} a_j \frac{d^jy}{dx^j} = f(x), \ d^0y/dx^0 \equiv y.$$

One method of calculating the particular integral which is frequently used is to differentiate the given equations enough additional times (say k times) to make it possible to find constants $b_0, b_1, \cdots b_k$ (if such constants exist, of course) such that $\sum_{j=0}^{k} b_j f^j(x) = 0$, and to use the resulting auxiliary equation $\sum_{j=0}^{n+k} A_j r^j = 0$. It is easily shown that this last equation can be factored into $(\sum_{j=0}^{k} b_j r^j)(\sum_{j=0}^{n} a_j r^j) = 0$, and hence that the particular integral can be found by using the equation $\sum_{j=0}^{k} b_j r^j = 0$.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Differential Equations. By Forest Ray Moulton. New York, The Macmillan Company, 1930. xv+395 pages. \$5.50.

This book is of very general interest and is at the same time a personal revelation of its author. It portrays Professor Moulton as one whose chief interests are practical, who regards differential equations primarily as tools in the solution of problems in astronomy and physics, but also as one who scorns the use of doubtful or unworthy tools and finds a keen satisfaction in the beautiful perfection of rigorous mathematical demonstration. It is this rare combination of application and theory, which sets this book apart.

In scope, the book differs from that of any other known to the reviewer.

Like Ince's book, it is limited to the study of ordinary differential equations. But unlike Ince it does not consider such problems as the characterization of differential equations by their singular points nor differential systems containing general boundary conditions as distinguished from initial conditions. Nor does it review the solution of differential equations by elementary methods. With these limitations it is quite comprehensive and presents most of the generally useful methods of solution. Numerous references to the eight appendices on topics in the Theory of Functions enable a good student to read the book without much more background in analysis than a year of advanced calculus.

Each existence proof of course does double duty. It is a proof and a careful one, but it is also a method or formula for the actual solution of a definite problem. To the author, perhaps, neither function seems more important than the other. It is doubtless to prevent the *proof* from overshadowing the *method* that theorems are rarely set apart and stated formally as theorems. Though this style does allow the theory and the formulae of solutions to develop coordinately, it increases the difficulty of reading the book and makes it less convenient for reference.

Chapter I, Generalities and Definitions, defines a normal system of differential equations:

$$\frac{dx_1}{dt} = f_1(x_1, \cdots, x_n; t),$$
$$\cdots\cdots\cdots\cdots$$
$$\frac{dx_n}{dt} = f_n(x_1, \cdots, x_n; t).$$

Such systems furnish the material for treatment in the book. Solutions and integrals of normal systems are defined and illustrated. This chapter, like the others, contains a list of suggestive problems.

Chapter II, Analytic Differential Equations, assumes the functions f_i to be analytic in all the variables, and adjoins the initial conditions $x_i(t_0) = a_i$. It sets up the formal solution in power series, and proves the convergence by the method of dominant functions.

Chapter III, Differential Equations Analytic in a Parameter, first sets up a solution as a power series in μ for the case of equations in the form $dx_i/dt = \mu f_i(x_j : \mu : t)$ and then considers more general systems.

Chapter IV, Variation of Parameters, is a general discussion of this method of transformation with special application to linear equations.

Chapter V, Integrals of Differential Equations, deals with the existence under suitable hypotheses of a fundamental set of integrals and with the use of known integrals in reducing the order of an equation.

Chapter VI, Analytic Implicit Functions, treats of the existence and formulation of solutions of systems of the form

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= c_1 y + \phi_1(x_1, \cdots, x_n; y) \\
&\cdots\cdots\cdots\cdots\cdots \\
a_{n1}x_1 + \cdots + a_{nn}x_n &= c_n y + \phi_n(x_1, \cdots, x_n; y)
\end{aligned}
$$

(where the ϕ_i are power series in the x_j and y and contain no terms of lower degree than the second) under various hypotheses on the determinant $|a_{ij}|$.

In Chapter VII, The Problem of Elliptic Motion, we find the first of the major applications which give the book its real character. The problem of two bodies is set up in the form of a differential system. The nature of the orbit and the period are first determined directly. Then the methods of Chapters II and III are used to form the complete solution as a power series in the variable of time, and also in terms of the eccentricity.

Chapter VIII, The Sine-amplitude Function, develops the characteristic properties of the sine-amplitude function partly from the corresponding elliptic integral but mainly as properties of the solution of the equivalent differential equation.

Chapter IX, The Deviations of Falling Bodies, gives the author's own discussion of the deviations of a falling body from a plumb line.

Chapter X, The Damped Gyroscope, emphasizes the necessity of choosing the method of solution to suit the problem. The method of variation of parameters is exemplified in studying the nutations of the axis.

In Chapter XI, The Method of Successive Integrations, the process of successive approximations is established in a general way by means of a variation in the form of the first integral.

Chapter XII, Numerical Solution of Differential Equations, is an exceedingly interesting chapter which expounds the author's own method of numerical solution previously developed in his work on Ballistics. The method, in brief, consists in approximating to the solution in a very small interval by the method of successive approximations, and then extrapolating to get initial values on an adjoining interval. For solutions of this type, the author's arrangement of the work is of prime importance.

In Chapter XIII, The Cauchy-Lipschitz Process, this classical method is explained and compared with the method of Chapter XI.

Chapter XIV, Generalities on Linear Differential Equations, defines a fundamental set of solutions. It also defines the equations for variations of the solution corresponding to variations in initial conditions.

Chapter XV, Linear Differential Equations with Constant Coefficients, is unusual in adopting the method of Nyswander rather than the customary treatment. The chief advantage is in simplifying the routine of actual calculation. The characteristic determinant is reduced in a novel manner to a normal form that is always applicable.

Chapter XVI, Applications of Linear Differential Equations, gives applications of the problem of stability of planetary orbits, and to an electric circuit with a periodic imposed electromotive force.

Chapter XVII, Linear Differential Equations with Periodic Coefficients, extends the methods of Chapter XV to the equations described in the title. Identities are introduced analogous to those of Nyswander used in Chapter XV, and the characteristic equation is reduced in a manner similar to the one employed there.

In Chapter XVIII, Differential Equations of Infinitely Many Variables, necessary restrictions are placed on the functions of the system, a formal solution is set up, and convergence established. Interesting applications are then made to an infinite universe extending in the one sense to subelectrons and on the other to super-galaxies of all orders. On the basis of postulates laid down it is established that the individual stars have determinable motions with respect to arbitrary axes. The system occupies infinite space in such a way that the sums of components of attraction on each star are finite.

In the main the details of the book have been well handled. It is not entirely free from lapses in notation and expression. For example, on page 98 the same letter is used in a single equation to represent the eccentricity of an ellipse and the Napierian base of logarithms. On page 117 the equations $t=\frac{1}{2}\sqrt{-1}\ P_2$ and $t=\frac{1}{2}P_1$ do not properly describe the lines in question. On page 249 the assumption should be $\delta=0$ instead of $\delta\neq 0$.

The book is a valuable and important contribution and is recommended to students of mathematics, physics, and astronomy. It will be relished by all who have an appetite for rigorous mathematical deduction flavored with the wholesome tang that comes from keeping close to a basis of physical application and numerical computation.

RAYMOND W. BRINK

Elêments de Trigonomêtrie Sphêrique. By G. Papelier, Paris, Librairie Vuibert, 1930. 162 pages. Paper, 20 francs.

This small inexpensive book contains a surprising amount of material which is presented in a most excellent manner. The first five chapters—which is less than half of the book—are devoted to general remarks on spherical triangles, relations among the elements of a spherical triangle, properties of spherical triangles, solutions of right spherical triangles, and solutions of triangles in general. All cases are carefully and fully discussed. A numerical example is generally given for the sake of fuller illustration. The material covered here is more extensive than that contained in the ordinary plane and spherical trigonometry text book.

The last five chapters are devoted to a discussion of the escribed, inscribed, and circumscribed circles of spherical triangles; medians, bisectors, and altitudes of spherical triangles; formulas relative to spherical excess; several diverse applications; and the solution of triangles which come outside the standard cases. These chapters all contain material that is of much general interest and would prove stimulating to any college student who is interested in further read-

ing in spherical trigonometry. Especially is this true of the material leading up to the theorems of Menelaus and Ceva.

The book is practically void of exercises, there being only eighteen, which are given at the end of the book along with a résumé of the most important formulae that have been developed.

H. M. HOSFORD

A LETTER TO THE EDITOR

DEAR SIR:

In the recent comment in your columns on my *Table Slide Rule* it was stated that this Rule requires a large space for efficient use. Permit me to point out that the dimensions of the plates are $8\frac{1}{2}'' \times 11''$, so that they will fit into standard size loose-leaf notebooks. As published the plates are not punched because there is no uniformity in the punching of notebook paper. They may however be punched locally so that they may fit the notebooks used.

When carried in a notebook the slide rule is immediately available and is found to be very convenient.

Sincerely yours,
J. P. BALLANTINE

June 2, 1932.

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D.C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide. All club activity manuscripts for the academic year 1932–1933 should be submitted for publication not later than June 1, 1933.

CLUB ACTIVITIES

A.

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research in the preparation of papers in the field of mathematical science to be presented at its regular meetings.

CHAPTER REPORTS
1931–1932

Pi Mu Epsilon of the University of California at Berkeley

The officers for the academic year 1931–1932 were elected by the members at their last regular meeting, on April 21, 1931. They were: Mr. W. L. Hutchings, Director; Miss C. M. Daly, Vice Director; Miss D. A. Wilson, Secretary; Mr. A. L. Buckman, Treasurer; Mr. R. M. Robinson, Librarian.

by John Dittrick; "Practical applications of mathematics to paths and projectiles" by Mr. Wendell A. Dwyer.

December 16, 1931: "Pascal's theorem" by Paul Schneider; "Brianchon's theorem" by Hugh M. Schwaab.

January 20, 1932: "Mathematics and music" by Joseph Solomonow; "Telegraph, telephone and mathematics" by Mr. Blakeslee of Northwestern Bell.

February 17, 1932: "Vector equations" by Olga Dyba; "Pythagoras" by Walter Smith; "The fundamental theorem of algebra" by Mr. John Fitzpatrick.

March 18, 1932: "Hyperbolic functions" by Professor Alvin K. Bettinger; "Poles and polars" by John Dittrick.

April 27, 1932: "Transcendentals" by Basil Lazure; "Families of curves and surfaces" by John Meters; "The nine-point circle" by Mr. Frank E. Marrin.

May 19, 1932: "Abel and Galois" by Glenn Rhoades; "Harmonic curves" by Mr. Zimmerman, S. J.; "Principles of astronomy" by Mr. Wendell A. Dwyer. This meeting was followed by a trip through the Creighton observatory with explanations of the apparatus by Mr. Dwyer.

JOSEPH SOLOMONOW, *Secretary*

The School of Crotona of Carleton College

The School of Crotona is a local honorary mathematics society which was organized three years ago for the purpose of stimulating an interest in the mathematical world.

To be eligible for membership, one must have completed at least one year of study in differential and integral calculus, as well as shown general mathematical ability. A comprehensive oral examination is given to qualified applicants by the faculty of the department of mathematics.

Three members, who constitute the Triad, are chosen from the Society each year to form the governing body of the Mathematics Club, a club whose meetings are open to all students interested in that subject. The members of the Triad this year are: William Conley, President; Frederick Schurmeier, Secretary-Treasurer; and Robert McCallum.

Other members of the School of Crotona are Philip Nason, Eleanor Walker, Impi Aho, and Edward Tomastic. Members of the Faculty are Dr. Gingrich and Dr. White.

The meetings and programs of the Mathematics Club were as follows:

October 2, 1931: "Symmetry in mathematics" by Dr. Gingrich of Carleton College, Chairman of the Department of Mathematics and Editor of Popular Astronomy.

November 19, 1931: "Hyperbolic functions" by Dr. Carlson, Professor of Mathematics, St. Olaf's College.

January 14, 1932: "Pascal's triangle" by Mr. William Conley, President of the Mathematics Club.

March 22, 1931: "Mathematics in business" by Dr. Larson of the Faculty of Carleton College.

April 12, 1932: "Mathematics of investment" by Dr. Hart, Professor of Mathematics in the University of Minnesota.

F. A. SCHURMEIER, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON, AND WM. FITCH CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to Wm. Fitch Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and de-

manding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E1. *Proposed by H. E. Slaught, University of Chicago.*

Find all the unknown digits, represented by x's, in this exact division.

```
x x x) x x x x x x x x (x 7 x x x
       x x x x
       -------
           x x x
           x x x
           -----
           x x x x
             x x x
           -------
                 x x x x
                 x x x x
                 -------
```

E2. *Proposed by R. K. Morley, Worcester Polytechnic Institute.*

In finding the volume cut from the sphere $x^2+y^2+z^2=9$ by the cylinder $x^2+y^2=3x$, one may use cylindrical coordinates and choose as element of volume a column of cross-section dr by $rd\theta$ and of height $z=(9-r^2)^{1/2}$. The volume appears computable either by

$$V_1 = 2\int_{-\pi/2}^{+\pi/2}\int_0^{3\cos\theta}(9-r^2)^{1/2}rdrd\theta,$$

where symmetry with respect to the xy-plane only is used, or by

$$V_2 = 4\int_0^{\pi/2}\int_0^{3\cos\theta}(9-r^2)^{1/2}rdrd\theta,$$

where symmetry with respect to the xz-plane is also used. But upon integration it appears that V_1 exceeds V_2 by twenty-four cubic units. Which (if either) is correct, and why?

E3. *Proposed by Wm. R. Ransom, Tufts College.*

Has the locus $y=x^x$ a highest point in the second quadrant?

E4. *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

What is the simplest way to cut a wooden block 1 ft. $\times$ 1 ft. $\times$ 2 ft. into pieces which may be reassembled into a solid cube?

E5. *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

How may the total surface of a sphere be divided into the largest possible number of congruent pieces, if each side of each piece is an arc of a great circle less than a quadrant?

Advanced Problems

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3566. *Proposed by V. Ivanoff, San Francisco, Calif.*

A perpendicular CM_0 is drawn to the line segment M_0M_n and the points $M_1, M_2, \cdots, M_{n-1}$ are taken on M_0M_n so that $\angle M_0CM_1 = \angle M_1CM_2 = \cdots = \angle M_{n-1}CM_n = \alpha$. By parallel displacement of the vectors $CM_i (i=2, 3, \cdots, n)$ a broken line $CM_1M_2'M_3' \cdots M_n'$ is formed with $M_1M_2' = CM_2$ and $M_{i-1}'M_i' = CM_i (i=3, 4, \cdots, n)$. Find the locus of the vertices of this broken line if $CM_0 \to 0$ and $\alpha \to 0$ in such a way that $CM_0/\alpha = K$, K constant.

3567. *Proposed by Leverett Davis, University of Washington.*

Find the line through a given point that shall form with two given lines a triangle of minimum area. Characterize the solution geometrically.

3568. *Proposed by Orrin Frink, Pennsylvania State College.*

What is the convex region of greatest area which contains a given triangle T but contains no triangle of greater area than T?.

3569. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Through a given line s to draw a plane cutting the faces of a given dihedral angle along the lines a, b, and the bisecting planes of this dihedral angle along the lines c, d, so that the lines c, d shall be the bisectors of the angles formed by a, b. The problem has, in general, two solutions. What are the special cases?

3570. *Proposed by Norman Anning, University of Michigan.*

Given five points in the same three-space but otherwise independent, construct the point in which the join of two of them intersects the plane of the other three.

3571. *Proposed by Lester R. Ford, Rice Institute.*

What should one pay in order to receive $1, 8, 27, \cdots, n^3, \cdots,$ dollars at the end of $1, 2, 3, \cdots, n, \cdots,$ years, the interest rate being six percent?

SOLUTIONS

3365. [1929, 168]. *Proposed by R. E. Gaines, University of Richmond.*

An open pan with a square bottom (a frustum of a pyramid) having a given

total surface is to be made so as to have a maximum content. Find its dimensions.

Solution by Otto Dunkel, Washington University.

The pans considered in this solution have the form of a frustum of a regular pyramid with a square base. Let x be the length of the side of the square base of the pan; y, that of the open top; z, the slant height of the lateral sides; h, the altitude of the pan; a^2, the superficial area; and V, its content. We have the following relations;

$$(1)\qquad \begin{aligned} a^2 &= x^2 + 2z(x+y);\ 4h^2(x+y)^2 = (a^2 - y^2)(a^2 + y^2 - 2x^2);\\ 6V &= [(a^2 - y^2)(a^2 + y^2 - 2x^2)]^{1/2}\frac{x^2 + xy + y^2}{x + y}. \end{aligned}$$

If a pan V_1 has $y<x$, then a pan V_2 of the same height whose base has the dimensions of the top of V_1 and whose top has the dimensions of the base of V_1 has the same content as V_1, but it has less superficial area. Hence there is a pan V_3 with a form similar to that of V_2 and with the area of V_1 whose content is greater than that of V_1. This shows that we need to consider only the cases $y \geqq x$. Since in such cases y^2 is the projection of a^2 on the base, we must have also $y \leqq a$. Hence we have a region defined by $a \geqq y \geqq x \geqq 0$; this region is an isosceles right triangle. Along the hypotenuse we have boxes, i.e., pans with vertical sides; along the leg for which $x=0$ we have the cases where the base reduces to a point, or funnel forms; while along the leg $y=a$ the height is zero and $V=0$. The maximum volumes in the first two cases will be denoted by V_b and V_f, respectively. We easily find

$$(2)\qquad \begin{aligned} V_b &= 2^{-1}3^{-3/2}a^3, \qquad x = y = 3^{-1/2}a;\\ V_f &= 2^{-1/2}3^{-7/4}a^3, \quad x = 0,\ y = 3^{-1/4}a;\\ &\qquad V_b < V_f. \end{aligned}$$

If we define $V=0$ for $x=y=0$, then V is a continuous function of x and y in the closed triangular region. It will appear later that V has values within the region greater than its maximum on the boundary; and hence V has an absolute maximum at some point, or points, within the region. At such points the partial derivatives V_x and V_y, which exist at all points within the region, must be zero. It will be shown that there is only one point within the region where the derivatives vanish simultaneously. Therefore this point must give an absolute maximum. From these considerations we see that we may reject all factors which vanish only on the boundary or outside the region in seeking the solution of $V_x = V_y = 0$ at an interior point. For convenience set

$$(3)\qquad \begin{aligned} R &= (a^2 - y^2)(a^2 + y^2 - 2x^2);\\ R_x &= -4x(a^2 - y^2);\ R_y = -4y(y^2 - x^2). \end{aligned}$$

The equations $V_x = V_y = 0$ give

$$
(4) \qquad \begin{aligned} R(x + 2y) - 2(a^2 - y^2)(x + y)(x^2 + xy + y^2) &= 0, \\ R(y + 2x) - 2(y^2 - x^2)(x + y)(x^2 + xy + y^2) &= 0. \end{aligned}
$$

It follows at once from these two equations that

$$
(5) \qquad (y + 2x)(a^2 - y^2) = (x + 2y)(y^2 - x^2).
$$

The first equation of (4) gives at once

$$
(6) \qquad (x + 2y)(a^2 + y^2 - 2x^2) = 2(x + y)(x^2 + xy + y^2);
$$

and adding (6) and (5) we have after slight reduction

$$
(7) \qquad a^2 = x^2 + xy + y^2.
$$

Eliminating a^2 from (5) by (7) we find

$$
(8) \qquad 2y^2 - 2xy - 3x^2 = 0.
$$

Set $y = rx$ in (8), then

$$
(9) \qquad 2r^2 = 2r + 3, \ 2r = 1 + 7^{1/2}, \ y = rx.
$$

The first equation in (9) is useful for reducing higher powers of r to linear expressions in r before inserting the numerical value of r. From (7) and (9) there results

$$
(10) \qquad \begin{aligned} x^2 &= \frac{a^2}{1 + r + r^2} = \frac{2a^2}{4r + 5} = \frac{2a^2(7 - 2 \cdot 7^{1/2})}{21}, \\ y^2 &= \frac{a^2(14 - 7^{1/2})}{21} \cdot \end{aligned}
$$

The first equation in (1) with (7) gives

$$
(11) \qquad z = \tfrac{1}{2}y.
$$

In order to obtain h it is simpler to return to the right triangle for h:

$$
(12) \qquad \begin{aligned} h^2 &= \left(\frac{y}{2}\right)^2 - \left(\frac{y - x}{2}\right)^2 = \tfrac{1}{4}x(2y - x) = \tfrac{1}{4}(2r - 1)x^2 = \tfrac{1}{4}7^{1/2}x^2, \\ &= \frac{a^2(7^{1/2} - 2)}{6} \cdot \end{aligned}
$$

We now find from (7) and (12) the maximum

$$
(13) \qquad V_p = \frac{a^3}{3}\left(\frac{7^{1/2} - 2}{6}\right)^{1/2}.
$$

It is easily shown that $V_p > V_f$. The numerical results are as follows:

$$x = .403378a, \qquad y = .735308a, \qquad z = .367654a,$$
$$V_p = .1093543a^3, \quad V_f = .103401a^3, \quad V_b = .096225a^3.$$

The sides of the pan make with the base an angle $\theta = 2 \text{ arc cot } 7^{1/4} = 63° \, 9' \, 55''$.

3469 [1931, 50]. *Proposed by V. M. Spunar, Chicago, Illinois.*

A constant length, PQ, is measured along the tangent at any point, P, on a curve; give a geometrical construction for the center of curvature of the locus of the point Q. Also if PQ' be measured equal to PQ in the opposite direction along the tangent, prove that the point P and the centers of curvature of the loci of Q and Q' lie in a straight line.

A Note by the Editors. A solution of this problem by R. Goormaghtigh has been printed in this Monthly [1932, 53–54], and the solver has recently sent in the generalization in the note below.

II. *Solution by R. Goormaghtigh, Bruges, Belgium.*

In our article "*Solution géométrique du problème fondamental de la Géométrie intrinsèque plane*" (*Mathesis*, 1929, p. 53) we have given a geometrical solution of the following general problem:

If a moving point Q has the coordinates x, y with respect to the tangent and the normal at a moving point P on a plane curve (P), x and y being functions of the inclination angle ϕ of the tangent to (P) at P with respect to a fixed axis, give a geometrical construction for the normal and the center of curvature at Q to the locus (Q) of Q.

Let x', y' and x'', y'' be the two first derivatives of x, y with respect to ϕ; let further C be the center of curvature of (P) at P and C' the center of curvature of the evolute of (P) at C.

The normal to (Q) at Q passes through the point T having x', y' as coordinates with respect to PC, CC'.

If S is the point having x'', y'' as coordinates with respect to the tangent and the normal to the curve (C'), second evolute of (P), the center of curvature q of (Q) at Q lies on the straight line joining the mid-point of TS to the point where QS meets the perpendicular erected on QT at T.

The point q can also be obtained as follows:

The point q is the mid-point of the distance between T and the harmonically conjugate of T with respect to Q and the projection of S on QT.

A Note by Otto Dunkel. Since *Mathesis* may not be accessible to many readers, as is the case with the writer, a proof of this theorem will be given, and vector analysis will be used to show the advantage of this form of analysis for curvature problems. Let $\boldsymbol{t}$ and $\boldsymbol{n}$ be the unit vector tangent and normal for (P) at P, where the positive direction of $\boldsymbol{t}$ is that of increasing arc length s and that of $\boldsymbol{n}$ is along $PC = \rho$; let $CC' = \rho_1$. The unit tangent and normal to (C) at C are $\boldsymbol{n}$ and $-\boldsymbol{t}$; for (C') at C', they are $-\boldsymbol{t}$ and $-\boldsymbol{n}$. It will be seen from a figure that

$$QT = (\rho + x' - y)\boldsymbol{n} - (x + y')\boldsymbol{t}; \tag{1}$$

$$ST = (\rho_1 - y' + x'')\boldsymbol{t} + (x' + y'')\boldsymbol{n}. \tag{2}$$

Let $\boldsymbol{r}$ and $\boldsymbol{r}_2$ be the vectors of P and Q drawn from a given origin, so that

$$\boldsymbol{r}_2 = \boldsymbol{r} + x\boldsymbol{t} + y\boldsymbol{n}. \tag{3}$$

After differentiation with respect to s, (3) gives

$$s_2'\boldsymbol{t}_2 = \boldsymbol{t} + \kappa(x'\boldsymbol{t} + y'\boldsymbol{n}) + \kappa(x\boldsymbol{n} - y\boldsymbol{t});$$

where the accents refer to differentiation with respect to ϕ only in the cases of x and y, while with other quantities they denote differentiation with respect to s; $\kappa = \rho^{-1} = d\phi/ds$; $s_2' = ds_2/ds$ is taken as positive; and where the unit vector tangent $\boldsymbol{t}_2$ has the direction of increasing arc s_2. The equation above may be written

$$s_2'\rho\boldsymbol{t}_2 = (\rho + x' - y)\boldsymbol{t} + (x + y')\boldsymbol{n}. \tag{4}$$

Since the scalar product $QT\cdot\boldsymbol{t}_2$ is easily seen from (1) and (4) to be zero, it follows that the vector QT is along the unit normal $\boldsymbol{n}_2$ to (Q) at Q. Let the absolute value of the right side of (1) and (4) be l. Then we may write

$$\begin{aligned} l\boldsymbol{t}_2 &= (\rho + x' - y)\boldsymbol{t} + (x + y')\boldsymbol{n}; \; s_2'\rho = l; \\ l\boldsymbol{n}_2 &= (\rho + x' - y)\boldsymbol{n} - (x + y')\boldsymbol{t}. \end{aligned} \tag{5}$$

In order to find the curvature $\kappa_2 = \rho_2^{-1}$ of (Q) at Q, we differentiate the first equation of (5) with respect to s. There results after slight changes

$$\begin{aligned} \rho l'\boldsymbol{t}_2 + \rho l\kappa_2 s_2'\boldsymbol{n}_2 &= (\rho + x' - y)\boldsymbol{n} - (x + y')\boldsymbol{t} \\ &\quad + (\rho\rho' + x'' - y')\boldsymbol{t} + (x' + y'')\boldsymbol{n}; \\ &= l\boldsymbol{n}_2 + ST; \end{aligned} \tag{6}$$

where $\rho_1 = \rho\rho'$. Taking the scalar product of $\boldsymbol{n}_2$ and the second form of (6) we have

$$l^2\kappa_2 = l + ST\cdot\boldsymbol{n}_2$$

Let the projection of S on QT be M, then the last equation becomes $l^2 = \rho_2(l + MT)$, or

$$\frac{\rho_2 - l}{\rho_2} = \frac{TM}{l}, \quad \text{or} \quad \frac{Tq}{Qq} = \frac{TM}{QT}, \tag{7}$$

where q is the center of curvature of (Q) at Q.

In order to derive the constructions for q, draw TH perpendicular to QT at T cutting QS in H, and produce TH to K so that $HK = TH$. Draw KS cutting QT in T'. It is obvious that TS, $T'H$ and MK meet in a point, and that Q, T, M, T' form an harmonic set of points. Draw Hq parallel to KS cutting QT' in q. Obviously Hq bisects ST, and q bisects TT'. It will be shown that the q thus

obtained is the desired center of curvature. From the figure we have

$$\frac{Tq}{Qq} = \frac{qT'}{Qq} = \frac{HS}{QH} = \frac{TM}{QT},$$

and hence this q is the same as in (7).

We obtain additional equations by taking the scalar product of $\boldsymbol{t}_2$ and (6). We have $\rho l' = ST \cdot \boldsymbol{t}_2 = NQ$, where N is the projection of S on the tangent at Q to (Q). This equation with $\rho s_2' = l$ gives

$$\frac{dl}{ds_2} = \frac{NQ}{QM}.$$

3482 [1931, 170]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

If the circumscribed and the inscribed spheres of a tetrahedron are concentric, the sum of the face angles of each trihedral angle of the tetrahedron is equal to two right angles.

Solution by Wallace Smith, New River State College, Montgomery, W. Va.

Sphere S is circumscribed about and sphere s is inscribed in the tetrahedron $ABCD$, each having the center O. Since planes ABD, BDC, ADC, and ABC are tangent to sphere s, they are at equal distances from the center of sphere S. Planes equidistant from the center of a sphere intersect the sphere in equal circles. Therefore, triangles ABD, ABC, ADC, and BCD are inscribed in equal circles. The chords AD, AB, and BD are each common to two equal circles. It follows at once that $\angle ACD = \angle ABD$, $\angle BCA = \angle BDA$, and $\angle BCD = \angle BAD$. Therefore, $\angle BCD + \angle BCA + \angle ACD$ equals $\angle BAD + \angle BDA + \angle ABD$ equals two right angles.

A similar proof applies to the other three vertices.

Therefore, the sum of the face angles of each trihedral angle of the tetrahedron is equal to two right angles.

Also solved by V. F. Ivanoff, A. Pelletier, J. Rosenbaum, Otto J. Ramler, and F. Underwood.

A Note by Otto Dunkel. In the above proof the fact that pairs of angles such as ACD and ABD are equal rather than supplementary follows from the fact that each of such face angles is acute. For, the center O of the inscribed sphere s is the intersection of the three planes bisecting the dihedral angles at AB, BC, CA. Thus, if the point of tangency of s with the face ABC is T_d, this point must lie within each angle of the triangle ABC, and, consequently, it must lie within the area of that triangle. Moreover, since O is also the center of S, T_d is the center of the circle circumscribing ABC and lying upon S. Since the center of the circumcircle lies within ABC, all its angles must be acute; and, similarly

all the face angles of the tetrahedron must be acute. It was proved above that the four circumcircles are equal.

Let the other three faces be rotated into the plane of ABC so that the vertex D falls at D', D'', D''', where A and D', B and D'', C and D''', lie, respectively, on opposite sides of BC, CA, AB. As shown above the interior angles at D', D'', D''' are equal, respectively, to the angles A, B, C of triangle ABC. Also $\angle D''AC=C'$, $\angle D'''AB=B'$, where $A+B'+C'=180°$; thus $D''AD'''$ is a straight line. Similarly, $D'''BD'$ and $D'CD''$ are straight lines. Hence $D'D''D'''$ is a triangle with C, A, B as the mid-points of its sides. Therefore BC is parallel to $D'''D''$, and $B'=B$, $C'=C$. Continuing in this way we see that the four triangles are congruent. Thus the tetrahedron $ABCD$ is such that its opposite edges are equal.

Such tetrahedrons may be constructed by taking any triangle $D'D''D'''$ which has only acute angles and reversing the above process.

3505 [1931, 408] *Proposed by J. Rosenbaum, Milford, Conn.*

Under what condition will the lines joining the vertices of a tetrahedron with the points of contact of the opposite faces and the insphere be concurrent?

Solution by the Proposer.

Let the vertices of the tetrahedron be A_1, A_2, A_3, A_4; the faces opposite them F_1, F_2, F_3, F_4; and the points of contact of these faces with the insphere O_1, O_2, O_3, O_4.

From the assumption that A_1O_1, A_2O_2, A_3O_3 and A_4O_4 are concurrent, follows that three lines such as A_1O_3, A_3O_1 and A_2A_4 are also concurrent. Accordingly, draw A_1O_3 and A_3O_1 each meeting A_2A_4 at P. Because O_1, O_2, etc. are the points of contact, the three segments A_1O_2, A_1O_3 and A_1O_4 are equal. From this it follows that the triangles $O_1A_2A_4$ and $O_3A_2A_4$ are congruent; from which in turn it follows that the triangles O_1A_2P and O_3A_2P are congruent; also the segments O_1P and O_3P are equal.

From the concurrency of the lines of the problem it can further be proved that four masses, m_1, m_2, m_3, m_4 can be placed at the vertices, A_1, A_2, A_3, A_4 such that O_1 is the center of mass of the masses at the vertices of F_1; O_2 of the masses at the vertices of F_2; etc., and also that the point P mentioned above is the center of m_2 and m_4.

Considering the face F_3, since O_3 is the center of m_4, m_1 and m_2, and since P is the center of m_2 and m_4,

$$A_1O_3/O_3P = (m_2 + m_4)/m_1. \tag{1}$$

In the same way, by considering the face F_1

$$A_3O_1/O_1P = (m_2 + m_4)/m_3. \tag{2}$$

Dividing (1) by (2), and taking account of the equality of O_3P and O_1P, we have

$$(3) \qquad A_1O_3/A_3O_1 = m_3/m_1.$$

Replacing A_1O_3 and A_3O_1 by A_1O_4 and A_3O_4, equation (3) becomes

$$(4) \qquad A_1O_4/A_3O_4 = m_3/m_1.$$

Considering now the face F_4 and drawing A_2O_4 cutting A_1A_3 at Q, we obtain, since Q is the center of m_1 and m_3,

$$(5) \qquad A_1Q/QA_3 = m_3/m_1.$$

From (4) and (5),

$$(6) \qquad A_1Q/QA_3 = A_1O_4/O_4A_3.$$

From this last equation follows that O_4Q, which is the same line as A_2O_4, bisects the angle $A_1O_4A_3$. Similarly A_1O_4 bisects angle $A_1O_4A_2$. Hence the angles $A_1O_4A_2$, $A_2O_4A_3$, and $A_3O_4A_1$ are 120 degrees.

This result applies also to the corresponding angles in the other faces.

Thus is obtained the conclusion: In order that the lines of the problem may be concurrent it is necessary that the point cf contact in each face be the isogonic point of the face. It can be proved that this condition is also sufficient.

The construction of such a tetrahedron from cardboard will show that it does not need to be regular.

Also solved by William Hoover and A. Pelletier.

3514 [1931, 462]. *Proposed by J. P. Ballantine, University of Washington.*

Let D_n denote a determinant of order n whose elements are all zeros and ones, and which has 2 ones and n-2 zeros in every row and column. Show that:

(a) For every n, $D_n = \pm 2^m$, where m and n are both even or both odd, or $D_n = 0$.

(b) If $D_5 = 0$, then two rows are identical, and conversely.

(c) If two rows of D_n are identical, then two columns are alike, and conversely.

(d) $3m \leqq n$.

(e) Show that property (b) does not hold except for D_2, D_3, and D_5.

Solution by Dwight F. Gunder, Colorado Agricultural College

By rearrangement of the order of the rows and the columns D_n reduces to;

$$D_n = \pm \begin{vmatrix} A_1 & 0 & 0 \cdot \cdot & 0 \\ 0 & A_2 & 0 \cdot \cdot & 0 \\ 0 & 0 & A_3 \cdot \cdot & 0 \\ \cdot & \cdot & \cdot \quad \cdot & \cdot \\ \cdot & \cdot & \cdot \cdot \cdot & A_m \end{vmatrix}$$

where the A_k are square arrays of order n_k $(n_k>1)$ with elements $a_{1,1}=a_{n_k,n_k}=1$, $a_{i+1,\,i}=a_{i,i+1}=1$ $(i=1, 2, \cdots, n_k-1)$ and all other elements are zero.

Any A_k is at once seen to simplify to B_k in which $b_{i,i}=(-1)^{i+1}$ for $i=1, \cdots, n_k-1$, and b_{n_k,n_k} equals 0 or 2 according as n_k is even or odd, $b_{i,i+1}=1$ $(i=1, \cdots, n_k-1)$, and all other elements are zero. This B_k expands at once and we have:

$$A_k = B_k = 0 \quad \text{for } n_k \equiv 0 \bmod 2,$$
$$A_k = B_k = 2 \quad \text{for } n_k \equiv 1 \bmod 4,$$
$$A_k = B_k = -2 \text{ for } n_k \equiv 3 \bmod 4.$$

Further, if n is odd $D_n\neq 0$ only when m and every n_k are odd, and if n is even, $D_n\neq 0$ only when m is even and every n_k is odd.

Hence,

(a) For n odd $D_n=\pm 2^m$ where m is odd, or $D_n=0$, and for n even $D_n=\pm 2^m$ where m is even, or $D_n=0$.

(b) After the first reduction above, D_5 will have only one A_k (of order five) or will have two A_k, one of order three and the other of order two. In the first case, $D_5\neq 0$, and in the second case $D_5=0$ and by definition of an A_k of order two we find that two rows, and two columns, of D_5 are identical. The converse is obvious.

(c) If two rows of D_n are identical the two ones must fall in corresponding columns, hence two columns are identical, and conversely.

(d) Since to be different from zero an A_k must have an order $n_k \geqq 3$, then obviously

$$n = \sum_{k=1}^{k=m} n_k \geqq 3m.$$

(e) By (b), $D_5=0$ necessitates two identical rows, and obviously $D_2=0$, and $D_3=0$ do likewise; but any $D_n=0$ $(n=4$ or $n>5)$ may be due to the appearance of an A_k of order $4p$ whose vanishing does not necessitate two identical rows in D_n. The converse part of (b) is of course always true.

Note by the Editors

$$D_3 = \pm \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \pm \begin{vmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = \mp 2.$$

Hence D_3 cannot be zero, and the part of (e) referring to D_3 seems to have no meaning; and for D_2 it is trivial.

Solutions were received from the following contributors after selections for publication had been made: William Hoover 291, 313; R. A. Johnson, 3458; A. Hershfield, 232; F. Underwood, 309, 313; H. S. Uhler, 313; L. B. Haughwout, 313; and V. F. Ivanoff, 313.

The following contributors should have been credited as follows: J. B. Reynolds, 3404, 3411, 3412, 3413, 3419, 3420, 3424, 3429; V. F. Ivanoff, 3405, 3406; Hassler Whitney, 3421; M. Morris, 3426; B. F. Yanney, 3432, 3434; Mabel S. Graham, 3425; A. S. Merrill, 3426; Paul Wernicke, 3432; A. Pelletier, 3432, 3433, 3434, 3436, 3437, 3438; Raymond Graves, 3434; H. A. Meyer, 3434; E. M. Berry, 3434, 3437; Milton Weinberger, 3434, 3437; J. W. Clawson, 3436, 3440; A. E. Johnson, 3436, 3437; and J. H. Neelley, 3440.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Wesleyan University has conferred an honorary doctorate of science on Professor Henry S. White of Vassar College.

Professor E. R. Hedrick, of the University of California at Los Angeles, has been appointed chairman of a committee on the teaching of mathematics in the colleges and universities of the North Central Association. Members of this committee include Professor H. E. Slaught of the University of Chicago, and Professor Dunham Jackson, of the University of Minnesota.

At the ninetieth meeting of the American Association for the Advancement of Science, which was held in Syracuse from June 20 to June 25, the following addresses, papers, and report of interest to mathematicians were presented:

A general session of the meeting on the teaching of mathematics was addressed by Professor W. B. Carver of Cornell University, Editor-in-chief of this Monthly, and by Professor E. R. Hedrick. Professor Carver pointed out some of the contributions of mathematics to science, and indicated how certain branches of mathematics which at first interested only mathematicians became later useful tools *for physicists and engineers*. He also emphasized the need for inspired science text-books and teachers of mathematics. Professor Hedrick made a plea for closer cooperation between mathematicians and educationists, and urged that state laws be passed requiring teachers of science to have had at least one college course in the science they teach.

Papers were read before Section A (mathematics) by Professors H. M. Gehman of the University of Buffalo, W. A. Hurwitz of Cornell University, and J. A. Shohat of the University of Pennsylvania. Professor Gehman developed several theorems on homeomorphic geometry, Professor Hurwitz discussed logical foundations for groups and fields, and Professor Shohat considered a number of problems in interpolation.

Section A met with section K (social and economic sciences) for a session on

mathematical statistics. In a paper by Professor C. F. Roos it was shown how the effect of past prices and the memory of past purchases lead to a demand function containing an integral either of past prices or past purchases, the two forms being equivalent. A report on mathematics needed in the social sciences, adopted by the Social Science Research Council after the deliberations of a committee, was read by Doctor Mordecai Ezekiel. Sections A and K (with the affiliated Econometric Society) of the Association approved the recommendations in this report that in preparation for work in the social sciences undergraduates should study logarithms, graphic methods, interpolation, equations and forms of simple curves, probability, and the elements of differential and integral calculus and curve fitting.

The following names were omitted from the list of doctorates conferred during 1931 as published in the June-July issue of the MONTHLY:

O. W. Albert, Washington (Seattle), June, minor in astronomy, *Relations between the projective and metric differential geometries of surfaces.*

T. C. Bumer, Ohio State, August, *Dynamical systems with hysteresis effects of the Fredholm type.*

Richard S. Burington, Ohio State University, June, *An invariantive classification of plane cubic curves under the affine group.*

L. S. Kennison, California Institute, June, minor in mathematical physics, *Linear functional equations on a composite range.*

Professor Oliver Dimon Kellog of Harvard University, a charter member of the Association, died suddenly while mountain climbing near Greenville, Maine, August 27, 1932, at the age of 54 years. He was connected with the department of mathematics at the University of Missouri, 1905–1920, and at Harvard University, 1920 to the time of his death. A complete statement of his scientific career will appear in due time.

Professor Ernest Julius Wilczynski died in Denver, Colorado, on September 15, 1932 at the age of 56. He had been incapacitated for active service since the summer of 1923. He was a charter member of the Mathematical Association of America and a loyal supporter of its aims and activities, as vice-president, council member, and contributor to the MONTHLY. He had been connected with the department of mathematics in the University of California, 1898–1907, University of Illinois, 1907–1910, and the University of Chicago, 1910 to the time of his death, with the title of Professor Emeritus of Mathematics in recent years.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Seventeenth Annual Meeting of the Association, Atlantic City, N.J., Dec. 27-30, 1932.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1932 and reported to the Secretary.

ILLINOIS, Urbana, May 6-7.
INDIANA, Indianapolis, May 6-7.
IOWA, Cedar Falls, April 29-30.
KANSAS, Topeka, Feb. 13.
KENTUCKY, Lexington, May.
LOUISIANA-MISSISSIPPI, Oxford, Miss., March 11-12.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Md., Dec. 3.
MICHIGAN, Ann Arbor, March 19.
MINNESOTA, River Falls, Wis., May 7.
MISSOURI.
NEBRASKA, Omaha, May 6-7.
OHIO, Columbus, Ohio, April 7.
PHILADELPHIA, Philadelphia, Pa., Nov. 26.
ROCKY MOUNTAIN, Laramie, Wyo., April 15-16.
SOUTHEASTERN, Gainesville, Fla., Mar. 18-19.
SOUTHERN CALIFORNIA, San Diego, March 26.
TEXAS, Austin, Jan. 30.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS.
THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXIX, 1932

NUMBER 9, NOVEMBER

PUBLISHED BY THE ASSOCIATION

MENASHA, WIS., AND ITHACA, N.Y.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members

$5.00 a Year, Single Copies 60 cents, to Others

THE SIXTEENTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The sixteenth summer meeting of the Mathematical Association of America was held, by invitation, at the University of California at Los Angeles on Monday and Tuesday, August 29 and 30, 1932, in conjunction with the summer meeting and colloquium of the American Mathematical Society. Two hundred and ten were present at the meetings, including the following ninety-two members of the Association:

Beatrice Aitchison, Johns Hopkins University
O. W. Albert, University of Redlands
L. D. Ames, University of Southern California
Clara L. Bacon, Goucher College
J. P. Ballantine, University of Washington
M. A. Basoco, University of Nebraska
Harry Bateman, California Institute of Technology
May M. Beenken, State Teachers College, Oshkosh, Wisconsin
Clifford Bell, University of California at Los Angeles
E. T. Bell, California Institute of Technology
B. A. Bernstein, University of California
R. W. Bolton, Glendale, Calif.
J. L. Botsford, California Institute of Technology
W. A. Bratton, Whitman College
A. L. Buckman, University of California
J. H. Bushey, Hunter College
Jessie R. Campbell, Hollywood Junior College
W. B. Carver, Cornell University
A. H. Clifford, California Institute of Technology
Myrtie Collier, Los Angeles
R. H. Conlee. Fairfield, Iowa
Lennie P. Copeland, Wellesley College
N. A. Court, University of Oklahoma
P. H. Daus, University of California at Los Angeles
G. G. Entz, Hollywood, Calif.
H. J. Ettlinger, University of Texas
Raymond Garver, University of California at Los Angeles
Harriet E. Glazier, University of California at Los Angeles
F. L. Griffin, Reed College
Lois W. Griffiths, Northwestern University
W. L. Hart, University of Minnesota
E. R. Hedrick, University of California at Los Angeles
J. D. Hill, Brown University
Jewell C. Hughes, Hunter College
G. H. Hunt, University of California at Los Angeles
J. W. Hurst, Montana State College
C. A. Hutchinson, University of Colorado
M. H. Ingraham, University of Wisconsin
Dunham Jackson, University of Minnesota
C. G. Jaeger, Pomona College
Glenn James, University of California at Los Angeles
C. M. Jensen, Macalester College
Mary N. Keith, University of Redlands
D. H. Lehmer, Altadena, Calif.
D. N. Lehmer, University of California
H. B. Leonard, University of Arizona
Jack Levine, Princeton University
G. R. Livingston, State Teachers College, San Diego, Calif.
Deca Lodwick, High School, Long Beach, Calif.
Mayme I. Logsdon, University of Chicago
Ada A. McClellan, High School, Long Beach, Calif.
W. H. McEwen, Mount Allison University
James McGiffert, Rensselaer Polytechnic Institute
Ruth G. Mason, Berkeley, Calif.
W. E. Mason, University of California at Los Angeles
Kate M. Meek, High School, Pasadena, Calif.
A. D. Michal, California Institute of Technology
E. L. Mickelson, State Teachers College, Silver City, New Mexico
W. E. Milne, University of Oregon
C. V. Newsom, University of New Mexico
W. B. Orange, Los Angeles Junior College
F. W. Owens, Pennsylvania State College

MRS. F. W. OWENS, State College, Pa.
T. S. PETERSON, Los Angeles, Calif.
E. J. PURCELL, Cornell University
TIBOR RADÓ, Ohio State University
LENA E. REYNOLDS, Junior College, Fullerton, Calif.
J. F. RITT, Columbia University
J. M. ROBB, Junior College, Taft, Calif.
A. A. SHAW, University of Arizona
G. E. F. SHERWOOD, University of California at Los Angeles
H. M. SHOWMAN, University of California at Los Angeles
MARCUS SKARSTEDT, Whittier College
L. L. SMAIL, Lehigh University
CLARA E. SMITH, Wellesley College
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CLYDE WOLFE, California Institute of Technology
EUPHEMIA R. WORTHINGTON, University of California at Los Angeles

The meetings were held on the campus of the University of California at Los Angeles. The majority of visiting mathematicians were housed in Mira Hershey Hall and had their dining room and social rooms there, thus making possible more personal contacts than could otherwise be attained. The committee on arrangements planned carefully in advance and due to the efforts of the local members of the committee, Professors E. R. Hedrick and W. M. Whyburn, the details of registration and service throughout the week were carried out smoothly.

Vice President and Provost Ernest C. Moore was host at a luncheon on Tuesday in Kerckhoff Hall on the campus. The same afternoon a tea was held in the lounge of Kerckhoff Hall in honor of the visiting ladies.

The joint dinner was held on Wednesday evening at the Beverly Hills Hotel. Professor E. T. Bell acted as toastmaster. Professor Hedrick welcomed the visitors to Southern California and Professor Ingraham spoke for the Society. Miss Jewell C. Hughes and Professor W. L. Hart contributed in a lighter vein.

The excursion to Pasadena and Mount Wilson was held Thursday afternoon and evening. After luncheon at the Athenaeum at the California Institute of Technology, there were exhibits given at the high tension laboratory and the aeronautical laboratory before the journey up the mountain. The staff of the Mount Wilson Observatory most graciously exhibited both the 100 inch and the 60 inch telescopes.

The American Mathematical Society held its thirty-eighth summer meeting and sixteenth colloquium from Tuesday to Friday, the colloquium lectures being given by Professor J. F. Ritt of Columbia University on "Differential equations from the algebraic standpoint." On Wednesday morning Professor D. N. Leh-

mer of the University of California gave an address by invitation on "The continued fraction representing cubic and higher irrationalities," and on Thursday morning Professor Tibor Radó of Ohio State University gave an address by invitation on "Recent work in the problem of Plateau." Sessions for the reading of papers were held on Tuesday and Wednesday afternoons and Friday morning.

The Mathematical Association held sessions Monday afternoon and Tuesday morning, President E. T. Bell presiding at both sessions. The Association is under obligations to the committee consisting of Professors W. M. Whyburn (chairman), E. T. Bell and R. M. Winger for the preparation of the instructive program. Abstracts of some of the papers follow, numbered in accordance with the numbers of the papers. Due to a cloudburst in the mountains, Professor C. A. Hutchinson failed to arrive in time for the meetings and his paper was read by title.

FIRST SESSION OF THE ASSOCIATION

1. "Solid college geometry" by Professor N. A. Court, University of Oklahoma.

2. "The master's thesis" by Professor Mayme I. Logsdon, University of Chicago.

3. "Graeffe's method" by Professor C. A. Hutchinson, University of Colorado.

4. "Collegiate mathematics needed in the social sciences." A report of a committee of the Social Science Research Council, read by H. R. Tolley, Director of the Giannini Foundation of Agricultural Economics, University of California.

1. Professor Court made a plea for a course in synthetic solid geometry such as he is now giving to first year graduate students at the University of Oklahoma. He enumerated a number of relatively simple, but not commonly known, properties of the tetrahedron, which are readily proved by synthetic methods, these proofs having decided advantages over corresponding analytical proofs.

3. Professor Hutchinson pointed out that recent activity in various fields of applied science has renewed interest in the practical numerical solution of algebraic equations. The method due to Dandelin and Graeffe, published a century ago, is well adapted to the purpose. It has these advantages: (1) all roots, real and complex, are determined simultaneously; (2) the character of the roots appears early in the process used for their evaluation; (3) a calculating machine or multiplication tables can be used conveniently.

The process consists in deriving from the given equation a new one whose roots are high powers of the roots of the given equation. The roots of the transformed equation, being widely separated in absolute value, are then easily determined. Details of special cases are given in the paper.

4. This report will appear in full in an early number of this MONTHLY. The report called for a course including the following topics: logarithms, graphs (as a tool in the study of tabulated data), interpolation, equations and forms of

curves, probability, elements of differential and integral calculus, and curve fitting, these topics to be presented with particular reference to their application to the social sciences; these topics to be covered in a three-hour course running through two, or, at most, three semesters.

The discussion of this paper was quite enthusiastic and developed two different viewpoints, as indicated below in the abstract prepared by Professor James.

It was the sense of those speaking in behalf of the committee that such a selection of topics would decidedly shorten the time required to cover our ordinary courses extending through the elementary calculus. It was argued that experience in teaching brief courses in analysis justified this position. This viewpoint seemed to presuppose an intake ability on the part of the students, which if not constant is certainly not dependent upon the conventional sequences of topics.

On the contrary, there were those who felt that the whole question was qualitative rather than quantitative, that it was a question of developing sufficient mathematical maturity to digest the topics desired, that is, a question of growth which requires time. They pointed out that with our present curricula they usually found that students required more time than was allotted them to really understand differentiation, for instance. Attention was called to the fact that many topics omitted from the report were actually bound up theoretically with those selected, hence their deletions might actually increase the students' difficulties.

The attitude of the members of the association was to prepare these points as questions rather than as criticisms. All agreed that we should be receptive in this matter which was primarily the business of the social sciences.

Second Session of the Association

1. "Series of orthogonal polynomials" by Professor Dunham Jackson, University of Minnesota.

2. "Mathematical preparation for a student of modern physical science" by Professor Linus Pauling, California Institute of Technology, by invitation.

3. "On the foundations of commutative algebra" by Professor H. S. Vandiver, University of Texas.

1. Professor Jackson's paper is concerned with systems of polynomials orthogonal over an interval with respect to a given weight function, and in particular with the convergence of expansions in series of such polynomials, giving an account of such properties as are readily deducible by methods comparable with those used in elementary presentations of the theory of Fourier series. Convergence is treated on the basis of the Christoffel-Darboux formula under the assumption that the weight function is such that the polynomials $p_n(x)$ of the normalized orthogonal system are bounded with respect to n for a specified value or range of values of x, and for a broader class of weight functions on the basis of general theorems on approximation by means of polynomials, leading to an estimate of the order of magnitude of the coefficients.

3. Professor Vandiver's paper treats extensions of the theorem in abstract algebra which replaces the theorem in ordinary algebra to the effect that the polynomial $f(x)$ with given rational coefficients vanishes for at least one complex value of x. If $\phi(x)$ is a polynomial with rational coefficients irreducible in the rational field, then residue classes are defined, with respect to a modulus $\phi(x)$, which when combined by the operations addition, subtraction, multiplication and division give a field which is isomorphic with the field generated upon assuming the existence of a quantity α such that $\phi(\alpha)=0$. Indeterminates are explicitly defined and the above mentioned notions extended to algebraic rings.

Meeting of the Board of Trustees

A meeting of the board of trustees was held on Monday evening.

The following twenty-four persons were elected to membership on applications duly certified:

S. R. Baker, A.B.(Ursinus) York, Pa.

Jessie W. Boyce, A.M.(Minnesota) Instr., State Teachers Coll., Wayne, Nebr.

B. R. Brick, B.S.(Minnesota) Instr., Washington Univ., St. Louis, Mo.

J. H. Butchart, Ph.D.(Illinois) Asst., Univ. of Illinois, Urbana, Ill.

A. H. Clifford, A.B.(Yale) Student, California Inst. of Tech.

G. S. Cook, A.M.(Kansas) Instr., State School of Mines, Rapid City, S. D.

L. L. Cruise, M.S.(Kansas State Coll., Fort Hays) Teacher, Jr. Coll., Rochester, Minn.

P. S. Donchian, A.B. (Yale) Vice-President and Genl. Mgr., Donchian Rug Co., Hartford, Conn.

E. A. Drago, B. S.(Cooper Union) Mariners Harbor, Staten Island, N. Y.

J. N. Eastham, Ph.D.(Catholic Univ.) Head of Dept., Nazareth Coll., Rochester, N. Y.

O. C. Edwards, M.E.(Pennsylvania State Coll.) Asst. Prof., Mech. Eng., Genl. Extension Div., Univ. of Minnesota, Minneapolis, Minn.

M. C. Erwin, Head of Dept., High School, Reynolds, Ind.

R. E. Gadske, M.S.(Northwestern) Teacher, High School, Carbondale, Ill.

Sister Mary Aloysius, A.M.(Chicago). Instr., Coll. of St. Catherine, St. Paul, Minn.

Sister Mary Felice, Ph.D.(Catholic Univ. of America) Prof., Mount Mary Coll., Milwaukee, Wis.

D. B. Perry, A.B.(Stanford) Grad. Student, Stanford Univ.; Menlo Park, Calif.

L. J. Quaid, B.S. in E.E.(Illinois) Instr. in Drawing, Univ. of Minnesota, Minneapolis, Minn.

K. E. Rosinger, Ph.D.(Harvard). Research Fellow, Harvard Univ., Cambridge, Mass.

R. F. Schnepp, Ph.D.(Fribourg) Prof. and Head of Dept., St. Mary's Univ. of San Antonio, San Antonio, Texas.

C. N. Stokes, Ph.D.(Minnesota) Asst. Prof., Education; Head of Math. Dept., Univ. High School, Minneapolis, Minn.

A. J. Strane, Eng. of Mines(Minnesota) Head of Dept., Duluth Jr. Coll., Duluth, Minn.

Helen E. Ternandt, B.S.(Northwestern) Chicago, Ill.

Beatrice M. Velten, B.Sc.(New Jersey Coll. for Women) Actuarial Clerk, Metropolitan Life Ins. Co., Rahway, N. J.

E. B. Worthington, A.M.(Franklin and Marshall) Instr., Franklin and Marshall Academy, Lancaster, Pa.

The trustees voted (1) to authorize the business manager to sign the revised contract for printing the Monthly; (2) to approve the addition of J. H. Weaver to the Board of Associate Editors to have charge of advertising matters and also editing the Department of News and Notices; (3) to appoint as a Committee on Arrangements for the summer meeting in Chicago for 1933, Professors H. S.

Everett, E. J. Moulton, Mark Ingraham, and as a Committee on Program, Professors W. D. Cairns, Mayme I. Logsdon and H. W. March, with instructions to report at the Christmas meeting of the Association; (4) to appoint a Committee on Arrangements for the Christmas meeting at Atlantic City, a Committee on Program, and a nominating committee for the coming election.

P. H. DAUS, *Acting Secretary*

THE SPRING MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual (Spring) meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the University of Maryland on Saturday, May 7, 1932. About eighty attended.

The following forty-one members were present: Beatrice Aitchison, G. F. Alrich, Ethel M. Anderton, R. M. Ashmun, G. A. Bingley, Archie Blake, C. C. Bramble, W. D. Cairns, Paul Capron, Tobias Dantzig, Alexander Dillingham, J. A. Duerksen, P. F. Federico, Michael Goldberg, Patricia Gosnell, Harry Gwinner, W. M. Hamilton, F. E. Johnston, L. M. Kells, A. E. Landry, C. L. Leiper, G. A. Lyle, Florence M. Mears, F. D. Murnaghan, O. J. Ramler, C. H. Rawlins, J. N. Rice, H. M. Robert, Jr., R. E. Root, J. B. Scarborough, Maurice Scheier, J. T. Spann, T. H. Taliaferro, J. H. Taylor, Mildred E. Taylor, F. M. Weida, Paul Wernicke, J. W. Williamson, E. W. Woolard, R. C. Yates, Oscar Zariski. These represented the following institutions: Goucher College, Johns Hopkins University, St. John's College, University of Maryland, U. S. Naval Academy and Postgraduate School, Mary Baldwin College, Catholic University, George Washington University, U. S. Coast and Geodetic Survey, International Boundary Commission, Bureau of Ordnance, U. S. N., U. S. Naval Observatory, U. S. Patent Office, Oberlin College.

The meeting was distinguished by the presence of Professor Richard Courant of Göttingen and Secretary Cairns. The morning session opened with an address of welcome from President Pearson of the University of Maryland and a message of good will and coöperation from Secretary Cairns of the Association. Chairman Woolard presided. The afternoon session opened with an address by Professor T. H. Taliaferro, Dean of the College of the University of Maryland.

At the business meeting it was voted to hold the next meeting of the Section at Johns Hopkins University on the first Saturday of next December. Officers were elected for the ensuing year as follows: Chairman, Paul Capron, U. S. Naval Academy; Secretary, F. M. Weida, George Washington University; other members of the Executive Committee: Tobias Dantzig, University of Maryland and G. T. Whyburn, Johns Hopkins University.

The following papers were presented.

At the morning session:

1. "A second note on the celestial sphere" by Professor Frank Morley, by invitation.

most are real, while the other three are pure imaginary. Eddington's proof is strictly algebraic. The author gives a geometric and a shorter proof of Eddington's theorem, by making use of known facts concerning abelian groups of involutory collineations in space.

With each matrix E_i there is associated a collineation Γ_i in the projective complex space, and *vice versa* Γ_i determines the matrix E_i to within a factor. The condition $E_i^2=-1$ says that $\Gamma^2=1$; that is, Γ_i is involutory, and therefore is either an harmonic homology or an harmonic biaxial collineation. The condition $E_iE_j=-E_jE_i$ says that $\Gamma_i\Gamma_j=\Gamma_j\Gamma_i$. The n collineations Γ_i belong to an abelian group G of involutory collineations. The abelian group cannot be the tetrahedral group, because the matrices of the collineations of this group can be reduced simultaneously to the diagonal form and therefore are not anticommutative. Therefore the group G is the group G_{16} of biaxial harmonic collineations, associated with a quadric surface Q. If Q is given in the canonical form $x_1x_4-x_2x_3=0$, it appears that the matrices E_i are direct products of two two-row matrices, and this representation of the matrices E_i leads immediately to the proof of Eddington's theorem.

7. Definitions with illustrations were given of open and closed sets, of the notions of connectedness and local connectedness, compactness and separability, of simple continuous arcs, of G_δ sets and of accessibility and of regular accessibility. With these as a basis, a very general condition for regular accessibility was stated. A necessary and sufficient condition that a point P be regularly accessible from a connected, locally connected G_δ set G, is that the set $G+P$ be connected and locally connected. Most of the known theorems on accessibility can be shown to be special cases of this proposition.

8. In the representation of the lines of space by the Grassmanian hyperquadric V_4^2 in S_5, the lines of a congruence are given by the points of a surface F on V_4^2. The study of the surfaces of V_4^2 yields results on the congruences of lines in space. Thus the congruences of bisecants to rational space curves are represented by the well known surfaces of Veronese. Translating the known properties of these surfaces into line geometry of space, we obtain the more important properties of these congruences.

PAUL CAPRON, *Secretary*

THE SIXTEENTH ANNUAL MEETING OF THE KENTUCKY SECTION

The sixteenth annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky on Saturday, May 14, 1932. Ill health during the year compelled Professor Smith Park, Eastern State Teachers College, to resign as chairman of the Section on April 6, 1932. Dean P. P. Boyd, University of Kentucky, kindly consented to act as chairman, and members quickly responded to the call for an emergency program.

The attendance was forty-eight including the following twenty members of the Association: N. B. Allison, W. E. Baxter, P. P. Boyd, L. W. Cohen, J. M. Davis, A. R. Fehn, Charles Hatfield, S. F. Hendricks, W. R. Hutcherson, E. R. Keller, C. G. Latimer, Elizabeth LeStourgeon, Buena C. Mathias, W. L. Moore, Sister Charles Mary Morrison, Sallie Pence, D. W. Pugsley, E. L. Rees, Guy Stevenson, H. M. Yarbrough.

The Secretary was instructed to write a letter of deep regret and sympathy to Professor Smith Park. Professor Fehn reported that the Kentucky Section of the Mathematical Association received its "charter" as an affiliated member of the National Council of Teachers of Mathematics, May 1, 1932. The purpose in affiliating with the National Council is to make contact with teachers of secondary and elementary mathematics to further the interests of mathematics in the State. The committee, consisting of P. P. Boyd, chairman, Guy Stevenson and H. M. Yarbrough, to consider ways and means of organizing a branch of the National Council was continued. The following officers were elected for the coming year: Chairman, Charles Hatfield, Georgetown College; Secretary, A. R. Fehn, Centre College.

The following program was presented:

1. "The triangles in-and-circum-scribed to the biflecnodal rational quartic" by Sister Charles Mary Morrison, Nazareth College.
2. "A group of transformations" by Professor C. G. Latimer, University of Kentucky.
3. "A simple problem in mapping" by Professor Guy Stevenson, University of Louisville.
4. "A theorem concerning perfect points in algebraic geometry" by Professor W. R. Hutcherson, Berea College.
5. "The problem of coloring maps" by Professor L. W. Cohen, University of Kentucky.
6. "Problems involving linkages" by Sallie E. Pence, University of Kentucky.
7. "Experiments in laboratory mathematics" by Professor W. L. Moore, University of Louisville.
8. "How to reduce failures in freshman mathematics" by Professor M. C. Brown, University of Kentucky, by invitation.

Abstracts of three of the papers follow:

1. Four possible contours for a biflecnodal rational quartic may be imagined, namely, the form with three crunodes, with three acnodes, with two crunodes; and one crunode. We consider the four mentioned types to discover in each case where the curve proves to be real, the nature of the possible eight triangles. The study shows that

(1) a quartic with three acnodes cannot have a biflecnode.

(2) a rational quartic with a biflecnode cannot have a real in- and circumscribed triangle.

We then depart from the symmetry of the curve and consider a quartic with merely a flecnode and show that a real in- and circumscribed triangle may be obtained.

3. The three distinct points, z_1, z_2, z_3, in the complex z-plane are mapped into the three distinct points, w_1, w_2, w_3, in the w-plane by means of the transformation

$$\frac{(w - w_3)(w_1 - w_2)}{(w - w_1)(w_3 - w_2)} = \frac{(z - z_3)(z_1 - z_2)}{(z - z_1)(z_3 - z_2)},$$

which may be written in the form

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}, \tag{1}$$

where

$$\begin{aligned}
\alpha &= w_1(z_1 - z_2)(w_2 - w_3) - w_3(w_1 - w_2)(z_2 - z_3),\\
\beta &= z_1 w_3(z_2 - z_3)(w_1 - w_2) - w_1 z_3(z_1 - z_2)(w_2 - w_3),\\
\gamma &= (z_1 - z_2)(w_2 - w_3) - (w_1 - w_2)(z_2 - z_3),\\
\delta &= z_1(z_2 - z_3)(w_1 - w_2) - z_3(z_1 - z_2)(w_2 - w_3).
\end{aligned}$$

The point $z = g \equiv -\delta/\gamma$ is outside the circle c_1 passing through the points z_1, z_2, z_3, if the order of these points around c_1 is the reverse to that of w_1, w_2, w_3, around the circle c_2 passing through these points, and consequently all points outside c_1 are mapped into points outside c_2, and all points inside c_1 are transformed into points inside c_2. If the order of the points on c_1 is the same as that on c_2 then g is inside c_1 and then all points inside (outside) c_1 are mapped into points outside (inside) c_2 by the transformation (1).

4. Let an algebraic surface be invariant under a cyclic birational transformation T of period n. A point and its images under T constitute an involution. When this involution has only a finite number of fixed points, they are singular points on the surface. The configuration of the surface near each such point is discussed, and it was proved that if $n=2$, every fixed point is a perfect point. (For a definition of perfect point see W. R. Hutcherson, Maps of certain cyclic involutions on two dimensional carriers, Bulletin of the American Mathematical Society, volume 37, 1931, pp. 759–765.)

A. R. FEHN, *Secretary*

NOTATION OF DECIMAL FRACTIONS IN BOHEMIA

By QUIDO VETTER, Carolina University, Prague

In the year 1923 the late Professor Florian Cajori asked me why it is that in Austria and Bohemia the fractional part of a decimal fraction is separated from the integral part by a raised point, while in other parts of the Continent an ordinary period, or more often a comma, is used. Professor Cajori was then writing his valuable History of Mathematical Notations and his inquiry was

an evidence of the care he took in his work. I am sorry to say, however, that I was at that time prevented by other work from investigating the matter with the care that it deserved, and it is only recently that I have been able to consider it.

Few scientific works on mathematics were written in the Czech regions in the 17th century. Moreover, there was no reason why decimal fractions should get into such arithmetics as appeared, for the measures and weights as well as the coinage were not on a decimal scale, with the exception of those used by surveyors. It is for this reason that Wácslav Joseff Wessel ý, in his work *Gruntownj Počátek Mathematického Vmĕnj* (The Fundamental Beginnings of the Mathematical Art, Prague, 1734, p. 57), in speaking of measures of lengths separates the integers from their "brychy" or "shorter partes" by a period (.) and then places an arc before the right-hand figure, thus: 7.563(3, which stands for 7 rods, 5 feet, 6 inches, 3 grains. It is therefore apparent that the period is not looked upon as a decimal point, but simply as a separatrix of various units. For example, he writes 51.26.42(2 when giving a measure of an angle, meaning what we should express as $51°26'42''$.

Father Josef Stĕpán Schmidt (1720–1783) gave three ways of writing a decimal fraction. In his *Tabulae mathematicae* (Prague 1757, p. 38) he writes $6.4'.5.''3'''$, or $6453'''$, or 6,453 to mean what in the United States is expressed by 6.453.

Finally Father Fr. Zeno, a Jesuit, in his *Elementa algebrae, geometriae, trigonometriae* (Prague, 1769, p. 37), writes the decimal fraction 3.879 as $3°8'7''9'''$.

A new movement in the scientific life of Prague was developed in the 18th century by certain Liberals who formed a private society later known as Royal Society of Arts. This society published semiannually the Gelehrte Nachrichten, and later the Abhandlungen einer Privatgesellschaft in Böhmen zur Aufnahme der Mathematik, vaterländischen Geschichte und Naturwissenschaften, Abhandlungen der böhmischen Gesellschaft der Wissenschaften in Prag, Abhandlungen der königlichen böhmischen Gesellschaft der Wissenschaften, and various other papers and proceedings and a Gazette.

These publications show that the decimal fraction was indicated by a comma at the end of the 18th century as well as in the first third of the 19th century. Only in the Abhandlungen for 1780, in one of the tables we find the form 5.70, while in another volume (1785) of the same publication, A. Ferkeles uses 0.04347826086. In the second volume of *Philosophiae naturalis principia mathematica*, by Newton, which was published 1785 by the most prominent Czech mathematician of the time, Father Jan Tesánek, S. J. (1728–1788), there occurs in a table (p. 236) the form 0.000001, and a similar usage is seen in a work by Father Stanislav Vydra, S.J., (1741–1804), on p. 96 of his *Počátkové Arytmetyky* (Beginnings of Arithmetics) published by Lad. Jandera in the year 1806. In this work the author gives a systematic treatment of decimal fractions, stating that 6789,2345 may be written 6789.2345, but he does not make any further use of this symbol. He states that decimal fractions were used by sur-

veyors for the reason that their measures were divided decimally. Of the other books dealing with the same subject we may mention two works of Father Lad. Jos. Jandera (1776–1857), a monk in the Strahov monastery, his *Prima calculi exponentialis elementa* (Prague, 1817) where he uses (p. 29) the form 0,6666, and his *Beiträge zu einer leichten und gründlichen Behandlung einiger Lehren der Arithmetik* (Prague, 1830), as on p. 179, where he writes 428,5000000. Mention may also be made of the *Mémoire sur la dispersion de la lumière*, by A. L. Cauchy, published in Prague by the Czech Royal Society in the year 1836, where we find (p. 62) the number 1,330935.

This notation got into the practical life, as we see in a handbook for traders, written by J. Gunz, F. C. Nelkenbrecher's *Taschenbuch der neuesten Münz- Maass- und Gewichtsverfassung aller Länder und Oerter*, ihrer Wechselarten, Usi, Respettage, öffentlichen Banken, Messen und andrer zur Handlung gehörigen Anstalten und Gegenstände (Prague, 1809) where a decimal is written as 30, 48.

At the end of the 18th century in Abhandlungen of the Czech Royal Society the decimal comma is used, as also in the works of Ant. Strnad, S.J., director of the Prague observatory, (1747–1799).

This symbolism continued to be used in the Abhandlungen up to 1850. The decimal comma was adopted by the first head-master of the Technical University in Prague, a well-known scientist, Fr. Jos. Gerstner (1756–1832), as is seen in his *Theorie der Wellen* (1804) and the *Über die ob. Wasserräder*, and by such scholars as Jos. Steinmann (1819), Jos. Jüttner (1833), K. Kreil (1798–1862), the director of the observatory in Prague, K. Fritsch (1812–1897), and Professor Fr. Ad. Petřina (1799–1855). On the other hand, in 1806 another Prague Astronomer, a Premonstrant, Al. David (1757–1836) used the period in his *Längenunterschied zwischen Prag und Breslau*, as in 6.64 (p. 74), but in his other works he adopted the usual continental symbol of a comma.

In the years 1830–1840 the decimal point began to appear again as a period, especially in the works of those authors who had spent some time in the Alpine countries. For example, Jak. Fil. Kulik (1793–1863), a professor at the University in Prague, used the period in his first study published in the Abhandlungen (1832), as did Al. David, Ad. Bittner (1777–1844), Professor Fr. Hyn. Kaj. Hallaschka, and Jos. J. Böhm (1807–1868), who spent some years in Innsbrück as a Professor of mathematics.

The year 1837 marks a new stage in the history of decimal symbols in Bohemia. At that time Professor Christian Doppler, who taught at the technical university in Vienna, was called to Prague. In his first essay, "*Versuch einer analytischen Behandlung etc.*," published in the Abhandlungen (1839) he used the raised period, as the English writers do at present, as in 16·66025. This symbol was popularized by another book, his *Arithmetik und Algebra* (Prague, 1844), in which he explained the use of decimal fractions, using forms like 0·543. His symbolism was adopted by Vil. Matzka (1798–1891), a Professor at the Prague Technical University, in his *Elementarlehre von den Logarithmen* (Prague, 1850), and by many other writers, especially by the authors of the

arithmetics for the elementary schools and of manuals for the secondary schools. Among these authors was Fr. Močník (1814–1892), who in his *Theorie der numerischen Gleichungen* (Vienna, 1839), used the comma, but in the second edition of his *Lehrbuch der Arithmetik für das Untergymnasium* (Vienna, 1851) we find the raised decimal points only, as in 17·7482. His *Počtářství praktické* (Practical Calculation, Prague, 1853) was translated into Czech by Ant. Skřivan, and the translator used the period, although the Italian edition of his algebra for the secondary schools, the *Trattato di algebra pel ginnasio superiore* (Vienna, 1854) by P. Magrini, we find the comma again. In the Czech versions I have found the point above until the appearance of the *Anleitung zum Rechnen für Unter-Realschulen in Kaisertum Oesterreich*, Mit eingeschalteter Terminologie in böhmischer Sprache (Prague, 1858).

In the meantime an important reform of currency was introduced in Austria and together with it the use of the raised point became common. A monetary convention having been concluded between Austria and the other countries of the German Confederation, a new currency was introduced in 1858. In the place of the conventional florin (Konventions-Gulden) which was equal 60 kreutzers, a new florin, equal to 100 kreutzers was introduced. The new money naturally made decimal fractions an essential feature of elementary education, the decimal point being usually the raised period. We find, however, in a booklet printed in 1858, the form 136,5 (5/10 = 50/100) with this note: "The first figure after the comma is a decimal fraction." In Matzka's *Bequemste Tafeln zur wechselweisen Umrechnung des alten und neuen österreichischen Geldes* (Prague, 1858) the author speaks of a "Decimalstrich" and "Decimalpunkt," and writes: "2˙78′6 = 2 Neugulden $78\frac{6}{10}$ Neukreutzer." An explanation of the new currency, including the operation with decimals, was published by Močník and was immediately translated into Czech as *Klíc k novému řádu mincovnímu*—(A Key to the New currency, Vienna, 1858) and similar works appeared at about the same time.

Since that time the decimal part of the fraction has been separated from the integral one by a point, usually written above, although occasionally other symbols were given, as in Jos. Soukup's arithmetic in (Prague 1869), where the author remarks, "Decimal fractions are marked off from their units by a point or better by an inverted decimal comma (Decimalpunkt, Decimalstrich)" that is, 555 and 55 hundredths of a florian is given as 555˙55 or 555′55.

To summarize the matter we see that at the end of the 18th century and at the beginning of the 19th century with us, as generally on the Continent, the decimals were separated from their integers by a comma below. In the years 1830–1840, the raised decimal point was introduced from Austria; and this symbol, under the influence of the centralized schools and the new currency, continued until the second half of the century.

A LAPLACIAN EQUATION

By E. T. BELL, California Institute of Technology

1. *Introduction.* For $n=2$, Laplace's equation in n dimensions

$$\nabla_n u = 0, \quad \nabla_n \equiv \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

has the operator ∇_n completely factorable in the field of complex numbers, so that the general solution for $n=2$ is readily found. For $n=3, 4$, ∇_n is not factorable in a (commutative) field, but is in quaternions; for $n>4$ it is not factorable in any linear associative algebra. Thus the usual method of solving $\nabla_2 u=0$ is incapable of generalization. The well known general solution of Whittaker for $n=3$ depends upon other considerations.

A similar situation in the arithmetic theory of forms suggests a generalization of $\nabla_2 u=0$ other than $\nabla_n u=0$, $n>2$, in which the procedure for $n=2$ goes through for any n. As a special case of such equations has recently received some attention[1] in connection with a certain hypergeometric equation of the third order, originally considered by Clausen,[2] we shall briefly discuss the most general extension of $\nabla_2 u=0$ possible of the type indicated. Humbert's equation is[3]

$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial y^3} + \frac{\partial^3 u}{\partial z^3} - 3\frac{\partial^3 u}{\partial x \partial y \partial z} = 0.$$

Although the equations discussed are of no apparent significance in physics (except possibly those of the second order), certain special instances of them, in particular those in which the operator is a norm, as is the case in Humbert's equation, have interesting connections with algebraic number fields.

The equations considered have constant coefficients. No similar discussion for equations with variable coefficients is possible, as the linear differential operators concerned do not then commute.

2. *Canonical form of equation.* In this section the summation convention of the tensor calculus is used: a repeated index, as i in $a_i b_{ij}$, denotes a summation over the values $1, \cdots, n (n>1)$ of i, $a_i b_{ij} = a_1 b_{1j} + \cdots a_n b_{nj}$. The c_{ij} are constants in the field of complex numbers; $|c_{ij}|$, $\neq 0$, is the determinant whose element in row i, column j is c_{ij}, and c^{ij} denotes the cofactor of c_{ij} divided by $|c_{ij}|$. Hence $c_{ij}c^{ik} = \delta_j^k$, where δ_j^k is a Kronecker delta, $\delta_j^j = 1$, $\delta_j^k = 0$, $j \neq k$.

Let $P(x_1, \cdots, x_n)$ be any homogeneous polynomial of degree n which is completely factorable in the form

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[1] Pierre Humbert, Atti del Congresso Internazionale dei Matematiche, (VI), 1928, vol. 3, pp. 53–57.

[2] Crelle's Journal, vol. 3, p. 89.

[3] The solution of this equation occurs incidentally in Forsyth's Treatise on Differential Equations, p. 449, Ex. 4.

$$P(x_1, \cdots, x_n) = \prod_{j=1}^{n}(c_{ji}x_i).$$

The equation to be discussed is

$$(1) \qquad P\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)u = 0.$$

Transforming to new variables $\xi^j(j=1, \cdots, n)$, we write $\xi^j = a^{ji}x_i$, where j, i in ξ^j, a^{ji} are superscripts, and obtain

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial \xi^j}\frac{d\xi^j}{dx_i}, = a^{ji}\frac{\partial}{\partial \xi^j}.$$

Hence, if

$$\frac{\partial}{\partial \xi^j} \equiv c_{ji}\frac{\partial}{\partial x_i},$$

we have

$$\frac{\partial}{\partial \xi^j} = c_{ji}a^{ki}\frac{\partial}{\partial \xi^k}, \quad c_{ji}a^{ki} = \delta_j^{\,k},$$

and therefore

$$c^{jl}c_{ji}a^{ki} = c^{jl}\delta_j^{\,k}, \quad a^{kl} = c^{kl}.$$

Thus (1) is transformed to

$$(2) \qquad \frac{\partial^n u}{\partial \xi^1 \cdots \partial \xi^n} = 0, \quad \xi^j \equiv c^{ij}x_i.$$

Denote by ξ_i the vector of $n-1$ components obtained from $(\xi^1, \cdots, \xi^n)$ by suppressing ξ^i. The general solution of (1) is, by (2),

$$(3) \qquad u = f_1(\xi_1) + \cdots + f_n(\xi_n),$$

where $f_1, \cdots, f_n$ are arbitrary functions possessing derivatives of the orders demanded by (2).

3. *Particular polynomial solutions.* We shall consider only polynomials in one variable. It is readily seen that each step in what follows can be suitably modified to provide polynomial solutions in more than one variable.

By (3), any constant multiple of $\log \xi^1 + \cdots + \log \xi^n$ is a solution of (1). Hence, and by the linearity of (1), the following are solutions

$$\frac{\partial^s}{\partial x_1^{\,s}}V(x_1, \cdots, x_n) \quad (s = 0, 1, 2, \cdots),$$

where

$$V(x_1, \cdots, x_n) \equiv \log Q(x_1, \cdots, x_n), \quad Q(x_1, \cdots, x_n) \equiv \prod_{j=1}^{n}(c^{ij}x_i).$$

Without loss of generality we may assume the coefficient of the highest power of a particular variable, say x_1, to be 1. In what follows we consider V, Q particularly as functions of x_1, and indicate this by writing $V(x_1)$, $Q(x_1)$.

To obtain polynomial solutions connected with an ordinary linear equation, we use a device which Humbert attributes to Giulotto. From the above, the functions

$$V_s(x_1) \equiv \frac{1}{s!}\left[\frac{\partial^s V(x_1+h)}{\partial h^s}\right]_{h=0} \quad (s = 0, 1, \cdots)$$

are solutions of (1), and

$$V(x_1 + h, x_2, \cdots, x_n) = \sum_{s=0}^{\infty} h^s V_s(x_1, \cdots, x_n).$$

Further, we have

$$Q(x_1 + h) = \sum_{j=0}^{r} \frac{h^j}{j!} Q_j(x_1),$$

where r is the degree of $Q(x_1, \cdots, x_n) \equiv Q(x_1)$ in x_1, and $Q_j(x_1)$ denotes the jth derivative of $Q(x_1)$ with respect to x_1, so that $Q_0(x_1) = Q(x_1)$, $Q_r(x_1) = r!$. The functions $V_s(x_1)$ are now reduced to polynomials in x_1 by constraining $(x_1, \cdots, x_n)$ to an appropriate locus in n-space.

Let $j_1, \cdots, j_a$ be all those values of $j > 0$ for which $Q_j(x_1)$ is not independent of all of $x_2, \cdots, x_n$. Then the locus in question has as its equations

$$Q_0(x_1) = 1, \quad Q_{j_p}(x_i) = 0 \quad (p = 1, \cdots, a).$$

On this locus we have

$$Q(x_1 + h) = 1 + \sum_{j=1}^{r-a-1} h^{s_j} A_{s_j}(x_1) + h^r,$$

where the A's are polynomials in x_1 alone, and the s_j are such that $0 < s_j < r$, $s_i \neq s_j$ $(i \neq j)$. The $V_s(x_1)$ are now polynomials in x_1 alone, say $W_s(x_1)$, and their generating identity is

$$\log(1 + * + h^r) = \sum_{s=0}^{\infty} h^s W_s(x_1),$$

where the * denotes the sum on the right of the expansion for $Q(x_1+h)$ above.

4. *Remarks.* For Humbert's equation, the polynomials satisfy an equation of hypergeometric type of the third order. By the usual standard processes for obtaining the differential equation satisfied by functions whose generating identity is given, for example as in the case of spherical harmonics or Bessel functions, it is a simple matter to find the differential equation for the $W_s(x_1)$ for any given $P(x_1, \cdots, x_n)$. But it does not seem to be so simple to impose necessary and sufficient restrictions upon the general completely factorable P to give as the final equation one of hypergeometric type. It should be clear from general principles that the device outlined above will lead for Humbert's equation to a hypergeometric equation; possibly it is. But the complexity of the mere algebra in the classical papers (Pochhammer, Goursat, and others) on higher hypergeometric equations seems to preclude any simple general solution.

THE DIRECTION COSINES OF A p-SPACE IN EUCLIDEAN n-SPACE[1]

By STEWART S. CAIRNS, Lehigh University

1. *Analytic definition of the cosines.* Let $(y_1, \cdots y_n)$ be rectangular cartesian coordinates in euclidean n-space. A p-space, L_p, in the n-space may be defined by a set of equations

$$y_i = \sum_{j=1}^{p} a_{ij}u_j + y_i^0 \qquad (i = 1, \cdots, n) \tag{1.1}$$

where the u's are parameters and the matrix $\|a_{ij}\|$ is of rank p. We will refer to $\|a_{ij}\|$ as a *matrix of* L_p. The p-rowed determinants of this matrix will be called *direction components of* L_p and denoted thus:

$$\delta_{i_1\cdots i_p} = \begin{vmatrix} a_{i_11} & \cdots & a_{i_1p} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{i_p1} & \cdots & a_{i_pp} \end{vmatrix} \qquad (i_1 < i_2 < \cdots < i_p). \tag{1.2}$$

The following quantities will be called the *direction cosines of* L_p:

$$\gamma_{i_1\cdots i_p} = \frac{\delta_{i_1\cdots i_p}}{(\sum' \delta^2_{i_1\cdots i_p})^{1/2}}, \tag{1.3}$$

where $\sum'$ indicates a summation over all the sets $(i_p, \cdots i_p)$. L_p has $n!/[(n-p)!\,p!]$ direction cosines.

2. *A preliminary theorem on matrices.*

Lemma. *In the matrix* $\|a_{ij}\|$ *of* §1, *let*

$$A_{st} = \sum_{i=1}^{n} a_{is}a_{it} \qquad (s = 1, \cdots p;\ t = 1, \cdots p). \tag{2.1}$$

Then

$$\sum' \delta^2_{i_1\cdots i_p} = \sum''(-1)^{[j_1\cdots j_p]}(A_{1j_1} \cdots A_{pj_p}), \tag{2.2}$$

where $\sum'$ *has the same meaning as in equation* (1.3); $\sum''$ *indicates a summation over all permutations,* $(j_1, \cdots j_p)$, *of* $(1, \cdots p)$, *and* $[j_1, \cdots j_p]$ *denotes the number of inversions in* $(j_1, \cdots j_p)$.

By definition,

$$\sum' \delta^2_{i_1\cdots i_p} \equiv \sum''' \sum''(-1)^{[j_1\cdots j_p]}(a_{i_11}a_{i_1j_1})(a_{i_22}a_{i_2j_2}) \cdots (a_{i_pp})(a_{i_pj_p}), \tag{2.3}$$

where $\sum'''$ indicates a summation over all selections, $(i_1, \cdots i_p)$, of p distinct numbers in any order from the set $(1, \cdots n)$. By equation (2.1),

$$\begin{aligned} &\sum''(-1)^{[j_1\cdots j_p]}(A_{1j_1} \cdots A_{pj_p}) \\ &\qquad = \sum''(-1)^{[j_1\cdots j_p]} \sum{}^{\text{iv}}(a_{i_11}a_{i_1j_1}) \cdots (a_{i_pp}a_{i_pj_p}), \end{aligned} \tag{2.4}$$

[1] Presented to the American Mathematical Society, March 26, 1932.

where $\sum^{\text{iv}}$ indicates a summation as the i's run independently through the values $(1, \cdots n)$. The terms of (2.4) for which the i's are all distinct are precisely the terms of the expansion (2.3). Consider a term of (2.4) in which a particular subset of the i's have all the same value. By permuting the corresponding j's, we obtain a number of numerically equal terms, half with plus signs and half with minus signs. Hence the terms of the expansion (2.4) which are not in (2.3) add up to zero. This proves the lemma.

Theorem.[1] *If, in any matrix*

$$\|a_{ij}\| \quad (i = 1, \cdots n; j = 1, \cdots p),$$

we have

$$\sum_{i=1}^{n} a_{is}a_{it} = \begin{cases} 0 \text{ when } s \neq t \\ 1 \text{ when } s = t \end{cases} \quad (s = 1, \cdots p; t = 1, \cdots p), \tag{2.5}$$

then the sum of the squares of all the $n!/[(n-p)!p!]$ distinct p-rowed determinants of $\|a_{ij}\|$ is unity.

This follows at once from the lemma, for, in view of (2.5), the only non-zero term on the right in the expansion (2.2) is $A_{11} \cdots A_{pp} = +1$.

3. *A geometric interpretation of the cosines.*

Theorem. *Employing the notation of* §1, *L_p contains a line in some $(n-p)$-space normal to the coördinate $(y_{i_1} \cdots y_{i_p})$-space if and only if $\gamma_{i_1 \cdots i_p} = 0$. If $\gamma_{i_1 \cdots i_p} \neq 0$, let V denote a p-dimensional volume on L_p and $V_{i_1 \cdots i_p}$ the volume into which V projects on the $(y_{i_1} \cdots y_{i_p})$-space. Then*

$$\left| \gamma_{i_1 \cdots i_p} \right| = \frac{V_{i_1 \cdots i_p}}{V}. \tag{3.1}$$

The first part of the theorem may be easily verified analytically. For the second part, consider a matrix $\|a_{ij}\|$ of L_p which satisfies the hypothesis of the theorem[2] in §2. This condition on $\|a_{ij}\|$ permits us to regard the u's of equation (1.1) as rectangular cartesian coordinates on L_p with the same unit distance as the y-coordinates. Furthermore, by the theorem of §2, together with equation (1.3),

$$\gamma_{i_1 \cdots i_p} = \delta_{i_1 \cdots i_p}. \tag{3.2}$$

If V denote the volume of a p-dimensional region, R, on L_p, then

$$V = \int_{R'} \cdots \int du_1 \cdots du_p, \tag{3.3}$$

and if $V_{i_1 \cdots i_p}$ denote the volume of the projection, R', of R on the $(y_{i_1} \cdots y_{i_p})$-space, then

[1] For $n = p$, this is a known theorem on determinants. See, for example, Goursat's *Cours d'Analyse*, vol. I (1924), §51.

[2] Such a matrix is henceforth called a *normal matrix of L_p*.

$$V_{i_1\cdots i_n} = \int_R \cdots \int dy_{i_1} \cdots dy_{i_p}. \tag{3.4}$$

Now a point $(u_1, \cdots u_p)$ on L_p projects into the point on the $(y_{i_1} \cdots y_{i_p})$-space for which

$$y_i = \sum_{j=1}^{p} a_{ij}u_j + y_i^0 \qquad (i = i_1, \cdots, i_p). \tag{3.5}$$

Using equations (3.5) to define a change of variables in the integral of equation (3.4), we find, regarding volumes as positive,

$$\begin{aligned} V_{i_1\cdots i_p} &= \int_{R'} \cdots \int \left| \delta_{i_1\cdots i_p} \right| du_1 \cdots du_p \\ &= \left| \gamma_{i_1\cdots i_p} \right| V \end{aligned} \tag{3.6}$$

by equations (3.2) and (3.3). Since this result was obtained by means of a special parametric representation of L_p, it should be remarked, to complete the argument, that the direction cosines are independent[1] of the particular parametric representation.

4. *Identities among the direction cosines.* If $\gamma_{1\cdots p} \neq 0$, a matrix of L_p may be written in the form

$$\left\| \begin{matrix} a_0 & 0 & \cdots & 0 \\ 0 & a_0 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & a_0 \\ a_{p+1,1} & & \cdots & a_{p+1,p} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{n1} & & \cdots & a_{np} \end{matrix} \right\| \tag{4.1}$$

where a_0 has such a value that $\sum' \delta^2_{i_1\cdots i_p} = 1$ [See §1]. Then, by definition of the direction cosines,

$$\left\{ \begin{aligned} \gamma_{1\cdots p} &= a_0^p \\ \gamma_{i_1\cdots i_{p-k} j_1 \cdots j_k} &= (-1)^{[i_1\cdots i_{p-k} e_1 \cdots e_k]} a_0^{p-k} \left| a_{st} \right| \end{aligned} \right. \tag{4.2}$$

where (1) $(i_1 \cdots i_{p-k})$ is any selection of $(p-k)$ numbers, in order, from $(1, \cdots p)$; (2) $(j_1, \cdots j_k)$ is any selection of k numbers, in order, from $(p+1, \cdots n)$; and (3) $|a_{st}|$ is the kth order determinant of the a's where $s = j_1 \cdots j_k$ and t runs through the complement $(e_1 \cdots e_k)$, in order, of $(i_1 \cdots i_{p-k})$ with respect to $(1, \cdots p)$. In the above, k is any positive integer not greater than either p or $(n-p)$. Letting $k=1$, we find

[1] At most, a change of parameters can simultaneously change the signs of all the γ's. This follows from the definitions in §1.

$$(4.3)\qquad a_{st} = (-1)^{t-p} a_0^{1-p} \gamma_{1\cdots t-1, t+1 \cdots p, s} \qquad (t = 1, \cdots p;\ s = p+1, \cdots n).$$

Hence, substituting in (4.2), we obtain the identities[1]

$$(4.4)\qquad \gamma_{i_1 \cdots i_{p-k} j_1 \cdots j_k} = (-1)^{[i_1 \cdots i_{p-k} e_1 \cdots e_k]} \gamma_{1\cdots p}^{1-k} \left| (-1)^{t-p} \gamma'_{ts} \right|,$$

where

$$(4.5)\qquad \gamma'_{ts} = \gamma_{1\cdots t-1, t+1 \cdots p, s}$$

and $|(-1)^{t-p}\gamma'_{ts}|$ is the kth order determinant with the same range of values for t and s as in the determinant of equation (4.2). With the identity

$$(4.6)\qquad \sum{}' \gamma_{i_1 \cdots i_n}^2 = 1$$

we have a set of $n!/[(n-p)!p!] - p(n-p)$ independent identities among the direction cosines of a p-space in n-space. This is a complete set, for L_p has $n!/[(n-p)!p!]$ direction cosines, and there are $\infty^{p(n-p)}$ p-spaces through a point in n-space. If $\gamma_{1\cdots p} = 0$, a similar set of identities may be found by using, instead of $\gamma_{1\cdots p}$, any non-zero direction cosine of L_p.

We call attention here to the following readily verified facts:

(I) *If* $\gamma_{1\cdots p} \neq 0$, *the direction of* L_p *is uniquely determined by the* $p(n-p)+1$ *direction cosines* $\gamma_{1\cdots p}$; $\gamma_{1\cdots t-1,\ t+1 \cdots p, s}$ (*where* $t = 1, \cdots p$; $s = p+1, \cdots n$), *or by the corresponding direction components.* [See (4.2) and (4.3) above.]

(II) *Any set* $\delta_{1\cdots p} \neq 0$; $\delta_{1\cdots t-1,\ t+1\cdots p, s}$ (*where* $t = 1, \cdots p$; $s = p+1, \cdots n$) *of* $p(n-p)+1$ *numbers is a set of independent direction components of some* p-*space,* L_p. *The remaining direction components of* L_p, *and hence its direction cosines, are given by equations* (4.4), *read with* δ *in place of* γ.

5. *Conditions for parallelism and perpendicularity.* To illustrate the use of our direction cosines, we will give a complete discussion of parallelism and perpendicularity.

Our definition of direction cosines, if applied to lines in n-space, coincides with the usual definition. Let $(c_1, \cdots c_n)$ and $(c_1', \cdots c_n')$ be direction cosines of two lines, L_1 and L_1'. Then it is known that L_1 is parallel to L_1' if and only if $c_i = \epsilon c_i'$ $(i = 1, \cdots n)$ where $\epsilon = +1$ or -1, and that L_1 is perpendicular to L_1' if and only if

$$(5.1)\qquad \sum_{i=1}^{n} c_i c_i' = 0.$$

We will call a p-space, L_p, and a q-space, L_q $(q \leqq p)$, *parallel* if every line on L_q is parallel to some line on L_p. We will call them *perpendicular* if each line on either of them is perpendicular to every line on the other. Perpendicularity implies $p+q \leqq n$.

[1] These were found, for determinants, by K. Th. Vahlen, *Ueber die Relationen zwischen den Determinanten einer Matrix*, Crelle's Journal, vol. 112 (1893), pp. 306–310.

(I) *A necessary and sufficient condition*[1] *that* L_p *and* L_q *be parallel* ($q \leqq p$) *is that some* q *linearly independent lines on* L_q *be parallel to* q *lines on* L_p. *The statement still holds if we restrict the lines on* L_q *to be a system of rectangular axes.*

(II) *A necessary and sufficient condition that* L_p *and* L_q *be perpendicular is that each of* p *linearly independent lines on* L_p *be perpendicular to each of* q *linearly independent lines on* L_q. *The statement still holds if we restrict each set of lines to be a system of rectangular axes.*

(*A*) *Two* p-*spaces*, L_p *and* L_p', *with direction cosines* $(\gamma_{i_1 \cdots i_p})$ *and* $(\gamma'_{i_1 \cdots i_p})$ *respectively, are parallel if and only if*

$$\gamma_{i_1 \cdots i_n} = \epsilon \gamma'_{i_1 \cdots i_p} \tag{5.2}$$

where $\epsilon = +1$ *or* -1.

For, the condition in (I) may in this case be interpreted analytically to mean that some matrix of L_p is also a matrix of L_p' [cf. §1]. The resulting sets of direction cosines are therefore equal. The factor ϵ appears because the signs of all the direction cosines of a p-space can be simultaneously changed by a change of parameters.

(*B*) *A* p-*space*, L_p, *with direction cosines* $(\gamma_{i_1 \cdots i_p})$ *and a* q-*space*, $L_q(q<p)$, *with direction cosines* $(\gamma'_{i_1 \cdots i_q})$ *are parallel if and only if there exists a set* $(c_{i_1 \cdots i_{p-q}})$ *of direction cosines*[2] *such that*

$$\gamma_{i_1 \cdots i_p} = \sum (-1)^{[j_1 \cdots j_{p-q},\, k_1 \cdots k_q]} c_{j_1 \cdots j_{p-q}} \gamma'_{k_1 \cdots k_q} \tag{5.3}$$

where the summation is over all permutations $(j_1, \cdots j_{p-q};\ k_1, \cdots k_q)$ *of* $(i_1, \cdots i_p)$, *with* $(j_1, \cdots j_{p-q})$ *and* $(k_1, \cdots k_q)$ *each in numerical order.*

For, condition (I) is equivalent to the condition that the first q rows of some normal matrix [§3] of L_p should form a normal matrix of L_q. The last $(p-q)$ rows are then a normal matrix of some $(p-q)$-space, L_{p-q}, whose direction cosines $(c_{i_1 \cdots i_{p-q}})$ satisfy (5.3). This becomes clear if one expand the determinant $\delta_{i_1 \cdots i_p} = \gamma_{i_1 \cdots i_p}$ [See (3.2)] by the Laplace development[3] using complementary minors from the first q and the last $(p-q)$ columns.

(*C*) *A* p-*space*, L_p, *with direction cosines* $(\gamma_{i_1 \cdots i_p})$ *is perpendicular to an* $(n-p)$-*space*, L_{n-p}, *with direction cosines* $(\gamma'_{i_1 \cdots i_{n-p}})$ *provided*[4]

$$\gamma_{i_1 \cdots i_p} = \epsilon (-1)^{[i_1 \cdots i_p,\, j_1 \cdots j_{n-p}]} \gamma'_{j_1 \cdots j_{n-p}}, \tag{5.4}$$

where $\epsilon = +1$ *or* -1, *and* $(i_1 \cdots i_p, j_1 \cdots j_{n-p})$ *is a permutation of* $(1, \cdots n)$ *with* $(i_1 \cdots i_p)$ *and* $(j_1 \cdots j_{n-p})$ *each in numerical order.*

We may interpret the condition in (II) above to mean that there exists a

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[1] For substantiation of this statement and the following one, see P. H. Schoute, *Mehrdimensionale Geometrie* I, §§2, 3, 6.

[2] That is, a set of numbers satisfying the necessary relationships [cf §4] among the direction cosines of a $(p-q)$-space.

[3] See, for example, H. W. Turnbull, *The theory of determinants, matrices, and invariants* (1929), Ch. II, §5.

[4] Formally the same relationships hold between the homogeneous point and space coordinates of a $(p-1)$-space in $(n-1)$-space. See Weitzenböck, *Invariantentheorie* (1923), pp. 68–72.

normal matrix, $\|a_{i1} \cdots a_{ip}\|$, of L_p and a normal matrix, $\|b_{i1} \cdots b_{i,n-p}\|$, of L_{n-p}, such that the square matrix $\|a_{i1} \cdots a_{ip}\ b_{i1} \cdots b_{i,n-p}\|$ corresponds to a transformation to a new system of rectangular cartesian coordinates in our n-space [See §3]. The determinant of this matrix therefore has the value ± 1. Let it be expanded by the Laplace development using complementary minors of the first p and the last $(n-p)$ columns. This gives us a necessary condition for perpendicularity:

$$\sum(-1)^{[i_1\cdots i_p j_1 \cdots j_{n-p}]}\gamma_{i_1\cdots i_p}\gamma'_{j_1\cdots j_{n-p}} = \pm 1. \tag{5.5}$$

Another condition equivalent to (II) above is that we be able to write matrices of L_p and L_{n-p}, respectively, in the forms

$$\left\|\begin{matrix} 1 & 0\cdots 0 \\ 0 & 1\cdots 0 \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ 0 & 0\cdots 1 \\ A_{11} & \cdots A_{1p} \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ A_{n-p,1} & \cdots A_{n-p,p} \end{matrix}\right\| \quad \text{and} \quad \left\|\begin{matrix} A_{11} & \cdots A_{n-p,1} \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ A_{1p} & \cdots A_{n-p,p} \\ -1 & 0\cdots 0 \\ 0 & -1\cdots 0 \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ 0 & 0\cdots -1 \end{matrix}\right\|. \tag{5.6}$$

For, any two columns, one from each matrix, are direction components of two perpendicular lines, one on L_p, one on L_{n-p} [cf (5.1)]. The above matrix of L_p assumes only that $\gamma_{1\cdots p}\neq 0$, which involves no loss of generality. From (5.6) we could determine the direction components $(\delta_{i_1\cdots i_p})$ of L_p in terms of those of L_{n-p}, and hence deduce necessary and sufficient conditions for perpendicularity. It is somewhat simpler, however, to note from (5.6) the numerical equality $|\delta_{i_1\cdots i_p}| = |\delta'_{j_1\cdots j_p}|$ [See (C) for the subscripts]. Hence

$$|\gamma_{i_1\cdots i_p}| = |\gamma'_{j_1\cdots j_{n-p}}|. \tag{5.7}$$

The correct sign may now be determined by comparing equations (5.5) and (4.6). This establishes condition (5.4).

(*D*) *A q-space, L_q, with direction cosines $(\gamma'_{i_1\cdots i_q})$ is perpendicular to an $(n-p)$-space, L_{n-p} $(q<p)$, with direction cosines $(\gamma_{i_1\cdots i_{n-p}})$ if and only if there exists a set $(c_{i_1\cdots i_{p-q}})$ of direction cosines such that*

$$(-1)^{[m_1\cdots m_{n-p}, i_1\cdots i_p]}\gamma_{m_1\cdots m_{n-p}} = \sum(-1)^{[j_1\cdots j_{p-q}\, k_1\cdots k_q]}c_{j_1\cdots j_{p-q}}\gamma'_{k_1\cdots k_q} \tag{5.8}$$

where (1) $(m_1\cdots m_{n-p},\ i_1\cdots i_p)$ *is any permutation of* $(1,\cdots n)$ with $(m_1\cdots n_{n-p})$ *and* $(i_1\cdots i_p)$ *both in numerical order and* (2) *the remaining notation has precisely the same meaning as in equation* (5.3).

For, let L_p be perpendicular to L_{n-p}. Then L_q is parallel to L_p. Our conclusion now follows from (B) and (C).

One can similarly deduce from (B) and (C) conditions that a given p-space, L_p, and a given q-space, L_q, contain, respectively, a pair of parallel (or perpendicular) spaces, L_r and L_s, thus completing a discussion of the various possibilities with respect to perpendicularity and parallelism.

CHARACTERISTIC PROPERTIES OF THE EUCLIDEAN LENGTH INTEGRAL

By LINCOLN LA PAZ, Ohio State University

For a non-singular problem of minimizing an integral

$$I = \int_{x_1}^{x_2} f(x, y, z, y', z')dx \tag{1}$$

a direction $dx:dy:dz$ is said to be *transversal* to the extremal through the point (x, y, z) in the direction $1:y':z'$ in case

$$(f - y'f_{y'} - z'f_{z'})dx + f_{y'}dy + f_{z'}dz = 0. \tag{2}$$

In (2) the arguments in f and its partial derivatives are (x, y, z, y', z'). If, in particular, $f \equiv (1+y'^2+z'^2)^{1/2}$ the extremals are straight lines and the transversality condition (2) reduces to

$$dx + y'dy + z'dz = 0, \tag{3}$$

so that a direction transversal to an extremal is orthogonal to it.

Conversely, by employing the lemma that the most general integral (1) for which transversality is equivalent to orthogonality has an integrand function of the special form $f(x, y, z, y', z') = g(x, y, z)(1+y'^2+z'^2)^{1/2}$, it can be shown that the Euclidean length integral is the only one for which transversality is orthogonality and the extremals are straight lines.[1] It is the purpose of the present note to give a proof of this converse theorem which is independent of the lemma just referred to and which illustrates a method believed to be of interest.

The most general integrand function f of a problem of mimimizing the integral (1) for which the extremals are straight lines was first found by Hamel[2] but has been determined in more satisfactory fashion in a paper recently published by D. R. Davis[3] who shows that f must have the form

$$f = g(x, y, z, y', z') + (d/dx)t(x, y, z), \tag{4}$$

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[1] For a proof of the corresponding lemma and theorem in space of $(n+1)$-dimensions see the author's papers *Problems of the Calculus of variations with prescribed transversality conditions*, Bulletin of the American Mathematical Society, vol. 36 (1930), p. 680 and *The Euler equations of problems of the calculus of variations with prescribed transversality conditions*, Proceedings of the National Academy of Sciences, vol. 17 (1931), p. 461.

[2] Hamel, G., *Über die Geometrieen, in denen die Geraden die Kürzesten sind*, Mathematische Annalen, vol. 57 (1903), p. 255.

[3] Davis, D. R., *The inverse problem of the calculus of variations in higher space*, Transactions of the American Mathematical Society, vol. 30 (1928), p. 724.

where t is an *arbitrary* function of (x, y, z) and where g is a *particular* solution of the system

$$\begin{gathered} g_{y'y'} = P(y', z', u, v), \quad g_{y'z'} = Q(y', z', u, v), \quad g_{z'z'} = R(y', z', u, v), \\ u = y - y'x, \; v = z - z'x, \end{gathered} \tag{5}$$

in which $PR - Q^2 \neq 0$ and

$$P_z = Q_y, \quad Q_z = R_y; \quad P_{z'} = Q_{y'}, \quad Q_{z'} = R_{y'}, \tag{6}$$

but P, Q and R are otherwise *arbitrary* functions of their arguments.

For an integrand of the form (4) the transversality condition (2) becomes

$$(g - y'g_{y'} - z'g_{z'} + t_x)dx + (g_{y'} + t_y)dy + (g_{z'} + t_z)dz = 0. \tag{7}$$

Hence if transversality is to be equivalent to orthogonality the equations (7) and (3) must define the same plane of directions $(dx:dy:dz)$ for all sets (x, y, z, y', z') in the fundamental region R of the integral (1). Hence g and t must satisfy relations of the form

$$\begin{aligned} g - y'g_{y'} - z'g_{z'} + t_x &= h, \\ g_{y'} + t_y &= hy', \\ g_{z'} + t_z &= hz', \end{aligned} \tag{8}$$

in which $h \neq 0$ is a function of (x, y, z, y', z').

If h is eliminated between these equations it is found that g and t must satisfy the relations

$$\begin{aligned} g_{y'} + t_y &= y'(g - y'g_{y'} - z'g_{z'} + t_x), \\ g_{z'} + t_z &= z'(g - y'g_{y'} - z'g_{z'} + t_x). \end{aligned} \tag{9}$$

If (9_1) is differentiated partially with respect to y' and (9_2) partially with respect to z' it is possible to show from the resulting equations that $(1+y'^2)P = (1+z'^2)R$. Similarly by differentiating (9_1) with respect to z' and (9_2) with respect to y' the second and third relations in (10) below are obtained.

$$\begin{aligned} (1 + y'^2)P \qquad\qquad\qquad - (1 + z'^2)R &= 0, \\ y'z'P + (1 + z'^2)Q \qquad\qquad\qquad &= 0, \\ (1 + y'^2)Q + \qquad y'z'R &= 0. \end{aligned} \tag{10}$$

Since the rank of the matrix of coefficients of the system (10) is found to be two, this system admits a single solution (p, q, r) to which all other solutions (P, Q, R) are proportional. An obvious solution of (10) is:

$$p = (1 + z'^2), \quad q = - y'z', \quad r = (1 + y'^2). \tag{11}$$

Hence the most general solution of (10) for $P(y', z', u, v)$, $Q(y', z', u, v)$ and $R(y', z', u, v)$ is

$$P = k(1 + z'^2), \quad Q = -ky'z', \quad R = k(1 + y'^2), \tag{12}$$

where k is an arbitrary function of y', z', u, v.

However, it can be shown that if the functions P, Q, R in (12) are to satisfy the first condition (6) then k is free of u and v, for the relations $P_z = Q_y$, $Q_z = R_y$ require

$$\begin{aligned} (1 + z'^2)k_v &= -y'z'k_u, \\ -y'z'k_v &= (1 + y'^2)k_u, \end{aligned} \tag{13}$$

and these equations are seen to imply that $k_u = k_v = 0$ since the determinant of the coefficients

$$\delta = \begin{vmatrix} 1 + y'^2 & y'z' \\ y'z' & 1 + z'^2 \end{vmatrix} \tag{14}$$

is different from zero.

The form of $k(y', z')$ has yet to be determined so that the solutions (12) satisfy also the second set of relations $P_{z'} = Q_{y'}$, $Q_{z'} = R_{y'}$ in (6). Imposing these conditions on (12) we find that k must satisfy the partial differential equations

$$\begin{aligned} (1 + y'^2)\partial k/\partial y' + y'z'\partial k/\partial z' + 3ky' &= 0, \\ y'z'\partial k/\partial y' + (1 + z'^2)\partial k/\partial z' + 3kz' &= 0. \end{aligned} \tag{15}$$

To integrate (15) transform this system into a homogeneous system in the first partial derivatives of a function $G(y', z', k)$ with $\partial G/\partial k \neq 0$ which defines k by means of the relation $G = \text{constant}$. The resulting system

$$\begin{aligned} U_1G &= (1 + y'^2)\partial G/\partial y' + y'z'\partial G/\partial z' - 3ky'\partial G/\partial k = 0, \\ U_2G &= y'z'\partial G/\partial y' + (1 + z'^2)\partial G/\partial z' - 3kz'\partial G/\partial k = 0, \end{aligned} \tag{16}$$

is a complete system, for the equations (16) are independent since the matrix of coefficients contains the second order determinant δ which is different from zero; and, moreover, the commutator

$$(U_1U_2)G = z'\partial G/\partial y' - y'\partial G/\partial z' \tag{17}$$

is seen to be a linear combination of U_1G and U_2G.

A particular solution of (16) is

$$G = k(1 + y'^2 + z'^2)^{3/2}. \tag{18}$$

Hence, since the general integral of a complete system of two equations in three independent variables is an arbitrary function of a single particular integral of

the system,[1] we readily find from the general solution of (16) that the most general non-singular solution k of equations (15) is

$$k = m(1 + y'^2 + z'^2)^{-3/2}, \tag{19}$$

where m is a constant. Therefore the most general solution (P, Q, R) of (10) which satisfies all of the conditions (6) is (12), where k has the value (19) with $m \neq 0$.

In view of equations (5) it is now seen that a particular solution for the function g is

$$g = m(1 + y'^2 + z'^2)^{1/2}. \tag{20}$$

The arbitrary function t in (4) must now be so determined that t and the function g in (20) satisfy (9). On substituting for g from (20) the system (9) reduces to

$$t_y = y't_x, \quad t_z = z't_x. \tag{21}$$

Hence, since t is a function of x, y and z alone,

$$t_x = t_y = t_z = 0, \tag{22}$$

and consequently the most general solution for the integrand function f in (4) is given by

$$f = m(1 + y'^2 + z'^2)^{1/2}, \quad m \neq 0. \tag{23}$$

Since two problems (1) for which the integrand functions differ only by a constant factor m are not regarded as distinct it is seen that the Euclidean length integral gives the only problem (1) for which transversality is orthogonality and the extremals are straight lines.

A CLASSIFICATION OF QUADRICS IN AFFINE n-SPACE BY MEANS OF ARITHMETIC INVARIANTS

By RICHARD S. BURINGTON, Case School of Applied Science

1. *Introduction.* The classification of real conics under the euclidean group has been discussed in this Monthly by MacDuffee,[2] Paradiso,[3] and Franklin,[4] with the use of algebraic invariants.

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[1] Goursat, *Leçons sur l'Intégration des Équations aux Dérivées Partielles du Premier Ordre* (1921), p. 71.

[2] C. C. MacDuffee, *Euclidean Invariants of Plane Second Degree Curves*, American Mathematical Monthly, Vol. 33, (1926), pp. 243–252.

[3] L. J. Paradiso, *A Classification of Second Degree Loci in Space*, American Mathematical Monthly, Vol. 33, (1926), pp. 406–418.

[4] P. Franklin, *The Classification of Quadrics in Euclidean n-space by Means of Covariants*, American Mathematical Monthly, Vol. 34, (1927), pp. 453–467.

In some of their euclidean canonical forms, there are infinitely many quadrics, all of which are equivalent under the affine group, and not equivalent to a quadric of any other canonical form. Thus the problem of the separation into types is a problem in real affine geometry.

It is the purpose of this paper to show that the matrix A of the quadric has four arithmetic invariants under the real affine group which are sufficient to give a complete separation of the quadrics into types. These invariants are the ranks and signatures R_1, S_1, R_2, S_2, of A and B, where B is the matrix A with the last row and column deleted. The types obtained by this means coincide exactly with those previously obtained, and the labor[4] in applying the theory to a given quadric is slight. The types of real quadrics for $n=2$, 3, and 4 are listed in detail.

2. *Invariants.* We consider the symmetric matrix

$$A \equiv (a_{ij}),\ i, j = 1, \cdots, n+1, \tag{2.1}$$

of the real quadric

$$F = \sum_{j=1}^{n+1} a_{ij}x_i x_j,\quad x_{n+1} = 1, \tag{2.2}$$

under the real affine non-singular transformations

$$x_i = \sum_{j=1}^{n+1} b_{ij}x_j',\ i = 1, \cdots, n+1,\ b_{n+1,i} = \delta_{i,n+1}\cdot b_{n+1,n+1}, \tag{2.3}$$

whose matrix is

$$T \equiv \begin{pmatrix} b_{11} & \cdots & b_{1n} & b_{1,n+1} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ b_{n1} & \cdots & b_{nn} & b_{n,n+1} \\ 0 & \cdots & 0 & b_{n+1,n+1} \end{pmatrix},\quad d(T) \neq 0, \tag{2.4}$$

where $\delta_{i,n+1}=1$ if $i=n+1$, $\delta_{i,n+1}=0$ if $i\neq n+1$, and $d(T)$ is the determinant of T.

Under T, A becomes

$$\overline{A} = T'\cdot A\cdot T, \tag{2.5}$$

where T' is the transpose of T. If $\overline{B}$ is $\overline{A}$ with the last row and column deleted, then

$$\overline{B} = S'\cdot B\cdot S, \tag{2.6}$$

where S is T with the last row and column deleted and B is A with the last row and column deleted. Thus B is an *invariant matrix* of A, for we can get $\overline{B}$ either

[4] For easy methods of obtaining S_1 and S_2, see Bôcher, *Higher Algebra*, pp. 146–147; or Dickson, *Modern Algebraic Theories*, p. 88.

(i) by transforming and then deleting the last row and column, or (ii) by deleting the last row and column of A and T and then transforming.

Let the *ranks* and *signatures*[1] of A and B be denoted by R_1, S_1, and R_2, S_2, respectively. If A can not be reduced by a T to a diagonal matrix,[2] then S_1 is meaningless.

As is well known, R_1, S_1; R_2, S_2, *are arithmetic invariants of A and B under T, and hence of A and F.*

3. *Reduction to Canonical Forms.* We assume that F is of second degree, i.e., $R_2 \neq 0$. From the theory of quadratic forms,[3] we know that there exists a T such that

$$\overline{A} = T' \cdot A \cdot T = \begin{pmatrix} E & \phi \\ \theta & H \end{pmatrix}, \tag{3.1}$$

where $E \equiv (a_{ij})$ is a diagonal matrix with $a_{ij}=0$, $i \neq j$; $a_{ii}=1$ for $i=1, \cdots, P_2$ where $2P_2=(R_2+S_2)$; $a_{ii}=-1$ for $i=(P_2+1), \cdots, R_2$; $a_{ii}=0$ for $i=(R_2+1), \cdots, (n-1)$; ϕ and θ are rectangular matrices all of whose elements are zero, with $(n-1)$ rows and 2 rows, respectively, and

$$H \equiv \begin{pmatrix} a_{nn} & a_{n,n+1} \\ a_{n+1,n} & a_{n+1,n+1} \end{pmatrix}.$$

Suppose $R_1 = r+1$ or r, $R_2 = r > 0$. If $a_{nn} \neq 0$, the transformation[4] (2.4) with $b_{ij} = \delta_{ij}$, except $b_{n,n+1} = -a_{n,n+1}/a_{nn}$ reduces A to a diagonal matrix which can be written

$$A \equiv (a_{ij}), \quad R_2 = r, \quad R_1 = r+1 \text{ or } r, \tag{3.2}$$

where (a_{ij}) is the $E \equiv (a_{ii})$ of (3.1) with R_1, S_1, $n+2$ substituted for R_2, S_2, and n. If $a_{nn}=0$, interchange a_{nn} and a non-zero element a_{kk} of E and proceed as before.

If $R_2 = r = R_1 - 2$, the transformation T with $b_{ij} = \delta_{ij}$, except $b_{nn} = 1/a_{n,n+1}$, $b_{n,n+1} = -a_{n+1,n+1}/2a_{n,n+1}$ reduces (3.1) to the form

$$A \equiv \begin{pmatrix} E & \phi \\ \theta & P \end{pmatrix}, \quad P \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R_2 = r = R_1 - 2. \tag{3.3}$$

It is easy to show that transformation (2.4) is insufficient to reduce (3.3) to form (3.2). Such a matrix (3.3) is called *parabolic.*

From (3.1) we note $0 \leqq R_1 - R_2 \leqq 2$, $0 \leqq |S_1 - S_2| \leqq 1$. Also, we see

Theorem I. A necessary and sufficient condition that A be parabolic is $R_1 - R_2 = 2$.

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[1] See Bôcher, loc. cit., p. 146.

[2] Dickson, *Algebras and their Arithmetics*, p. 173.

[3] See Bôcher, *Higher Algebra*, p. 146 and pp. 171–3; and also Dickson, *Modern Algebraic Theories*, p. 72; for notation in (3.1) see pp. 251–2.

[4] Bieberbach-Bauer, *Algebra*, pp. 87–8.

The following theorems establish the fact that the invariants R_1, R_2, S_1, S_2 form a *complete system* of arithemetic invariants. We have proved

Theorem II. The matrix A of F can be reduced by a real non-singular affine linear transformation to a diagonal canonical form (3.2) *for which*

$$F_1 = x_1^2 + x_2^2 \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_q^2, \textit{ if } R_1 - R_2 \neq 2,$$

and to a canonical form (3.3) *for which*

$$F = F_1 + 2x_n \cdot x_{n+1}, \textit{ if } R_1 - R_2 = 2.$$

We state the fundamental[1]

Theorem III. A necessary and sufficient condition that two forms F_1 and F_2 be equivalent with regard to transformation (2.3) *is that A_1 and A_2 have the same ranks and signatures R_1, R_2, S_1, S_2.*

In view of these theorems, *we classify the quadrics F according to R_1, R_2, S_1, S_2, as indicated in Table* 1. The signatures occuring in the table are positive or zero. A similar table could be built for S_1 and S_2 negative.

4. *Application to the locus $F=0$.* It is evident that this mode of classification permits us some choice in our canonical forms for the locus $F=0$. For example, $x^2-y^2-1=0$, $y^2-x^2-1=0$, $x^2-y^2+1=0$, with corresponding diagonal matrices (written horizontally) $(1\ -1\ -1)$, $(-1\ 1\ -1)$, $(1\ -1\ 1)$, could be canonical forms for the hyperbola. The possible number of canonical forms for $F=0$ and A is finite.

The loci $F=0$ and $(-F)=0$ are equivalent. But the signatures for F are the negatives of those for $(-F)$. Hence, in dealing with a classification of loci $F=0$, we consider only the numerical values of the signatures.

We divide the loci $F=0$ according to R_1, R_2, $|S_1|$, $|S_2|$, as indicated in Table 1.

In the selection of canonical forms we agree to use (3.2) and (3.3) with the understanding that $a_{11}=1$, (if F makes $a_{11}=-1$ use $-F$), and that $a_{n+1,n+1}=-1$ if $|S_1|$ leaves a choice in the matter.

With $F=0$, theorems II and III hold with S_1 and S_2 replaced by $|S_1|$ and $|S_2|$ respectively.

5. Tables 2, 3 and 4 show the classification for $n=2, 3, 4$.

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[1] Bôcher, *Higher Algebra*, p. 148, Theorem 2.

Table 1. CLASSIFICATION OF QUADRICS IN N-SPACE

R_1	R_2	S_1	S_2	R_1	R_2	S_1	S_2
$n+1$	n	$n+1$	n	r	r	r	r
		$n-1$	n			$r-2$	$r-2$
		$n-1$	$n-2$			·	·
		$n-3$	$n-2$			·	·
		·	·			·	·
		·	·				
		·	·	r	$r-1$	r	$r-1$
		$n-k+1$	$n-k$			$r-2$	$r-1$
		$n-k-1$	$n-k$			$r-2$	$r-3$
		·	·			·	·
		·	·			·	·
		·	·			·	·
$n+1$	$n-1$		$n-1$	r	$r-2$		$r-2$
			$n-3$				$r-4$
			·	·	·	·	·
			·	·	·	·	·
			·	·	·	·	·
n	n	n	n	·	·	·	·
				·	·	·	·
		$n-2$	$n-2$	1	1	1	1
		·	·				
		·	·				
		·	·				
·	·	·	·				
·	·	·	·				
·	·	·	·				

Table 2, $n=2$

R_1	R_2	$\|S_1\|$	$\|S_2\|$	*Canonical Form*	*Name*
3	2	3	2	(1 1 1)	imag. ellipse
3	2	1	2	(1 1 −1)	ellipse
3	2	1	0	(1 −1 −1)	hyperbola
3	1		1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	parabola
2	2	2	2	(1 1 0)	point
2	2	0	0	(1 −1 0)	pair intersecting lines
2	1	2	1	(1 0 1)	imag. parallel lines
2	1	0	1	(1 0 −1)	real parallel lines
1	1	1	1	(1 0 0)	coincident lines

Table 3, $n=3$

R_1	R_2	$\|S_1\|$	$\|S_2\|$	*Canonical Form*	*Name*
4	3	4	3	$(1\ 1\ 1\ 1)$	imaginary ellipsoid
4	3	2	3	$(1\ 1\ 1\ -1)$	real ellipsoid
4	3	2	1	$(1\ -1\ -1\ -1)$	hyperboloid of two sheets
4	3	0	1	$(1\ 1\ -1\ -1)$	hyperboloid of one sheet
3	3	3	3	$(1\ 1\ 1\ 0)$	point locus (imag. cone)
3	3	1	1	$(1\ 1\ -1\ 0)$	real cone
4	2		2	$\left(1\ \ 1\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$	elliptic paraboloid
4	2		0	$\left(1\ -1\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$	hyperbolic paraboloid
3	2	3	2	$(1\ 1\ 0\ 1)$	imaginary cylinder
3	2	1	2	$(1\ 1\ 0\ -1)$	elliptic cylinder
3	2	1	0	$(1\ -1\ 0\ -1)$	hyperbolic cylinder
3	1		1	$\left(1\ 0\ 0\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$	parabolic cylinder
2	2	2	2	$(1\ 1\ 0\ 0)$	single line
2	2	0	0	$(1\ -1\ 0\ 0)$	intersecting planes
2	1	2	1	$(1\ 0\ 0\ 1)$	two imag. parallel planes
2	1	0	1	$(1\ 0\ 0\ -1)$	two real parallel planes
1	1	1	1	$(1\ 0\ 0\ 0)$	two coincident planes

Table 4, $n=4$

R_1	R_2	$\|S_1\|$	$\|S_2\|$	*Can. Form*	R_1	R_2	$\|S_1\|$	$\|S_2\|$	*Can. Form*
5	4	5	4	$(1\ 1\ 1\ 1\ 1)$	4	4	4	4	$(1\ 1\ 1\ 1\ 0)$
5	4	3	4	$(1\ 1\ 1\ 1\ -1)$	4	4	2	2	$(1\ 1\ 1\ -1\ 0)$
5	4	3	2	$(1\ -1\ -1\ -1\ -1)$	4	4	0	0	$(1\ 1\ -1\ -1\ 0)$
5	4	1	2	$(1\ 1\ 1\ -1\ -1)$	4	3	4	3	$(1\ 1\ 1\ 0\ 1)$
5	4	1	0	$(1\ -1\ 1\ -1\ -1)$	4	3	2	3	$(1\ 1\ 1\ 0\ -1)$
5	3		3	$\left(1\ \ 1\ 1\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$	3	2	1	2	$(1\ 1\ 0\ 0\ -1)$
5	3		1	$\left(1\ 1\ -1\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$	3	2	1	0	$(1\ -1\ 0\ 0\ 1)$
4	3	2	1	$(1\ 1\ -1\ 0\ 1)$	3	1		1	$\left(1\ 0\ 0\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$
4	3	0	1	$(1\ 1\ -1\ 0\ -1)$	2	2	2	2	$(1\ 1\ 0\ 0\ 0)$
4	2		2	$\left(1\ 1\ \ 0\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$	2	2	0	0	$(1\ -1\ 0\ 0\ 0)$
4	2		0	$\left(1\ -1\ 0\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$	2	1	2	1	$(1\ 0\ 0\ 0\ 1)$
3	3	3	3	$(1\ 1\ 1\ 0\ 0)$	2	1	0	1	$(1\ 0\ 0\ 0\ -1)$
3	3	1	1	$(1\ 1\ -1\ 0\ 0)$	1	1	1	1	$(1\ 0\ 0\ 0\ 0)$
3	2	3	2	$(1\ 1\ 0\ 0\ 1)$					

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

A SQUARE ROOT METHOD AND CONTINUED FRACTIONS

By RAYMOND GARVER, University of California at Los Angeles.

A number of writers have studied a rather interesting method for approximating square roots which may be described briefly as follows: If a is an approximation to the square root of N, then $(a+N/a)/2$ is a better approximation. It was used by Heron of Alexandria about 200 A.D., and by several writers in the middle ages, but the first complete investigation of the method seems to be due to Bouton.[1] James[2] studied and generalized the device, in 1924, and mentioned that it was equivalent to the old approximation, $(a^2+b)^{\frac{1}{2}}=a+b/(2a)$, known as early as the tenth century. Both Bouton and James gave upper limits to the error committed in stopping after a certain number of approximations. Very recently Boys[3] has again described the method. In his paper he keeps the successive approximations in fractional form (assuming N is an integer or fraction), whereas the earlier writers seem to have reduced them to decimals.

I should like to indicate a relation between this method and the theory of continued fractions. This relation seems to make it possible to employ the method to best advantage, and gives at once a very simple upper limit to the error of any approximation.

The theorem in continued fractions which applies is the following, which may be found in Chrystal's Algebra,[4]

If c is the number of partial quotients in the cycle of the continued fraction which represents $\sqrt{N}/M$, p_n/q_n is the nth convergent to the continued fraction, and m is any positive integer, then

$$\frac{p_{2mc}}{q_{2mc}}=\frac{p_{mc}^2+(N/M^2)q_{mc}^2}{2p_{mc}q_{mc}}.$$

For the moment take $M=1$. If our approximation to $\sqrt{N}$ is the rational fraction a/b, the iterated, or second, approximation is $(a/b+Nb/a)/2$, which reduces to $(a^2+Nb^2)/2ab$. This shows that if a/b is a convergent of the form p_{mc}/q_{mc}, the iterative method steps ahead to another convergent twice as far out in the series of convergents, and so on. At each step we have the best possible rational approximation p_n/q_n of all fractions having denominators not greater than q_n, and we know at once that p_n/q_n is not in error by more than $1/q_n^2$

[1] Annals of Mathematics, ser. 2, vol. 10, 1909, 167–172.

[2] American Mathematical Monthly, vol. 31, 1924, 471–475.

[3] Mathematical Gazette, vol. 16, 1932, 111–115.

[4] Part 2, 2nd ed., 1926, page 468.

and, in fact, by not more than $1/(a_{n+1}q_n^2)$, where a_{n+1} is the $(n+1)$st partial quotient. Further, we can tell immediately whether p_n/q_n is too laige or too small.

If we have $c=1$, which is the case when N is one more than a perfect square we may use any convergent as our first approximation. Thus

$$\sqrt{2} = 1 + \frac{1}{2\,+}\,\frac{1}{2\,+}\cdots,$$

and two iterations on 3/2 give 577/408, which is too large (since it is an even convergent) by an amount not greater than $1/2\cdot 408^2$, or about .000003. Peculiarly enough, an iteration on 4/3, which is not a convergent, gives 17/12, which is. This is not usually the case.

When $c=2$, which holds when N is an integer one less than a perfect square, or of the form A^2+A, our theorem would seem to indicate that we should start with an even convergent. It still is true, however, that any convergent iterates to give the convergent twice as far out in the series, though I shall not prove this. Thus, in approximating

$$\sqrt{3} = 1 + \frac{1}{1\,+}\,\frac{1}{2\,+}\,\frac{1}{1\,+}\,\frac{1}{2\,+}\cdots,$$

a single iteration on the fifth convergent, 19/11, gives the tenth convergent, 362/209, which can not be too large by more than 1 1/4 in the fifth decimal place. There is not, however, any advantage in starting with an odd convergent; the second convergent, 2, is perfectly satisfactory as a first approximation, though some writers have stressed the desirability of getting a close first approximation. Without the aid of continued fractions, a person might take 17/10 as an easily found and close approximation; an iteration on it gives 589/340. This is not, however, as close to $\sqrt{3}$ as is 97/56, the eighth convergent, or the result of interating twice on 2.

If c is greater than 2, the continued fraction expansion may not be easily found. If it is, the method of this note may well be used. Thus,

$$\sqrt{7} = 2 + \frac{1}{1\,+}\,\frac{1}{1\,+}\,\frac{1}{1\,+}\,\frac{1}{4\,+}\cdots.$$

That is, $c=4$, and it would be desirable to take the fourth convergent, 8/3, as the first approximation. One iteration gives 127/48, which is too large by not more than 1 in the fourth decimal place. For values of c greater than 2, some convergents which are not of the form p_{mc}/q_{mc} may iterate to give convergents, and some non-convergents iterate to convergents. It does not seem easy to obtain general results covering these situations.

The approximation of the square root of a fraction requires no further discussion, since Chrystal's discussion and theorem cover this case. The only reason for writing the fraction in the form N/M^2 is that it makes the continued fraction expansion easier to obtain. In the actual iterations this form need not be used.

Note, added in proof. The usual upper limit for the error, $1/q_nq_{n+1}$, is not used in this paper, since it would usually not be easy to set up. It is possible, however by using another property of continued fractions to obtain a better upper limit than the one used here. For, one or more iterations on a convergent p_{mc}/q_{mc} will give a convergent of the form p_{2sc}/q_{2sc}, s an integer. And it is well known that $p_{2sc}^2-Nq_{2sc}^2=1$. Now, writing p_{2sc}/q_{2sc} without subscripts for convenience, we have the error $E=p/q-\sqrt{N}=1/q(p+q\sqrt{N})$. Since p/q is an even convergent it is greater than $\sqrt{N}$. Replacing p by $q\sqrt{N}$ in the expression for E then gives an upper limit $1/2q^2\sqrt{N}$ for the error. To make this suitable for computation $\sqrt{N}$ may be replaced by any available approximation less than $\sqrt{N}$; Nq/p is a satisfactory approximation of this type. For this value the upper limit is equivalent to Bouton's, but it is in a simpler form; if a better approximation is available, a better upper limit will result.

A very simple lower limit for E can be found by replacing $q\sqrt{N}$ by p, which is larger than it. E is at once seen to be greater than $1/2pq$, which is equivalent to Bouton's lower limit but is considerably easier to use.

These two limits may be used together very advantageously.

On Equilateral Triangles

By R. GOORMAGHTIGH, Bruges, Belgium

By a method similar to J. R. Musselman's (This Monthly, 1932, p. 290), it will be easy to prove the following theorem, generalizing several properties given in Musselman's paper and also a theorem proposed by W. H. Echols (This Monthly, vol. 39, 1932, p. 46):

Let $A_iB_iC_i$ be n equilateral coplanar triangles of the same rotation-sense $(i=1, 2, \cdots, n)$; at the vertices of each of the triangles $A_iB_iC_i$ are placed weights p_i; then the centers of gravity of the weights placed at the points A_i, at the points B_i and at the points C_i form an equilateral triangle.

When $n=2$, we find that the points dividing A_1A_2, B_1B_2, C_1C_2 in the same ratio form an equilateral triangle.

When $n=3$, the points having the same barycentric coordinates in the triangles $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ form an equilateral triangle.

The first part of Professor Echol's theorem will be obtained when $n=4$, $A_2\equiv A_1$, $B_3\equiv B_1$, $C_4\equiv C_1$ and $p_2=p_3=p_4=-p_1$; the center of gravity of the weights placed at A_1, A_2, A_3, A_4, is then the mid-point of A_2A_4. The second part is a particular case of the property found above for $n=2$.

Concerning the Solution of Certain Types of Equations

By E. C. KENNEDY, Texas College of Mines

To solve equations of the type

$$(X+a)^n(X+b)^m(X+c)^p\cdots=K,$$

let X_0 be an estimated approximation to one of the real roots of the equation.

Set $X=X_0(1+u)$ and obtain

$$n\log\left[(a+X_0)\left(1+\frac{X_0u}{a+X_0}\right)\right]+m\log\left[(b+X_0)\left(1+\frac{X_0u}{b+X_0}\right)\right]\cdots=\log K$$

Since $\log(1+t)=t$, approximately, for small values of t, we obtain a much better approximation, X_1, from $X_1=V/M+X_0$, where

$$V=\log K-n\log(a+X_0)-m\log(b+X_0)\cdots$$

$$M=\frac{n}{\mathrm{a}+X_0}+\frac{m}{b+X_0}\cdots.$$

A five place table usually yields results correct to five places upon the first or second application. To obtain ten place accuracy a ten place table would probably be needed. However, even for great accuracy we do not need the log of a number of more than four or five digits—which we can read directly from any one of several well known tables. Also the method does not necessitate evaluating an antilog. Usually the quickest way to find X_0 (to about three places) is to interpolate using a log-log slide rule. Then one application of the formula should give the result correct to about six places.

Let us illustrate the advantages of this formula by

Example I. Solve

$$\frac{(X+1)^{.511}(X+3)^{1.4}}{(X+2)^{.3}}=20.4.$$

Here $n=.511$, $m=1.4$, $p=-.3$, $K=20.4$, $a=1$, $b=3$, $c=2$. Taking $X_0=4$, we write at once

$$V=\log 20.4-.511\log 5-1.4\log 7+.3\log 6=.00637$$

$$M=.511/5+1.4/7-.3/6=.2522$$

and $X_1=.00637/.2522+4=4.0253$, the first approximation. Using $X_1=4.025$ a second approximation, X_2, is readily found to be 4.02529231.

Example II. Solve $X^3+AX^2+BX=K_1$.

To solve this equation we put it in either one of the two forms

$$X(X+R_1)(X+R_2)=K_1, \text{ or}$$

$$X^2(X+A)(X-K_1/B)^{-1}=-B,$$

(where R_1 and R_2 are easily determined, being roots of a quadratic) and proceed as in I. The latter form is perhaps preferable. To illustrate let us solve

$$X^3+2X^2+3X=91 \text{ or}$$

$$X^2(X+2)(X-91/3)^{-1}=-3.$$

Here $K=-3$, $n=2$, $m=1$, $p=-1$, $a=0$, $b=2$, $c=-91/3$. Taking $X_0=3.7$, we have

$$V = \log 79.9 - 2\log 3.7 - \log 5.7 = .02365,$$
$$M = 2/3.7 + 1/5.7 + 3/79.9 = .7534,$$

and

$$X_1 = .02365/.7534 + 3.7 = 3.7314 \text{ (Newton's method gives 3.7317)}$$
$$X_2 = 3.7314928.$$

The quintic $X^5+AX^4+BX^3+CX^2+DX=E$ may be solved by this scheme by putting it in the form

$$X^3(X+a)(X+b)(X+c)^{-1}(X+d)^{-1} = -C,$$

where $(X+a)(X+b)=X^2+AX+B$ and $-C(X+c)(X+d)=E-DX-CX^2$, provided these quadratics both have real zeros.

Example III. This method is especially useful on equations of the type $X^n+aX^{n-1}=K$. For example in solving $X^{11}+aX^{10}=K$ we set $X_1=X_0(1+u_1)$ and obtain

$$u_1\left(10+\frac{X_0}{a+X_0}\right) = \log\frac{K}{X^{10}(a+X_0)}\cdot$$

In the equation $X^{11}+X^{10}=2.5$, $a=1$ and $X_0=1$, say. We write immediately $u_1(10+\frac{1}{2})=\log\ 2.5/2=.22314$ whence $u_1=.0213$ and $X_1=1.0213$ (Newton's method gives 1.0238). Taking $X_1=1.021$ we write, without any preliminary calculations,

$$u_2(10+1.021/2.021) = \log\frac{2.5}{(1.021)^{10}(2.021)},$$

which gives X_2 correct to about eight places. We may obtain X_3, correct to about fifteen places, from $u_3=(V+T)/M$; where V is the right side of the equation above, M is the coefficient of u_2, and $T=[5+\frac{1}{2}(1.021/2.021)^2]u_2^2$ is obtained from the second term of the expansion $\log(1+t)=t-t^2/2\cdots$. To find X to this degree of accuracy by Horner's Method or by interpolation would be very laborious.

RECENT PUBLICATIONS

Edited by Roger A. Johnson, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

An Introduction to the Theory of Canonical Matrices. By H. W. Turnbull and A. C. Aitkin. London, Blackie and Sons, 1932. xiii+192 pages. 17 sh. 6d.

Integralgleichungen unter besonderer Berucksichtigung der Anwendungen. By G. Wiarda. Leipzig, B. G. Teubner, 1930. 184 pages. 8.64 marks.

Recent Developments in the Teaching of Geometry. By J. Shibli. State College, Pa., Published by the Author, 1932. x+252 pages. $2.25.

Introduction à la mécanique des fluides. By Adrien Foch. Paris, Librairie Armand Colin, 1932. vi+200 pages, 10 fr. 50.

Mathematische Grundlagen der Quantenmechanik. By J. von Neumann. Berlin, Julius Springer, 1932. 262 pages. Rm. 18.

Ergebnisse der Mathematik und ihrer Grenzgebiete, herausgegeben von der Schriftleitung des "Zentralblatt fur Mathematik." Erster Band. Berlin, Julius Springer, 1932.

1. *Knotentheorie*. By K. Reidemeister. vi+64 pages. Rm. 8.75.
2. *Graphische Kinematik und Kinetostatik*. By Karl Federhofer. vi+112 pages. Rm. 13.15.
3. *Lamésche—Mathieusche—und verwandte Funktionen in Physik und Technik*. By M. J. O. Strutt. viii+116 pages. Rm. 13.60.
4. *Die Methoden zur angenäherten Lösung von Eigenwertproblemen in der Elasktokinetik*. By K. Hohenemser. 90 pages. Rm. 13.60.
5. *Fastperiodische Funktionen*. By Harald Bohr. 96 pages. Rm. 11.40.

Plane and Spherical Trigonometry. By G. N. Bauer and W. E. Brooke. Third Revised Edition. New York, D. C. Heath and Company, 1932. xvi+236 pages; tables, iv+140 pages. $2.00.

David Hilbert, Gesammelte Abhandlungen. Erster Band, Zahlentheorie. Berlin, Julius Springer, 1932. xiv+540 pages. Rm. 48.

Georg Cantor, Gesammelte Abhandlungen. Edited by Ernst Zermelo. Berlin, Julius Springer, 1932. viii+486 pages. Rm. 48.

Mathematics of Finance. By Rietz, Crathorne, and Rietz. Revised Edition. New York, Henry Holt and Company, 1932. xvi+346 pages. $3.00.

Introduction to Trigonometry and Analytic Geometry. By E. B. Skinner. New New York, The Macmillan Company, 1932. xii+190 pages. $1.80.

An Arithmetic for Teachers. By W. F. Roantree and Mary S. Taylor. New York, The Macmillan Company, 1932. x+524 pages. $2.50.

Cambridge Tracts in Mathematics and Mathematical Physics. General Editors G. H. Hardy and E. Cunningham. Cambridge University Press, 1932. New York, The Macmillan Company.

No. 27. *Modular Invariants*. By D. E. Rutherford. viii+84 pages. $2.00.

No. 28. *Conformal Representation*. By C. Caratheodory. viii+106 pages. $2.25.

No. 29. *The Foundations of Differential Geometry*. By Oswald Veblen and J. H. C. Whitehead. x+98 pages. $1.75.

Statistical Methods for Research Workers. By R. A. Fisher, Fourth Edition, Revised and Enlarged. Edinburgh and London, Oliver and Boyd, 1932. xvi +308 pages, with tables. 15 shillings.

The Theory of Functions. By E. C. Titchmarsh. Oxford, The Clarendon Press, 1932. x+454 pages. $7.50.

Elementary Mathematical Analysis, with Tables. By Mayme I. Logsdon. Volume 1. New York, McGraw-Hill Book Company, 1932. xiv+312 pages, tables 32 pages. $2.25.

REVIEWS

Vorlesungen über Grundlagen der Geometrie. By Kurt Reidemeister. Die Grundlehren der Mathematischen Wissenschaften, volume 32. Berlin, Julius Springer, 1930. x+147 pages. 12.60 marks.

This new work in Springer's well known series brings us a picture of some of the more recent contributions to the study of the foundations of geometry, particularly those made by a group of German mathematicians to which Reidemeister and Blaschke belong.

Reidemeister's work is restricted almost entirely to the plane, and his chief aim seems to be the setting up of an axiomatic basis for the study of affine and projective geometries. Two distinct approaches to this task are considered—the analytic and the axiomatic, and they motivate the division of the book into two parts. In the first part Reidemeister discusses the foundations of algebra, and bases affine and projective geometries on them. In the second part, a purely axiomatic development of geometry is based on the concept of a 3-web (3-Gewebe), and its completeness and consistency are demonstrated by forming from it a number system satisfying the postulates embodied in the first part.

The opening chapter on groups of transformations deals abstractly with such fundamentals as the "congruence" of elements under a group of transformations, the "reference set" (Bezugsmenge), a set of elements invariant only under the identical transformation (as, e.g., a set containing four linearly independent points under projective transformations), and the "natural coördinates" of an element with respect to a basic reference set.

In the second chapter number systems are developed on an axiomatic basis. The main purpose here is to lay the foundations for later geometric developments, but under the guise of independence proofs for his system of postulates, Reidemeister exhibits a number of algebras which fail to satisfy one or another of the usual axioms. These algebras are of considerable interest on their own account, particularly one unilaterally distributive number system attributed to Dickson.

The construction of affine and projective geometries on the number system as a basis occupies the next chapter. Vectors are subjected only to right-handed multiplication by a scalar factor, and only right-handed linear dependence is considered; commutativity of multiplication is thus cast aside in the development. Consequently determinants are not used, and so opportunity is afforded the author to exhibit considerable ingenuity in his algebraic developments.

In the second part, Reidemeister returns to the beginning of his problem and attacks it from the geometrical angle. The working basis is the 3-web, three pencils of parallel lines in a plane, and this concept is developed axiomatically from the undefined terms "point," "line," and "incidence." Both non-homogeneous and homogeneous coördinates are introduced; to each of the pencils is assigned an ideal "improper" point as vertex, and the ideal points of their tri-

angle are also termed "improper." A fourth pencil, with vertex at a proper point, completes the basic picture.

Addition and multiplication, proportionality and equality, brought in in terms of operations on vectors, permit the introduction of vector ratios in the role of numbers. The introduction of a number of closure axioms to the concept of a 3-web turns out to be equivalent to the introduction of distributive and commutative laws. In this way affine geometry is developed, together with the accompanying algebra. As in the earlier part, the lack of necessity for commutative multiplication is noteworthy.

The book closes with a chapter on the development of projective geometry, in which cross-ratio takes a prominent part. Here again non-commutative multiplication plays a role—witness the fundamental theorem of projective geometry, essentially in the following form: If v is the cross-ratio of four collinear points, and v^* the corresponding cross-ratio of the transformed points, then there exists a number v_0 such that $v^* = v_0 v v_0^{-1}$.

The author has obviously put a great deal of time and effort into the details of his work, and should be complimented on his discussions at the beginning and end of each chapter and of the book. One criticism which might be made is that the desire for logical compactness is at times carried to extremes. The necessity for the reader to keep in mind so many aspects at once sometimes makes the book rather hard reading. In this connection an index would have been of great help.

About a dozen errors were noted in reading through the book. A few of these were obviously typographical, but a number of them which were algebraic must have been present in the original copy. As in most of the works of this series, the printing is a distinct credit to the publisher.

Both as an original composition and for its collection of interesting developments, this work is a distinct addition to geometrical literature, and should appeal strongly to anyone deeply interested in the logical foundations of geometry.

ROBIN ROBINSON

Mathematics of Finance. By B. H. Crenshaw, Z. M. Pirenian, and T. M. Simpson. New York, Prentice-Hall, Inc., 1930. xiv+383 pages. $3.75.

This book is a text designed for students of commerce and business administration and should prove suitable for a one-year course given in these departments. The text is divided into two parts, the first part providing sufficient background in algebra for the student to undertake the study of the latter part, the mathematics of finance and life insurance.

The first part is devoted to a survey of elementary algebra and its application to commercial problems, being a treatment of linear equations, ratio, proportion, percentage, quadratic equations, the binomial theorem, construction of graphs, and logarithms. The second part offers a rather comprehensive treatment of compound interest, the annuity certain, amortization and sinking fund

methods, valuation of bonds, and includes the theory of depreciation and the mathematics of life insurance. There is a reasonable supply of exercises with illustrative examples except in the chapter on the mathematics of depreciation.

Only fifteen pages are devoted to the treatment of life insurance, which is sufficient for the purpose of the book. The authors deviate in only one place from the accepted notation of the actuary. We find the calculation of net single premiums preceding that of the contingent annuity and in the subsequent calculation of the annual premium no mention is accorded the fact that it may be regarded as an annuity.

J. H. BUSHEY

La Gamme. By P. J. Richard. Paris, Hermann et Cie., 1930. viii+231 pages.

In this book, the author has set down the physical and mathematical foundations of the various musical scales which have been in use in the western world from the days of Pythagoras to the present, excluding the whole-tone scale exploited by Debussy and others. The reason for the omission of this familiar seven-tone scale can only be inferred from M. Richard's remark about a scale of eight tones separated by equal intervals, which, he says, "would give an impression of painful monotony." Originally written for his daughter, the book is full of information, presented in a clear and readable style, for students of music or mathematics and indeed for "everybody who will not be frightened by the dryness of the first chapter."

This first chapter is an arithmetical introduction in which the properties of fractions are dealt with at length followed by a very summary treatment of exponents and logarithms. The work in fractions goes beyond the elementary operations only to the extent of proving two theorems: (1) By adding numerators and denominators in two unequal fractions, a third is formed intermediate in value between the two given; and (2) If two fractions, a/b and c/d, are such that $ad-bc=1$, every fraction, whose value is intermediate, has terms larger than those of the originals. It is interesting to note that musicians may claim priority in the use of logarithmic language, for long before Napier they spoke of *adding* two intervals to produce a third, knowing that, in fact, it is the *product* of the frequency-ratios characterizing the component intervals which gives the ratio for the resultant interval. An interval is thus the logarithm of the ratio characterizing it.

Chapters II to VI construct and discuss the scales of Pythagoras, Aristoxenus, Zarlino, Delezenne, that known as the "musicians scale," and the tempered scale of Rameau and Bach, then, after a necessary discussion of acoustics, the "natural scale," based on harmonics. Chapter VII is given to the old Greek and Church modes and the modern minor mode. So far the discussion has been on the basis of vibrating strings, but Chapter VIII takes up sounding pipes, Chapter IX, concords and dissonances resulting from tones sounding simultaneously and, in conclusion, Chapter X, a very illuminating discussion on the

relation between the sensations of listening to an orchestra and the nice ratios which express the tones of the scale mathematically.

The author has proposed to avoid "all considerations of a purely aesthetic nature" but this, of course, is neither desirable nor possible for one whose interest is primarily musical. There are characteristic notes on futurist noise-orchestras (p. 118), "the melancholy gaiety of jazz" (p. 168), and the new tone-qualities, "perhaps a gain," resulting from the use of mutes (p. 188).

GORDON H. GRAVES

Le Calcul des Différences Finies et ses Applications. By Alfred Henry. Translated by A. Sallin. Paris, Hermann, 1932. 210 pages. 50 francs.

This book is a valuable exposition of the elements of the Theory of Finite Differences as this theory relates to problems in interpolation, finite summation, and approximate integration. It also includes a brief introduction to Differential and Integral Calculus and a chapter on Probability.

The arrangement of the work is as follows (the A, B, C headings are the reviewer's).

(A) Chapters 1–9 deal with the development of the Difference Calculus and applications (pp. 11–96).

(B) Chapters 10–19 treat Differential and Integral Calculus (pp. 97–170).

(C) Chapter 20 is on Approximate Integration and Chapter 21 gives a brief treatment of some typical problems in the Theory of Probability (pp. 171–210).

We believe that the main purpose of the author has been to produce a text which would enable the actuary without much mathematical background to gain sufficient knowledge of the Calculus of Differences so that, aided by a superficial knowledge of differential and integral calculus, he could intelligently handle a wide class of problems in interpolation, finite summation, and approximate integration. We believe that the book accomplishes this purpose admirably. Also we find in Chapter 21 a treatment of Probability along with an interesting set of problems. Chapter 21 seems to have been added so that the book would have more appeal to the potential reader. It is in no way connected with the Theory of Finite Differences.

Chapters 1–4 introduce the simpler formulas of interpolation. The difference $\Delta f(x)$ is defined as $f(x+h)-f(x)$ where one is to determine h by the context. This notation avoids confusion in a textbook introducing the beginner to methods of interpolation. We believe, however, that in theoretical work the notation

$$\underset{h}{\Delta} f(x) = [f(x+h) - f(x)]/h$$

is more satisfactory. The operator $Ef(x)=f(x+h)$ is employed frequently in proofs involving operational methods. Chapter 5 takes up the central difference interpolation formulae of Stirling, Bessel, Gauss and Everett, and gives an exceptionally clear discussion of them. Chapter 6 is on Inverse Interpolation.

The reviewer finds Chapter 7 on Finite Summation novel and ingenious and thinks that this topic is worthy of more consideration in present day teaching than it now receives. The author in an exercise presents the following method of summing a convergent alternating series

$$\phi(1) - \phi(2) + \phi(3) - \phi(4) + \cdots$$

The sum is given by

$$\tfrac{1}{2}\phi(1) - \tfrac{1}{4}\Delta\phi(1) + \tfrac{1}{8}\Delta^2\phi(1) - \cdots$$

Thus to evaluate the series

$$\frac{1}{10} - \frac{1}{11} + \frac{1}{12} - \cdots$$

to four decimal places, four terms of the difference series will suffice as against an odd 10,000 of the original series. There are several misprints in this chapter. In the table on page 76, the 2nd column should be headed Intégrales indéfinies, and in the 3rd and 4th rows of the table, the superscripts should be inclosed in parentheses or brackets to show that the functions represent factorials (defined on page 31). The $f(a)$ under the summation sign, line 11, p. 77, should read $f(x)$. The superscripts 1, 2, 3 in the 2nd line of Ex. 4, p. 80, should be subscripts.

Chapter 8 gives a brief discussion of Divided Differences[1] and Chapter 9 an intelligent treatment of Differences of Functions of Two Variables relative to interpolation.

Part (B) on the Integral and Differential Calculus is a brief exposition of basic technical results without much attempt to go below the surface in the theory. The reviewer believes that such a treatment might well be accompanied by frequent references which would put the beginner on his guard against the many pitfalls that one may encounter in using limiting processes. With the exception of Chapter 15, Part (B) is entirely unconnected with the Theory of Finite Differences. In Chapter 15, however, it is shown that the derivative and the difference of a function are related by the symbolical equation

$$\Delta = e^{hD} - 1,$$

where D is the operational symbol for differentiation. This relation is effectively used to obtain the Euler-Maclaurin summation formula in Chapter 20, although the justification of this use is not discussed. Chapter 20, on Approximate Integration, takes up, besides the Euler-Maclaurin formula, formulae of Woolhouse, Lubbock, Simpson, Weddle, G. H. Hardy, Lagrange, the Rule of Three Eighths and one other, not bearing a name.

As stated above, the last chapter, Chapter 21, is on Probability. Except for the errors in Chapter 7 noted above, there are few typographical errors.

B. F. KIMBALL

[1] cf. Whittaker & Robinson, *A Short Course in Interpolation*, London, Blackie & Son (1923), Chapter 2.

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

A.

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research in the preparation of papers in the field of mathematical science to be presented at its regular meetings.

CHAPTER REPORTS

1931–1932

Pi Mu Epsilon of Hunter College.

The Hunter College chapter of Pi Mu Epsilon has just finished another busy year, consisting of eight program meetings, two business meetings, and three social functions. At the program meetings during the first semester the topic discussed was "Maxima and minima." Eleven girls gave papers on various phases of the subject. The second semester topic was "Algebraic and transcendental numbers," and eleven girls reported on this topic also.

The chapter initiated twenty new members, making the membership for the year, including the faculty members, total fifty-three.

The October social function was a dinner for initiates. Many alumnae members attended. Interesting talks were given by former active members who told of their work in different fields. The February party was a luncheon. Professor E. R. Hedrick of the University of California at Los Angeles was our guest and speaker. The May party was an alumnae reunion and bridge party.

The officers for the year 1931–1932 were: Professor Jewell C. Hughes, Director; Sylvia Radlich, Vice Director; Rose Berger, Corresponding Secretary; Madeline Levin, Treasurer; Isabel Sklower, Recording Secretary.

ROSE BERGER, *Corresponding Secretary*

Pi Mu Epsilon of the University of California at Los Angeles.

We extend our best wishes to the other chapters of Pi Mu Epsilon. Our chapter presents the following account of its activities for the academic year 1931–1932:

Our officers were: Mr. Carroll P. Brady, Director; Miss Goldie Ivener, Vice Director; Mr. Hugh J. Hamilton, Secretary; Mr. Wendell E. Mason, Treasurer; Professor Raymond Garver, Librarian; Professor Clifford Bell, Faculty Advisor.

Twelve new members were elected on December 5, 1931, and three on April 30, 1932, so that the total membership in this chapter is now 117. Thirty-five of these are now at the University, thirteen being faculty members.

Since September there have been seven regular meetings and two business sessions. Two entertainments were given in honor of incoming members, an informal one on December 5, 1931, and a semi-formal one on April 30, 1932. Our chapter has also sponsored, in conjunction with the Mathematics Club of this University, a Christmas party on December 18, 1931, and a Beach party on June 14, 1932.

In order to stimulate interest in the Calculus, this chapter directs annually a Prize Calculus Examination, open to all students save members of Pi Mu Epsilon. The winner is awarded ten dollars.

The meetings and programs were as follows:

October 7, 1931: "Skew curves" by Mr. Hugh Hamilton.

November 18, 1931: "Some concepts of the calculus" by Professor G. E. F. Sherwood.

January 13, 1932: "An application of Euler's method for the summation of a certain class of series" by Mr. Carroll Brady.

March 9, 1932: "Business mathematics" by Miss Goldie Ivener.

April 6, 1932: "The problem of Apolonius" by Miss Jean Robb.

May 4, 1932: "The mathematics of rainbows" by Mr. R. Holloway.

June 1, 1932: "Researches on sound waves" by Mr. Reed Lawlor.

HUGH J. HAMILTON, *Secretary*

B.

LOCAL MATHEMATICS CLUBS

The Junior Mathematical Club of the University of Chicago

The officers, elected by vote of the club on June 10, 1931, for 1931–1932 were: Mr. Magnus R. Hestenes, President; Mr. Ralph D. James, Secretary-Treasurer; Miss Julia W. Bower, Social Chairman; Miss Frances H. Wiancko, Program Chairman.

The primary aim of the Junior Mathematical Club is to broaden the student's knowledge of mathematics and to supplement class room instruction by giving students an opportunity to present and to listen to papers on various phases of mathematics. The Club largely sponsors the social activities of the Department of Mathematics. Its activities are quite distinct and in addition to the work of the graduate research club. The membership is about seventy-five, any graduate or undergraduate student majoring in mathematics and mathematical astronomy being eligible to membership. Meetings are held fortnightly on Wednesday, a social half-hour preceding the presentation of the papers.

The meetings and programs were as follows:

October 7, 1931: Address of welcome to new students by Professor E. P. Lane; "The use of the library" by Professor L. M. Graves.

October 21, 1931: "Early developments of the calculus" by Dr. R. G. Sanger.

November 4, 1931: "Non-differentiable functions" by Dr. E. J. McShane.

November 18, 1931: "On the problem of moments for a finite interval" by Dr. I. Schoenberg.

December 2, 1931: "Non-Euclidean geometry" by Professor M. I. Logsdon.

December 16, 1931: "Elementary properties of ordinary linear differential equations of the second order" by Dr. W. T. Reid.

January 20, 1932: "Historical sketch of synthetic projective geometry" by Mr. P. Youtz.

February 3, 1932: "Some properties of orthogonal matrices and an application to analytic geometry of space" by Mr. R. C. Bullock.

February 17, 1932: "Classification of elements of a group" by Professor A. C. Lunn.

March 2, 1932: "Diophantus of Alexandria" by Mr. Ralph Hull.

March 30, 1932: "Mathematical models" by Professor E. P. Lane.

April 13, 1932: "Some problems in symmetric functions" by Miss Julia Bower.

April 27, 1932: "On minimum problems in geometry" by Dr. I. Schoenberg.

May 11, 1932: "Mathematics in the large" by Professor H. E. Slaught.

During this year, several bridge parties have been sponsored by the club.

RALPH D. JAMES, *Secretary-Treasurer*

The Mathematics Club of Hunter College.

During the fall term of the past year the program meetings of the Mathematics Club of Hunter College were devoted to the topic of Cartography. Several students reported on different types of

map construction, for example, Mercator's, Mollweide's, and Stereographic Projection. At one meeting, Mr. Nelson of the Rand McNally Company gave a summary of the types of map construction in actual use; and at the final meeting of the term, Professor Lehnerts, Head of the department of Geology at Hunter College, gave a talk on topographical maps. During the second term of the year, miscellaneous topics of mathematical interest were discussed. They were: (1) Geometrical constructions with a straight edge only; (2) The golden section; (3) The nine-point circle; (4) Spirals; (5) The conchoid of Nicomedes; and (6) Three Italian texts of the sixteenth and seventeenth centuries—translated by three students in the course in History of Mathematics.

In addition to the program meetings, several social affairs were held during the year. Each term, a party was given in honor of the new members of the Club. In the Fall, the Club at the Main Building was the guest of the Club at the New Buildings in the Bronx, and some two hundred were present at the party held there. At a Friday afternoon tea, Professor Walker told of some of her experiences on her trip around the world. Also, last Fall, there was a twelve mile Club hike attended by about fifteen students and two members of the Staff. The last party of the Club for the year was a Bridge party, held on the evening of April 30, 1932.

The officers for the year were: Irene Larson, President; Kathleen Downing, Vice President; Beatrice Jacobson, Secretary; Marian Moynahan, Treasurer; Marion Leary—Fall term, Helen Schroeder—Spring term, Publicity Manager; Laura Guggenbühl, Faculty Advisor.

BEATRICE JACOBSON, *Secretary*

The Mathematics Club of Boston University.

The Mathematics Club of Boston University, College of Liberal Arts was organized in 1922 to promote good fellowship among those interested in this science. This organization also affords the students and Professors an opportunity to discuss many interesting aspects of mathematics not covered in the regular courses offered by the college.

Under President Thomas Homkowycz several socials have been held during the past year. Early in the Fall an acquaintance party welcomed new members. A Christmas social and a theatre party in February constituted the winter program. The social activities were completed by an April Fool's party. But the serious side of a mathematics club has not been neglected. Papers have been given on the following subjects during the past year:

"Problems I have met" by Professor Elmer B. Mode; "Oriental numbers" by Miss Annie Hall; "The mathematics of the musical scale" by Mr. Lucien B. Taylor, Instructor in Physics; "Totients" by Miss Elizabeth E. Curtis; "The circular points at infinity" by Miss Ruth Irene Deffley; "Einstein" by Mr. Julius Miller; "The possible impossible" by Professor Robert E. Bruce, Faculty Advisor of the Club; and "Solutions of equations" by Mr. Edward Mann, Instructor of Mathematics at the College of the Eastern Nazarene.

The annual prize for the best paper was awarded to Mr. Miller.

The 1931-1932 Corps of Officers included: Mr. Thomas Homkowycz, President; Miss Gladys Knowlton, Vice President; Mr. Carleton Foss, Secretary; Miss Doris Atkinson, Assistant Secretary; Mr. Francis Blackwell, Treasurer; Miss Dorothy Brennan, Assistant Treasurer; Miss Eleanor Johnson and Miss Annie Hall, Executive Committee; Professor Robert E. Bruce, Faculty Advisor.

RUTH IRENE DEFFLEY, *Reporter*

The Dartmouth Mathematical Society.

The Dartmouth Mathematical Society was founded on October 6, 1930, "to make possible the presentation and discussion of subjects of general mathematical interest." Although primarily an undergraduate society, membership is open to anyone interested, upon payment of the regular dues of $1.00 per semester. There are at present thirteen members and frequently non-members are guests at the meetings.

Meetings are ordinarily held every second week during the college year. It is customary to serve refreshments during the discussion following the meeting.

The officers for 1931–1932 were: W. C. Johnson '33, President; C. Calmon '34, Vice President; and F. L. Engel '34, Secretary-treasurer.

The meetings and programs for the year were as follows:
October 5, 1931: "Fourth dimensional space" by Professor R. D. Beetle.
October 19, 1931: "The trisection of the angle" by F. L. Engel '34 and G. L. Engel '34.
November 2, 1931: "Mathematical concepts of infinity" by Dr. F. W. Perkins.
November 23, 1931: "The composition of simple harmonic motions" by G. F. Hull, Jr., '32.
December 14, 1931: "DeMontmort's problem" by Professor B. H. Brown.
February 15, 1932: "The development of the fundamental algebraic notations" by A. P. Bill '33.
February 29, 1932: "How mathematicians make generalizations" by Professor L. L. Silverman.
March 14, 1932: "The mathematics of ship building" by Dr. R. Robinson.
April 18, 1932: "Hyperbolic functions and the catenary" by W. C. Johnson '33.

FRANK L. ENGEL, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSEN AND W. F. CHENEY, JR.

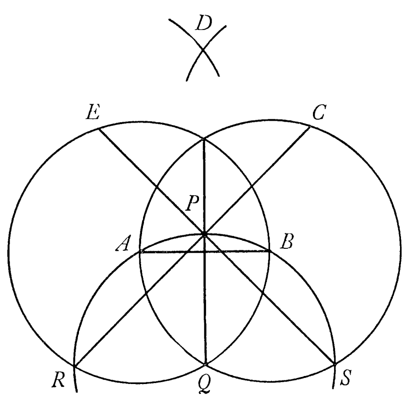
ELEMENTARY PROBLEMS

Send all communications about Elementary Problems to Wm. Fitch Cheney, Jr., Dept. Box 35 Storrs, Conn.

The Department of Elementary Problems and Solutions in the Monthly welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 6. *Proposed by W. R. Ransom, Tufts College.*



This construction was given in 1625 by Albrecht Dürer, the great engraver, for a regular pentagon, $ABCDE$, and it is still given in books on mechanical drawing. The circles are all drawn with the same radius, equal to the given length of the side AB, with centers at these points (in order) A, B, Q, C and E. Calculate the angle ABC to determine whether this is an exact or an approximate construction.

E 7. *Proposed by C. O. Oakley, Brown University.*

The following letters represent the digits of a problem in simple addition.

$$\begin{array}{ccccc} & S & E & N & D \\ & M & O & R & E \\ \hline M & O & N & E & Y \end{array}$$

Solve and show that the solution is unique.

E 8. *Proposed by Otto A. Spies, St. Paul, Minn.*

It is required to construct an inscriptible quadrilateral with ruler and compass, given the lengths of the four sides in order.

E 9. *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

The cylindrical tank on a railway tank-car is of length l feet and radius r feet. The car stands on a grade inclined Q degrees from the horizontal. The tank is being emptied, and it is desired to express the volume of liquid in the tank as a function of the vertical depth of liquid below the test-hole in the center of the top of the car.

This problem becomes more difficult if the tank has hemispherical ends.

E 10. *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

The x's below show the relative positions of the digits in a certain exact division. Find the divisor.

```
x x x ) x x x x x x x x ( x x x x x x
        x x x
        -----
        x x x x
          x x x
        -------------
              x x x x
                x x x
              -------
                x x x x
                x x x x
                -------
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Advanced Problems

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3572. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

The sum of the three bimedians of a tetrahedron (i.e., the lines joining the mid-points of the pairs of opposite edges) is less than one half and greater than one fourth of the sum of the edges of the tetrahedron.

3573. *Proposed by Samuel I. Jones, Nashville, Tenn.*

A hawk, eagle and sparrow are in the air. The eagle is 50 feet above the sparrow and the hawk is 100 feet below the sparrow. The sparrow flies straight forward in a horizontal line. Both hawk and eagle fly directly towards the sparrow. The hawk flies twice as fast as the sparrow. The hawk and eagle reach the sparrow at the same time. How far does each fly and at what rate does the eagle fly?

3574. *Proposed by Orrin Frink, Jr., Pennsylvania State College.*

A company wishes to establish five agencies at five points of a circular region. When this is done there will be a smallest number r such that every point of the circular region will be within a distance r of at least one of the agencies. How should the agencies be located so as to make r a minimum?

3575. *Proposed by Frank Morley, Johns Hopkins University.*

Given two circles in a Euclidean space, which are not interlaced and not co-spherical; show that there are four circles which touch both, and that these break into two pairs, the three pairs forming a symmetrical configuration.

3576. *Proposed by J. J. L. Hinrichsen, Iowa State College.*

Given any triangle with three line segments concurrent in a point interior to the triangle and joining each vertex to its opposite side; prove that the length of the longest of these three lines cannot be less than $\sqrt{3}/2$ times the length of the opposite side of the triangle.

3577. *Proposed by B. F. Kimball, Schenectady, N. Y.*

Given

$$v_n = \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1{}^r + x_2{}^r + \cdots + x_n{}^r}{(x_1 + x_2 + \cdots x_n)^q} dx_1 dx_2 \cdots dx_n,$$

where q is any real number and r is any real number greater than -1; show that

$$\lim_{n\to\infty} n^{q-1} v_n = 2^q (r+1)^{-1}.$$

Note. This is a generalization of problem 3408 [1930, 38] and problem 3460 [1930, 508].

3578. *Proposed by J. M. Feld, Brooklyn College.*

Two points are isogonal conjugates with respect to a triangle if and only if they are the foci of an inscribed conic.

SOLUTIONS

3508 [1931, 409]. *Proposed by the late Artemas Martin, Washington, D. C.*

The sides of a plane triangle are a, b, c. It is required to determine the radius of the circle circumscribing the escribed circles of this triangle. (See the Annals of Mathematics, March, 1894.)

Solution by R. Goormaghtigh, Bruges, Belgium

Let $A\ B\ C$ be the triangle, A', B', C' the mid-points of the sides, R and r the radii of the circum- and incircles, I, I_1, I_2, I_3, the in- and ex-centers of ABC, I' the in-center of $A'B'C'$, α the projection of A' on I_2I_3, $2s$ the sum of the sides of ABC.

The radical axis of the circles (I_2), (I_3) passes through A' and is perpendicular to I_2I_3; therefore I' is the radical center of (I_1), (I_2), (I_3). The power of A' with respect to (I_2), (I_3) being $(b+c)^2/4$, the power π of I' with respect to (I_1), (I_2), (I_3) is

$$(1)\qquad \begin{aligned}\pi &= (b+c)^2/4 - \overline{A'\alpha^2} + (A'\alpha - \tfrac{1}{2}AI)^2,\\ &= \tfrac{1}{4}(b+c)^2 - \tfrac{1}{2}(s-a)(b+c) + \tfrac{1}{4}(s-a)^2 + \tfrac{1}{4}r^2 = \tfrac{1}{4}(s^2+r^2).\end{aligned}$$

There are *eight* circles tangent to (I_1), (I_2), (I_3): the three sides, the nine point circle (O_9) and the inverses of these four circles, for the pole I' and the power π.

The power ω of I' with respect to (O_9) being $-\frac{1}{2}Rr$, the radius of the inverse of (O_9), which is the circle circumscribed to (I_1), (I_2), (I_3), will be

$$(2)\qquad \begin{aligned}R\pi/2\omega &= \tfrac{1}{4}(r + s^2/r)\\ &= \tfrac{1}{4}\left[s^{-1/2}(s-a)^{1/2}(s-b)^{1/2}(s-c)^{1/2} + s^{5/2}(s-a)^{-1/2}(s-b)^{-1/2}(s-c)^{-1/2}\right].\end{aligned}$$

It may be noted that the radii of the three other circles, inverse to the three sides, and also tangent to the escribed circles, are

$$(3)\qquad \tfrac{1}{4}a(r + s^2/r)(b+c)^{-1}, \cdots .$$

A Note by Otto Dunkel. The following amplification may be of aid in reading this interesting solution. Let $BC=a$ be tangent to the escribed circles (I_2) and (I_3) at A_2 and A_3. Then, since BA_2 and CA_3 are each equal to $s-a$, A' is also the mid-point of A_2A_3. Thus $A'A_2=\frac{1}{2}[2(s-a)+a]=\frac{1}{2}(b+c)$. Since I_2I_3 is the external bisector of angle A, it is perpendicular to the internal bisector AI. The triangle $A'B'C'$ has its sides parallel to the corresponding sides of ABC, and hence the internal bisector $A'I'$ of angle A' is parallel to AI and perpendicular to I_2I_3. It then follows that $A'\alpha$ passes through I', and that I' is the radical

center of (I_1), (I_2), (I_3). The perpendiculars from B and C to I_2I_3 have the respective lengths $c\cos\frac{1}{2}A$, $b\cos\frac{1}{2}A$, and hence $2A'\alpha=(b+c)\cos\frac{1}{2}A$. Also $AI\cos\frac{1}{2}A=s-a$, and, therefore,

$$2A'\alpha\cdot AI = (b+c)(s-a).$$

The circle orthogonal to the three escribed circles has I' for center and the length $\pi^{\frac{1}{2}}$ for radius. If r_2 is the radius of (I_2) we see from a figure the following equations

$$\begin{aligned}
\pi + r_2^2 &= (\alpha I_2)^2 + (\alpha I')^2,\\
(A'A_2)^2 + r_2^2 &= (\alpha I_2)^2 + (\alpha A')^2,\\
\pi &= (A'A_2)^2 - (\alpha A')^2 + (\alpha I')^2\\
&= (A'A_2)^2 - (\alpha A')^2 + (\alpha A' - I'A')^2.
\end{aligned}$$

From the similarity of ABC and $A'B'C'$ we have $AI=2A'I'$. Also $AI^2=r^2+(s-a)^2$. These results give the expression for π in (1).

The nine point circle (O_9) of ABC with center O_9 is the circumcircle of $A'B'C'$ and hence its radius is $\frac{1}{2}R$. Let d be the distance between O_9 and I', the centers of the circum- and in-circles of $A'B'C'$. This is one half of the corresponding distance for ABC, and hence a well known theorem gives for d

$$4d^2 = R^2 - 2rR.$$

The power ω of I' with respect to (O_9) is negative since I' lies within (O_9). It may be represented geometrically by constructing the chord TT' through I' perpendicular to O_9I'. Then

$$\begin{aligned}
\omega &= -(I'T)^2 = (I'O_9)^2 - (O_9T)^2\\
&= \tfrac{1}{4}(R^2 - 2rR) - \tfrac{1}{4}R^2 = -\tfrac{1}{2}rR.
\end{aligned}$$

Let the inverse of (O_9) with respect to the orthogonal circle above with radius $\pi^{\frac{1}{2}}$ be (O_9') with the center O_9'; let T and T' invert into $\overline{T}$ and $\overline{T}'$; let K be the pole of TT' with respect to (O_9); and let KI' cut (O_9) in M and N. Since K, M, I', N is an harmonic set of points, K inverts into the center of (O_9'). We then have in turn

$$I'K\cdot I'O_9' = \pi = I'T\cdot I'\overline{T};\ (I'T)^2 = O_9I'\cdot I'K;$$

$$\frac{I'O_9'}{I'O_9} = \frac{I'\overline{T}}{I'T}\,.$$

Hence $I'O_9T$ and $I'O_9'\overline{T}$ are similar triangles, and it then follows that

$$O_9'\overline{T} = O_9T\frac{I'\overline{T}}{I'T} = \frac{1}{2}\,\frac{R\pi}{(I'T)^2} = -\frac{1}{2}\,\frac{R\pi}{\omega} = \frac{\pi}{r}\,.$$

This gives the result in (2).

If h_a is the altitude of ABC from A, the distance between BC and $B'C'$ is $\frac{1}{2}h_a$. Let p_a be the distance of I' from BC; then $p_a=\frac{1}{2}h_a-\frac{1}{2}r$, since the distance of I' from $B'C'$ is one half that of I from BC. Also $h_a=2rsa^{-1}$ as we find by obtaining the area of ABC in two ways. Hence

$$p_a = \frac{r}{2a}(2s - a) = \frac{r(b + c)}{2a} \cdot$$

The side BC inverts into a circle with the diameter πp_a^{-1}, and hence the radius of this circle is

$$\frac{a\pi}{r(b + c)} \cdot$$

3512 [1931, 461]. *Proposed by J. Rosenbaum, Milford, Conn.*

Prove that in the tetrahedron of Problem 3482 (March, 1931) the common center of the two spheres is also the centroid of the tetrahedron.

Solution by the Proposer

Consider the tetrahedron $ABCD$ with the faces ABD, BCD, and CAD opened up to the positions ABD_1, BCD_2, and CAD_3 forming the plane figure $D_1BD_2CD_3A$. Since the sum of the face angles at A, B, and C is each two right angles, the figure $D_1BD_2CD_3A$ is a triangle. In this triangle, A, B, and C are the midpoints of the sides. Hence the faces of $ABCD$ are congruent, and thus are equivalent.

It will now be proved that in a tetrahedron with equivalent faces the centroid and the incenter coincide; Join the incenter to the vertices thus dividing the tetrahedron into four tetrahedra. The entire volume is the sum of the four volumes, or

$$Ah/3 = 4Ar/3, \tag{1}$$

where A is the area of each face, h is the altitude to a face and r is the inradius.

From (1) we have

$$r = h/4, \tag{2}$$

from which the theorem above readily follows, and the statement of the problem is thus proved.

3515 [1931, 462]. *Proposed by E. P. Bugdanoff, Harbin City, China.*

Give an elementary proof, without the use of the calculus, that the equation

$$2^x = 4 \cdot x$$

has two real roots only, and calculate each of them.

Determine the location of the imaginary roots, and compute the pair having the least absolute value.

Solution by Otto Dunkel, Washington University

Setting $x=0$, 1, 4, we find that one root is 4 and that a second root lies between 0 and 1. In order to prove that these are the only two real roots, it suffices to show that a straight line cuts the curve (C), $y=2^x$, in not more than two real points. It can be proved by elementary algebra that

$$(1)\qquad \frac{a^\alpha - 1}{\alpha} > \frac{a^\beta - 1}{\beta}, \quad a > 0,\ a \neq 1,\ \alpha > \beta > 0,$$

where α and β are any two real numbers. For each value of x there is only one point on (C), and y increases continuously with x. Let a straight line cut (C) at A and B with abscissas x_1 and x_2, $x_2>x_1$, and let P, with abscissa x, lie on (C) between A and B, then

$$2)\qquad \text{slope } AB = 2^{x_1}\frac{2^d - 1}{d}, \quad \text{slope } AP = 2^{x_1}\frac{2^e - 1}{e},$$

$$x_2 - x_1 = d, \quad x - x_1 = e.$$

It follows from (1) that slope $AB>$slope AP, and hence P lies below the secant AB. If P lies beyond B then the previous inequality is reversed, and P lies above the secant AB. If P lies to the left of A, we show in the same way that B lies above secant PA, and hence P lies above secant AB. This completes the proof that there cannot be more than two real roots, and the proof shows that (C) is concave upward. A simple method of calculating the remaining root r now follows from the rising and concave form of (C). Set

$$(3)\qquad y = 2^{x-2}.$$

If $r<x<4$, say $x=1$, then $y>r$ and $y<x$. Hence y is a closer approximation to r than x. Using this value of y for x ,we find a still better approximation y_1; setting $x=y_1$, we find y_2; etc. The root r may be approximated in this way to any degree of accuracy; but this method is not as rapid in the advanced stages of approximation as the so-called Newton's method. To four decimals $r=0.3099$.

The formula (1) is important, since from it may be deduced the derivatives of all powers of x, of the exponential functions, and of the logarithmic functions. The proof of this formula by algebra is not simple when compared with the proofs found in most of our texts for undergraduates.

In order to determine the imaginary roots we set $x=\rho e^{i\theta}$, $\theta\neq 0$, and find the equations

$$(4)\qquad \theta\cot\theta - \log_e\left(\frac{\theta}{\sin\theta}\right) - c = 0, \quad c = \log_e\left(\frac{4}{\log_e 2}\right) = 1.75280,$$

$$(5)\qquad \rho = \frac{\theta}{\log_e 2 \sin\theta}\,.$$

It will suffice to consider $\theta>0$; then from the above equations $\sin\theta>0$. Since $\theta/\sin\theta>1$, $\theta\cot\theta>c>1$, or $0<\tan\theta<\theta$; and this shows that no root of (4) lies

between 0 and $\pi/2$. Roots can occur only in the intervals $2k\pi \leqq \theta \leqq (4k+1)\pi/2$, $k=1, 2, \cdots$. If we set y equal to the left side of (4), then

$$\frac{dy}{d\theta} = -\theta^{-1}[(\theta \cot \theta - 1)^2 + \theta^2], \tag{6}$$

and therefore y decreases as θ increases. For $\theta = 2k\pi+$, $y = +\infty$; for $\theta = (4k+1)\pi/2$, $y<0$. This shows that there is one and only one root in each of the intervals. If θ_1 is a root in a given interval, then for $\theta = 2\pi+\theta_1$ in the next interval

$$y = (2\pi + \theta_1) \cot \theta_1 - \log_e\left(\frac{2\pi + \theta_1}{\sin \theta_1}\right) - c,$$

$$0 = \theta_1 \cot \theta_1 - \log_e\left(\frac{\theta_1}{\sin \theta_1}\right) - c, \tag{7}$$

$$y = 2\pi \cot \theta_1\left[1 - \frac{\tan \theta_1}{2\pi} \log\left(1 + \frac{2\pi}{\theta_1}\right)\right] > 2\pi \cot \theta_1\left[1 - \frac{\tan \theta_1}{\theta_1}\right]$$
$$> 2\pi \cot \theta_1[1 - c^{-1}] > 0.$$

Hence the root in the next interval lies nearer the right hand end of that interval than the root in the preceeding interval. It will appear later that the roots approach as a limit the right hand end of the intervals. It will be found by trial for the first interval that the root lies beyond $2\pi+\pi/3$, and hence for any root

$$2k\pi + \pi/3 < \theta < 2k\pi + \pi/2, \quad k = 1, 2, \cdots. \tag{8}$$

In order to examine the variation of ρ we consider the equation

$$\log_e 2 \cos \theta = \rho^{-1} \log_e (4\rho). \tag{9}$$

Since from (5) $\rho>1$, we see that the right side of (9) decreases as ρ increases. On the left side $\cos\theta$ decreases as we pass from the root in one interval to the root in the next, as has been shown above. Hence ρ increases as we pass from the root in one interval to that in the following interval.

In the order to calculate the roots in a simple manner, set $\theta = M\pi - \alpha$, $M = (4k+1)/2$; then

$$\tan \alpha = \frac{\log_e (M\pi - \alpha) - \log_e \cos \alpha + c}{M\pi - \alpha}, \quad 0 < \alpha < \pi/6. \tag{10}$$

From this expression we see that α approaches zero as M increases without limit. From the form of (10) we see that α may be approximated as follows: replace $\tan \alpha$ on the left by $\tan \alpha'$; for the first interval select any suitable trial value for α, say $\alpha = \pi/6$, and find α', $\alpha' = 27°46.5'$ approximately. Continue, using $\alpha = 27°46.5'$, and after a few steps we get $\alpha' = \alpha = 27°43'$ approximately. The convergence becomes, of course, more rapid as we proceed with the intervals. The results for the first five intervals for (ρ, α) are

$$(12.01,\ 0.4838);\ (20.95,\ 0.3099);\ (29.93,\ 0.2327);$$
$$(38.94,\ 0.1881);\ (47.96,\ 0.1601).$$

The root in the first interval is $10.63+5.59i$. To each such root there is a corresponding conjugate root.

Calculus has been used in (6) to save space; but it may be shown by trigonometry that $\theta\cot\theta$ decreases and $\theta/\sin\theta$ increases as θ increases in any of the given intervals. This gives the result used. Also the right side of (9) may be shown to decrease under the given condition by use of algebra; but the proof is rather longer than by the use of calculus.

Also solved by F. L. Wilmer.

3519[1931, 589]. *Proposed by Norman Anning, University of Michigan.*

AB is a fixed diameter and CD is a moving diameter of a given circle $ACBD$. E is the mid-point of arc BC. DE and AC intersect in P. Prove that DP is normal to the locus of P.

Solution by Mrs. Ruth B. Smith, Oberlin, Ohio

Choose the fixed circle $ABCD$ so that its center is at $(a, 0)$ and its radius is a. The coordinates of C are easily found to be $(2a\cos^2\alpha,\ a\sin 2\alpha)$, where $\alpha=\angle BAC$. The coordinates of D, $(2a\sin^2\alpha,\ -a\sin 2\alpha)$, easily follow from those of C. The coordinates of E are $[a(1+\cos\alpha),\ a\sin\alpha]$, and the equation of DE is then found to be

$$(1)\qquad (\sin 2\alpha+\sin\alpha)x-(\cos 2\alpha+\cos\alpha)y-a(2\sin\alpha+\sin 2\alpha)=0.$$

The coordinates of P are obtained by solving (1) with $y=x\tan\alpha$, and they are found to be

$$(2)\qquad x=2a\cos\alpha(1+\cos\alpha),\quad y=2a\sin\alpha(1+\cos\alpha).$$

The slope of the locus of P is

$$(3)\qquad \frac{dy}{dx}=-\frac{\cos\alpha+\cos 2\alpha}{\sin\alpha+\sin 2\alpha},$$

and, since it is the negative reciprocal of the slope of (1), the line DE is normal to the curve.

In Cartesian form the equation of the locus of P is

$$x^2+y^2-2ax=2a(x^2+y^2)^{1/2},$$

which is the equation of a cardioid with its cusp at the origin.

A Note by Otto Dunkel. If O is the center of the circle $ABCD$, OE is parallel to CP, and therefore $CP=2OE=AB$. Hence the locus of P is a cardioid. Since CP has a constant length, its instantaneous center of rotation I is the intersection of the normal to the circle CD and the normal PI to the locus of P. Since CP passes through the fixed point A, the perpendicular to PA at A passes through I. Hence $I\equiv D$, and DP is normal to the locus of P.

Also solved by R. P. Agnew, S. F. Bibb, A. D. Bradley, J. H. Butchart, C. S. Carlson, Mannis Charosh, Rufus Crane, N. C. Fisk, Edward Fleisher, R. Goormaghtigh, H. Grossman, L. S. Johnston, Roy MacKay, V. F. Murray, A. Pelletier, W. H. Rasche, H. D. Ruderman, William Sell, L. S. Shively, E. Siroky, Wallace Smith, F. Underwood, J. M. West, and Roscoe Woods.

3520 [1931, 589]. *Proposed by Elijah Swift, University of Vermont.*

To construct a triangle, given the base BC, the opposite angle, A, and the length of the bisector, t, of the angle A. Show the construction is possible with ruler and compass and give a simple construction.

Solution by Richard Morris, Rutgers University

On BC as chord construct a circle containing angle A in one of its segments. The mid-point, K, of the arc of the segment containing the supplement of angle A is a fixed point, and chord, BK, becomes known, since the length of bisector t is given. A perpendicular from K upon BC locates D on BC and E on the circle. The line AK would intersect BC at U. The right triangles BEK and BDK are similar, hence $(BK)^2 = KD \cdot KE$. Also triangles KAE and KDU are similar, and hence $KD \cdot KE = KA \cdot KU = (BK)^2$, or $KA(KA - t) = (BK)^2$. Thus

$$KA = \tfrac{1}{2}[t \pm \{t^2 + (2BK)^2\}^{1/2}].$$

Now construct a right triangle whose legs are t and $2BK$. The hypotenuse of this triangle plus t equals twice KA. With K as center and KA as radius the vertex A is located on the circumcircle. There will be 2, 1, or 0 solutions. Also the hypotenuse minus t equals twice KU. The radius KU locates U on BC and the line KU produced locates A on the circle. As before there will be 2, 1, or 0 solutions.

The same type of construction may be used to obtain the triangle if the external bisector is given.

The solution presented is a slight modification of that suggested in Casey's Sequel to Euclid, edition of 1886 page 80.

A Note by the Editors. There is no real construction if $2t \tan\frac{1}{2}A < BC$. If the inequality sign is replaced by the equality sign there is a single construction of an isoscoles triangle.

Also solved by Eugene Alliot, Brother Aurelius, L. M. Bauer, Mannis Charosh, Edward Fleisher, R. Goormaghtigh, H. Grossman, Theodore Lindquist, Mrs. Elizabeth Nixon, A. Pelletier, W. H. Rasche, H. D. Ruderman, William Sell, Wallace Smith, F. Underwood, Hymen Weisberg, J. M. West, and Roscoe Woods.

3521 [1932, 45]. *Proposed by J. M. West, Pennsylvania State College.*

Show that the equations of the tangents of the circle, $x^2 + y^2 = r^2$, through the external point (a, b) are

$$(ar \pm bt)x + (br \mp at)y = r(a^2 + b^2),$$

where t is the length of the tangents from the point (a, b) to the circle.

Solution by Dorothy McCoy, Belhaven College

The tangent to the circle $x^2+y^2=r^2$ at the point (α, β) on the circle is

$$\alpha x + \beta y = r^2. \tag{1}$$

The tangent at (α, β) will pass through (a, b) if

$$a\alpha + b\beta = r^2 \tag{2}$$

but (α, β) is on the circle; hence

$$\alpha^2 + \beta^2 = r^2. \tag{3}$$

The tangent is perpendicular to the radius at the point of tangency; hence

$$a^2 + b^2 - r^2 = t^2, \tag{4}$$

where t is the length of the tangent.

Solving (2) and (3) for α we get

$$\alpha = \frac{r}{a^2+b^2}\left\{ar \pm \left[r^2a^2 - (a^2+b^2)(r^2-b^2)\right]^{1/2}\right\}$$

which becomes by use of (4)

$$\alpha = \frac{r}{a^2+b^2}(ar \pm bt). \tag{5}$$

Substituting (5) in (2), we obtain

$$\beta = \frac{r}{a^2+b^2}(br \mp at).$$

Substituting these values of α and β in (1) we have the required equations of the tangents.

Also solved by E. F. Allen, H. T. R. Aude, George A. Baker, C. A. Barnhart, L. M. Bauer, S. F. Bibb, J. H. Butchart, W. B. Campbell, C. S. Carleton, A. G. Clark, J. M. Feld, S. E. Field, Edward Fleisher, J. W. Foust, H. G. Funkhouser, E. A. Goodhue, R. Goormaghtigh, D. F. Gunder, J. D. Hill, C. R. Hillard, H. S. Kaltenborn, J. J. Knox, Charlotte M. Lacy, C. H. Lehmann, J. D. Leith, Theodore Lindquist, Roy MacKay, J. B. Meyer, W. K. Morrill, W. V. Parker, A. Pelletier, A. R. Randall, A. W. Rankin, C. C. Richtmeyer, H. A. Robinson, H. D. Ruderman, Rafael Sanchez-Diaz, Hazel E. Schoonmaker, H. L. Schug, Wm. Sell, Mrs. Ruth B. Smyth, R. C. Staley, F. Underwood, Hymen Weisberg, Paul Wernicke, F. G. Williams, F. L. Wilmer, Roscoe Woods, and R. C. Yates.

3523 [1932, 45]. *Proposed by Dewey C. Duncan, University of California.*

Solve completely the equation

$$x^7 + Ax^6 + Bx^4 + Cx^2 + D = 0,$$

A, B, C, D, being unknown constants and given that the equation has a double root and one triple root.

Solution by F. Underwood, University College, Nottingham, England

Denote the roots of the given equation by

$$\alpha,\ \alpha,\ \alpha,\ \beta,\ \beta,\ \gamma,\ \delta.$$

CASE 1. When $D=0$, $\beta=0$, and the equation

$$x^5 + Ax^4 + Bx^2 + C = 0 \tag{2}$$

has roots α, α, α, γ, δ. Putting $\sum\alpha=-A$; $\sum\alpha\gamma=0$; $\sum\alpha\gamma\delta=-B$; $\sum a^2\gamma\delta=0$; $\alpha^3\gamma\delta=-C$, we find that $\gamma+\delta=-9\alpha/8$, $\gamma\delta=3\alpha^2/8$, $A=-15\alpha/8$, $B=5\alpha^3/4$, $C=-3\alpha^5/8$, and hence the equation,

$$8x^7 - 15\alpha x^6 + 10\alpha^3x^4 - 3\alpha^5x^2 = 0,$$

has roots α, α, α, 0, 0, $\alpha\{-9\pm(-15)^{1/2}\}/16$ for all values of α.

Case 2. When $D\neq 0$, the equation has no zero roots. Proceeding as before, but using only

$$\sum\alpha\beta = \sum\alpha\beta\gamma\delta = \sum\alpha^2\beta^2\gamma\delta = 0,$$

(i.e. the three relations which do not involve the constants A, B, C, D), and eliminating γ, δ we find that a necessary condition is

$$3\alpha^5 + 18\alpha^4\beta + 43\alpha^3\beta^2 + 48\alpha^2\beta^3 + 24\alpha\beta^4 + 4\beta^5 = 0,$$

which reduces to

$$(\alpha+\beta)^2(3\alpha^3 + 12\alpha^2\beta + 16\alpha\beta^2 + 4\beta^3) = 0.$$

It is found immediately that an equation of the type required cannot exist for $\alpha+\beta=0$, and so, putting $\alpha/\beta=z$, we have

$$3z^3 + 12z^2 + 16z + 4 = 0. \tag{3}$$

Substituting $z=y/3$, followed by $y=t-4$, this equation is reduced to $t^3-28=0$, so that the only real root of equation (3) is $\{(28)^{1/3}-4\}/3$, giving the only possible real value of α/β. If we regard this as giving α in terms of β, γ and δ are expressed in terms of β by means of the three relations previously used, and the other four relations then give A, B, C, D in terms of β, so that an equation of the required form is obtained in the general case.

It may be remarked that the actual evaluation of these constants necessarily involves long and clumsy work.

Note by the Editors. In regard to the last remark by the solver, it may be

added that the results do not appear to be of sufficient interest or importance to justify the labor of obtaining them.

3524 [1932, 46]. *Proposed by W. H. Echols, University of Virginia.*

At the corners of any equilateral triangle ABC let there be hinged three equilateral triangles ALM, BNO, CPQ of any sizes or positions.

Then will the midpoints of each of the sets of the three segments

$$(LQ,\ OP,\ MN),\quad (BQ,\ CN,\ OP),$$
$$(LB,\ OA,\ MN),\quad (AP,\ CM,\ LQ)$$

be the corners of an equilateral triangle.

This problem has been covered as a special case in a paper by J. R. Musselman (This MONTHLY, vol. 39, 1932, p. 290) and in the note by R. Goormaghtigh in this issue, p. 535. Also solved by E. Alliott, J. H. Butchart, W. Sell, Paul Wernicke, and F. L. Wilmer.

3526 [1932, 46]. *Proposed by R. E. Gaines, University of Richmond.*

If a triangle PQR be inscribed in the ellipse $x^2/a^2+y^2/b^2=1$, and if PQ and PR are tangent to the hyperbola, $x^2/a^2-y^2/b^2=1$, the locus of the pole of QR with respect to the ellipse is identical with the locus of the pole of QR with respect to the hyperbola.

Solution by Roscoe Woods, The State University of Iowa

Let the coordinates of P be (x_1, y_1). Suppose the equations of the lines PQ and PR to be $y-y_1=m_i(x-x_1)$ $(i=1, 2)$. Set $X=x-x_1$ and $Y=y-y_1$ and multiply these two equations together. We have

$$Y^2 - (m_1 + m_2)XY + m_1m_2X^2 = 0. \tag{1}$$

The line $y-y_1=m(x-x_1)$ will be tangent to the hyperbola $x^2/a^2-y^2/b^2=1$ when $y_1-mx_1=\pm(a^2m^2-b^2)^{1/2}$ or when $m^2(x_1^2-a^2)-2mx_1y_1+b^2+y_1^2=0$. If m_1 and m_2 are the roots of the equation just written, we may write equation (1) in the form

$$X^2(y_1^2 + b^2) - 2x_1y_1XY + Y^2(x_1^2 - a^2) = 0. \tag{2}$$

This is the equation of the lines PQ and PR which pass through the new origin $X=0$, $Y=0$.

Replacing x and y by $X+x_1$ and $Y+y_1$, respectively in the equation of the ellipse we have

$$b^2X^2 + a^2Y^2 + 2b^2x_1X + 2a^2y_1Y = 0. \tag{3}$$

Let the equation of QR be $AX+BY=C$. If we make equation (3) homogeneous by means of this equation, we have

$$X^2(b^2C + 2b^2x_1A) + Y^2(a^2C + 2a^2y_1B) + 2(b^2x_1B + a^2y_1A)XY = 0. \tag{4}$$

This equation is the equation of the pair of lines joining the origin ($X=0$, $Y=0$) and the intersections of the line QR and the ellipse. Hence the equations (2) and (4) represent the same lines and must be identical. Equating coefficients of like powers we have after a short calculation $A:B:C=x_1/2a^2:-3y_1/2b^2:2y_1^2/b^2$. Hence the equation of QR becomes, after replacing X and Y by their values $x-x_1$ and $y-y_1$ respectively,

$$xx_1/a^2 - 3yy_1/b^2 = 1. \tag{5}$$

The poles of the line (5) with respect to the ellipse and the hyperbola are $(x_1, -3y_1)$ and $(x_1, 3y_1)$ respectively. Since the point (x_1, y_1) is any point on the ellipse, the loci of these poles are identical. It is easily found to be $x^2/a^2+y^2/9b^2=1$. This proves the proposition.

Also solved by J. H. Butchart, W. B. Campbell, Rufus Crane, F. Underwood, and Paul Wernicke.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this Department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

The quadrennial International Congress of Mathematicians was held as Zurich, Switzerland, September 4–12, 1932. Twenty general lectures in variout fields of mathematics were given by invited representatives of ten different nationalities. The United States was represented in able addresses given by Professors Marston Morse of Harvard University and J. W. Alexander of Princeton University. Four half days were devoted to sectional meetings at which results of recent research were given in about 240 briefer papers. The full text of the invited lectures and extended abstracts of the contributed papers will appear in the printed volume of the Congress. The Congress was characterized by reports of extensive current research and by profitable personal contacts between mathematicians of different countries, rather than by announcements of any epoch-making discoveries.

The International Commission on the Teaching of Mathematics held important sessions under the presidency of Professor David Eugene Smith. Doctor Smith mapped out a program for the next few years. He urged the great need for the careful collection of information as to recent developments in mathematical teaching the world over, in continuation of the well-known series of books issued through the activity of the Commission beginning about 1907; he instanced in particular our ignorance of what is being done in connection with the tremendous experiment of the Russian people. Professor Jacques Hadamard is to serve as president of the Commission for the next four years.

A session was held of the International Mathematical Union. Because of the unsatisfactory character of its work and because of its decidedly political nature, representing as it does the governments rather than the scientific or-

ganizations, the Union has been subjected to severe criticism; and at the insistence of American, English and Danish mathematicians it was agreed to disband the Union and to ask the Congress to appoint a committee to investigate the need for a continuing body of the sort.

Approximately 650 mathematicians with 200 persons accompanying them came to the Congress from 41 countries. Of these there were 68 mathematicians from the United States and Canada, with 30 accompanying them. Elaborate provisions were made for the social entertainment of the visitors, including several half-days given to outings in that general region of Switzerland, a festal evening of welcoming speeches and spectacular dancing at the Stadttheater, and a low-cost excursion to the Jungfraujoch following the week of the Congress.

The Congress accepted the invitation to hold the next Congress at Oslo, Norway, four years hence.

A full account of the Congress appears in the Bulletin of the American Mathematical Society.

The first number of a new quarterly journal of mathematics, Scripta Mathematica, has just appeared. It is published by Yeshiva College at Amsterdam Avenue and 186th Street, New York. Jekuthiel Ginsburg is the editor. Raymond Clare Archibald, Cassius Jackson Keyser, Louis Karpinski, Gino Loria, Lao Genevra Simons and David Eugene Smith are the other members of the editorial staff. The journal is to be devoted to the interests of the philosophy, history, and expository treatment of mathematics.

Each number will contain approximately ninety-six pages of reading matter described by the editor as follows:

"The pages of the periodical will be devoted chiefly to the history and philosophy of mathematics. The expositorial treatment of mathematics will be included for the purpose of giving to the reader a knowledge of what is being done at present in various branches of the subject and of the history of mathematics in the making.—A special effort will be made to have the articles free from such technicalities as would repel the intelligent reader who has not had a thorough training in mathematics."

The American Institute of Physics announces that it will shortly begin publication of a new periodical, the Journal of Chemical Physics, under the editorship of Professor H. C. Urey, of Columbia University.

The American Mathematical Society appointed the following as its official delegates to the International Mathematical Congress at Zurich: Edward Kasner, C. N. Moore, R. G. D. Richardson, E. B. Stouffer, and J. D. Tamarkin.

Dr. Joseph Slepian, of the Westinghouse Electric and Manufacturing Company, has received a John Scott award, for his work in connection with gases and fundamental inventions involving these discoveries.

Professor R. A. Millikan has received an honorary doctorate of science from Harvard University.

Professor J. M. Thomas, of Duke University, has been elected chairman of the mathematics section of the North Carolina Academy of Sciences, and Professor Helen Barton, of North Carolina College, secretary of that section.

Professor Oswald Veblen, of Princeton University, represented the American Association for the Advancement of Science at the meeting of the British Association at York, August 31 to September 7, 1932.

Dr. O. J. Farrell has been appointed assistant professor of mathematics at Union College.

Assistant Professor J. H. Fithian has been promoted to an associate professorship of mathematics at the Newark College of Engineering.

Dr. H. H. Germond has been appointed assistant professor of mathematics at the University of Florida.

Assistant Professor R. F. Graesser, of the University of Arizona, has been promoted to an associate professorship of mathematics.

R. A. Hefner has been promoted to an assistant professorship at the Georgia School of Technology.

Assistant Professor L. D. Hemenway has been promoted to an associate professorship at Simmons College.

Dr. G. B. Price has been appointed instructor in mathematics at Union College.

Mr. A. E. Whitford, of the University of Wisconsin, has been appointed professor of mathematics at Alfred University.

Dr. Kamcheung Woo has been appointed professor of mathematics at Sun Yatsen University, Canton, China.

Professor W. M. Brodie, of the Virginia Polytechnic Institute, died in April, 1932 of pneumonia. He had been a member of the Association since 1920.

Professor C. N. Dickinson, of Hollins College, a charter member of the Association, died May 25, 1932.

Associate Professor James E. Donahue, of the University of Vermont, died suddenly of cerebral hemorrhage on August 13, 1932, at the age of fifty-two years. He was a charter member of the Association.

Dr. Harold Jacoby, retired professor of astronomy at Columbia University, died July 20, 1932, at the age of sixty-seven. Professor Jacoby was a member of the American Mathematical Society at its organization as the New York Mathematical Society, and was its first treasurer.

Alfred H. Jekel, president of the Colorado Clay and Mining Co., Boulder, a member of the Association since 1925, was killed in an automobile accident near Pine, Colo., March 8, 1932.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the Editor-in-Chief, W. B. Carver, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. Johnson, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the Secretary-Treasurer of the Association, W. D. Cairns, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the Secretary-Treasurer, W. D. Cairns, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Seventeenth Annual Meeting of the Association, Atlantic City, N.J., Dec. 27-30, 1932.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1932 and reported to the Secretary.

Illinois, Urbana, May 6-7.
Indiana, Indianapolis, May 6-7.
Iowa, Cedar Falls, April 29-30.
Kansas, Topeka, Feb. 13.
Kentucky, Lexington, May.
Louisiana-Mississippi, Oxford, Miss., March 11-12.
Maryland-District of Columbia-Virginia, Baltimore, Md., Dec. 3.
Michigan, Ann Arbor, March 19.
Minnesota, River Falls, Wis., May 7.
Missouri.
Nebraska, Omaha, May 6-7.
Ohio, Columbus, Ohio, April 7.
Philadelphia, Philadelphia, Pa., Nov. 26.
Rocky Mountain, Laramie, Wyo., April 15-16.
Southeastern, Gainesville, Fla., Mar. 18-19.
Southern California, San Diego, March 26.
Texas, Austin, Jan. 30.

Affiliated Organizations: The New England Association of Teachers of Mathematics. The National Council of Teachers of Mathematics.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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THE AMERICAN MATHEMATICAL MONTHLY

By H. S. EVERETT AND R. G. SANGER, The University of Chicago

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Numbers in black-face type refer to problems; those in light-face to pages.

NOTES AND NEWS

PERSONAL MENTION

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Volume XXXIX 1932:

P. 28, A Correction to vol. XXXV, p. 11.
P. 487, line 13, for "V. H. Ivanoff" read "V. F. Ivanoff."
P. 506, line 13, for "R. M. Ashmun" read "R. N. Ashmun."
P. 507, line 9, add: "by invitation;" and, line 11, omit "by invitation."
P. 510, line 4, for "Mathias" read "Mathis."
P. 537, fifth line from bottom, for "Aitkin" read "Aitken."
P. 547, sixth line from bottom, for "1625" read "1525."
P. 555, line 15, for "Smith" read "Smyth."

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXIX, 1932

NUMBER 10, DECEMBER

PUBLISHED BY THE ASSOCIATION

MENASHA, WIS., AND ITHACA, N.Y.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members

$5.00 a Year, Single Copies 60 cents, to Others

THE TWENTY-FIRST MEETING OF THE IOWA SECTION

The twenty-first meeting of the Iowa Section of the Mathematical Association of America was held with the Iowa Academy of Science at Iowa State Teachers College, Cedar Falls, Iowa, on April 29 and 30, 1932. The meetings were held in Room 214, Administration Building.

The attendance was about thirty-five, including the following twenty-one members of the Association: F. A. Brandner, L. M. Coffin, I. S. Condit, N. B. Conkwright, A. T. Craig, C. W. Emmons, Dora E. Kearney, F. M. McGaw, J. V. McKelvey, J. F. Reilly, H. L. Rietz, B. D. Roberts, Fred Robertson, W. J. Rusk, E. R. Smith, G. W. Snedecor, C. W. Strom, J. S. Turner, L. E. Ward, C. W. Wester, Roscoe Woods.

The Section chairman, Professor C. W. Strom, presided at both the Friday afternoon and Saturday morning sessions, relieved for a time by the vice-chairman, Professor B. D. Roberts. Dinner was enjoyed together Friday evening in the Crystal Room of Bartlett Hall, at the close of which the chairman of the local committee, Professor Condit, and the newly elected officers made brief remarks. The officers elected for 1932–1933 are as follows: Chairman, L. M. Coffin, Coe College; Vice-Chairman, Julia T. Colpitts, Iowa State College; Secretary-Treasurer, J. F. Reilly, University of Iowa.

The program consisted of fifteen papers, as follows:

1. "A generalization of the probability integral" by Professor E. R. Smith, Iowa State College.

2. "Discussion of the value of daily written work in mathematics" by Professor Roscoe Woods, University of Iowa.

3. "The utility of analysis of variance in biological research" by Professor G. W. Snedecor, Iowa State College.

4. "Analysis of variance in a $2\times n$ table with disproportionate frequencies" by Professor A. E. Brandt, Iowa State College, by invitation.

5. "A sampling test of the technique of analyzing variance in a $2\times n$ table with disproportionate frequencies" by Bernice Brown, Iowa State College, by invitation.

6. "Some conics with names" by Professor Roscoe Woods, University of Iowa.

7. "Complete systems under certain finite groups" (second paper) by Professor C. W. Strom, Luther College.

8. "The function idea for undergraduates" by Professor C. W. Wester, Iowa State Teachers College.

9. "A set of determinants analogous to those of Tchebychef for the factorization of large numbers" by Professor J. S. Turner, Iowa State College.

10. "A relation between graduate and undergraduate work" by Professor B. D. Roberts, Parsons College.

11. "A mapping problem related to the equilateral triangle" by Professor J. J. L. Hinrichsen, Iowa State College, by invitation.

12. "A graphical determination of the principal moments of inertia of a plane area, with some extensions" by Professor D. L. Holl, Iowa State College, by invitation.

13. "An introduction to the summation principle" by Professor J. V. McKelvey, Iowa State College.

14. "Cost of paying for a home by borrowing from a building and loan association compared with a six per cent mortgage loan" by Professor J. F. Reilly, University of Iowa.

15. "On the correlation of measurements under a constant law of error" by Doctor A. T. Craig, University of Iowa.

Abstracts of some of these papers follow:

1. In his paper Professor Smith showed that the integral of exp $(-ax^4-bx^3-cx^2-dx)$ from minus to plus infinity can be evaluated by means of a series whose terms are Heine polynomials. The expressions for the coefficients of the series involve Gamma functions. The rapid convergence of the series facilitates the numerical calculations.

2. In this paper Professor Woods not only discussed the possible attitudes of the teacher toward daily written work in mathematics but presented some devices whereby the time of grading written work may be lessened and the students discouraged from copying. Also the advantages both to the student and the instructor were emphasized.

4. The method of analysis of variance as originated by Dr. R. A. Fisher is based on equal cell frequencies. In his paper Professor Brandt offered a generalization of the Fisher formulas for a $2\times n$ classification in which the cell frequencies are disproportionate.

6. In mathematical literature there are several conics referred to by name, for example, the hyperbola of Apollonius. In this paper Professor Woods sets forth a large number of these conics with their most common definitions and names. As far as possible, references which give a good discussion of the conics have been selected.

7. Professor Strom presented a continuation of his paper given last year on "Complete systems under certain finite groups."

8. Professor Wester pointed out some reasons for the failure of so many undergraduates to understand and use the notion *function*, and suggested some simple means for removing some of the difficulties.

9. Tchebychef has given a number of determinants D suitable for the factorization of large integers (Journal de Mathématique, vol. 16, 1851, pp. 257–282). With these an *integral* representation of an integer m in the form x^2-Dy^2 is sought, where x lies within certain limits. With the present set of determinants Professor Turner seeks a representation of m in the form $ax^2-by^2=m$, where $ab=D$, and x lies within certain limits. As with Tchebychef, m is prime if and only if there is one such representation, and from two representations by the

same form, its factors can be found. However, the limits are smaller than those of Tchebychef, and x, y may be either integers, or fractions with denominator 2. These determinants are: 5, 21, 77, 165, 285, 357, 437, 957, 1085, 1365, 2397, 2805, 4485, 7917, 8645, 26565. In each case $T^2-D=4$. If $ax^2-by^2=m$ is possible where $ab=D$, there is a representation with

$$\left(\frac{m}{a}\right)^{1/2} \leqq x \leqq \left(\frac{(T+2)m}{4a}\right)^{1/2}.$$

10. In his paper Professor Roberts suggested that the graduate attitude be introduced into elementary courses, and he made a plea for more vital teaching.

11. Consider the equilateral triangle ABC. With A as center and the altitude of the triangle as radius, swing an arc $A'B''$ from the center A' of BC to the point B'' on the side AC. Let S denote the area $A'B''C$ and let P be a point of S. Form a triangle ABP and construct two lines $AR_i (i=1, 2)$ and PG of length $(\overline{AB}\sqrt{3})/2$ extending from the vertices A, P to their opposite sides BP, AB, where angle BGP is acute. If the points of intersection are denoted by $Q_i (i=1, 2)$, then Q_i are defined to be the transforms of the point P.

Professor Hinrichsen obtained the equations of transformation and studied the transforms of certain simple families of curves. Then he considered the transformation from Q_1 to Q_2. He found this transformation to be similar to a reflection in a straight line. It is one-to-one except along the straight line segment AB'' which is carried into the point A.

12. In this paper Professor Holl gave a simple geometric representation of moments and products of inertia of a plane area with the aid of a circle instead of the ellipse. He represented the scalar invariants as geometric invariants, and made extensions to curvatures of normal sections of a surface, also to stresses, strains and moments in elastic theory.

13. In this paper Professor McKelvey presents a detailed schedule of lessons covering approximately the first two weeks' instruction on the Definite Integral.

15. Let successive sets of n observations each be made upon a variable which is subject to a constant law of error and denote by x, y, z, $x \leqq y \leqq z$, the pth, qth and rth measurements of the sets. Doctor Craig derived the correlation function $V=\phi(x, y, z)$ and showed that the regression surfaces are planes. He proved that the correlation between the pth and rth measurements within one set of observations is equal to that between the pth measurement of one set and the rth measurement of a second set, the two sets having identical qth measurements. The correlation function $W=\psi(u, v)$ of $u=z-y$ and $v=y-x$ was also given. By assigning appropriate values to p, q, r, the correlation between various averages is readily determined.

J. F. REILLY, *Secretary-Treasurer*

THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The sixteenth regular meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the University of Wyoming, Laramie, Wyoming, on Friday and Saturday, April 15 and 16, 1932. There were three sessions, Professor O. H. Rechard, chairman of the Section, presiding at each.

The attendance was thirty-five, including the following thirteen members of the Association: Jack Britton, Pauline F. Folk, G. W. Gorrell, C. A. Hutchinson, M. H. Ingraham, A. J. Kempner, Claribel Kendall, A. J. Lewis, S. L. Macdonald, A. S. McMaster, O. H. Rechard.

At the business meeting the following officers were elected for the ensuing year: Chairman, A. G. Clark, Colorado Agricultural College; Vice-Chairman, W. V. Lovitt, Colorado College; Secretary, A. J. Lewis, University of Denver.

Members of the Association and friends were guests of the University of Wyoming at a dinner on the evening of April 15. The principal speakers at the dinner were President A. G. Crane, of the University of Wyoming, and Professor S. L. Macdonald, of the Colorado Agricultural College. The Section was fortunate in having Professor M. H. Ingraham, of the University of Wisconsin, present as guest speaker.

The following seven papers were read:

1. "Operational calculus" by Professor C. A. Hutchinson, University of Colorado.
2. "Contributions of mathematics to life insurance" by Professor G. W. Gorrell, University of Denver.
3. "The Baire classification of functions" by Professor O. H. Rechard, University of Wyoming.
4. "The development of the postulational method in mathematics" by Professor M. H. Ingraham, University of Wisconsin.
5. "The teaching of calculus" by Professor S. L. Macdonald, Colorado Agricultural College.
6. "Geometric progressions" by Professor A. J. Kempner, University of Colorado.
7. "Controversial topics in mathematical logic" by Professor M. H. Ingraham, University of Wisconsin.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. This paper is expository in character, and presents the salient features of the operational calculus, as applied to the solution of problems in electrical engineering.

2. This paper gives a brief history of the problem of life insurance and shows the role mathematics has played in its development.

3. In this paper, Professor Rechard presents the Baire method of classifying functions as given by Baire in his dissertation *"Sur les functions de variables*

réelles" published in "Annali di Matematica Pura et Applicata" in 1899. Functions of classes one, two, and three are exhibited and the classical proofs are given that the classification can be correct to any number α of the first or second class, but is not exhaustive. Comparison is made between the Baire, Young, and Sierpinski methods of classification. Some of the questions which need to be answered before a necessary and sufficient condition can be found for a function to be of class two, Baire, are suggested.

4. This paper discusses the historical development of the postulational method, and the major characteristics and uses of this method. Especial attention is paid to the use of the method for generalization, in which case the postulates should be non-categorical, and for establishing isomorphisms in which case they should be categorical. As illustrations, the postulates for a field, for Euclidean geometry, and Huntington postulates for an arithmetic mean are used. It is pointed out that at least logic is generally assumed as a background for sets of mathematical postulates.

5. It is the belief of the writer that in an elementary course in calculus, definitions, principles and descriptive matter should be reduced to a minimum consistent with clearness and rigor. It is maintained in this paper that most text books are at fault in this particular. The paper maintains that the derivative is not a rate, it is not a slope. The derivative is the limit of a ratio and is an entirely abstract concept. Rate and slope are merely properties of the derivative. By making clear that in certain cases a distinction is necessary between the limit of a ratio and the ratio of the limits the writer holds that the definition of the derivative may be clarified, which is seldom done by text book writers.

7. This paper discusses some of the current attempts to examine and explain the relation of logic to mathematics. In particular three schools of thought are considered. 1) The school led by Russell which attempts to define all mathematics in logical terms. 2) The school led by Brouwer which makes mathematics prior to logic and places stringent limitations on the use of classical logic. 3) The school led by Hilbert which is interested in the formal structure of mathematics and the questions of formal consistency, and studies mathematics as a set of marks on paper which are made in accordance with certain rules. The attempt is made to give a sympathetic discussion of each of these three points of view.

A. J. LEWIS, *Secretary*

ERNEST JULIUS WILCZYNSKI

Ernest Julius Wilczynski was born in Hamburg, Germany, on November 13, 1876, and died in Denver, Colorado, on September 14, 1932, after a lingering illness of about ten years. With respect to his original contributions to existing mathematical knowledge, his influence on the development of mathematical institutions in the United States, his interest in the promotion of good teaching,

and his heroic example of how a great man can meet and endure physical disability, he set a shining example and proved himself to be a leader of whom we may well be proud.

Wilczynski's early life and education can be briefly sketched. After he had gone to elementary school but a short time in Hamburg, his family migrated to the United States and settled in Chicago. Here Wilczynski graduated from the North Division High School. Then, with the assistance of an uncle, he returned to Germany for the purpose of studying at the University of Berlin. His interest lay in the fields of mathematics, physics and mathematical astronomy, and he specialized in the latter. He received the degree of doctor of philosophy from the University of Berlin in 1897, while still in his twenty-first year, the subject of his thesis being "Hydrodynamische Untersuchungen mit Anwendung auf die Theorie der Sonnenrotation."

Wilczynski returned to the United States after receiving his degree, and for a year was a computer in the Office of the Nautical Almanac at Washington. Then in 1898 he secured a post as instructor in mathematics at the University of California, and thereafter his advancement was rapid. At California he served until 1907 as instructor, assistant professor, and associate professor, with an interruption from 1903 to 1905 when he was in Europe as research assistant and associate of the Carnegie Institution of Washington. He was associate professor at the University of Illinois from 1907 to 1910, and at the University of Chicago from 1910 to 1914. He became professor of mathematics at Chicago in 1914, and was made professor emeritus in 1926 when it was clear after three years of ill health that he would never be able to resume his active duties.

In spite of the fact that his greatest reputation was made as a geometer, Wilczynski began his scientific career as a mathematical astronomer and made notable contributions in this field. Then his interest turned to projective differential geometry. He developed a new method in this domain, and his energy in applying it soon won for him wide recognition as the creator of a new mathematical discipline. Toward the end of his career he was contributing to the theory of functions of a complex variable.

It must not be thought that Wilczynski became a great research investigator at the expense of interest or proficiency in teaching. In fact he was a very clear and effective teacher. He had a beautiful English style, and his lectures both to graduate and undergraduate students were models of elegant mathematical exposition. His interest in instruction at the undergraduate level is exemplified by the two texts that he wrote for freshmen, one a trigonometry and the other a college algebra. In the trigonometry, especially, his fondness for the heuristic method of presentation found ample scope. It was a tenet of his that the first prerequisite for effective instruction was that the teacher should win the confidence of his students and should make them come to feel thoroughly at ease in his class room. After all, perhaps the most eloquent comment on his qualities as a teacher is the fact that his advanced courses were most popular with graduate students of mature years who were themselves experienced teachers, coupled

with the fact that Wilczynski directed the theses of twenty-five candidates for the doctor's degree, besides innumerable master's theses.

Wilczynski was interested in the affairs of several scientific organizations and was honored by them from time to time in various ways. He was at one time vice-president of the American Mathematical Society, and a member of the council of the Mathematical Association of America. He was a lecturer at the New Haven Colloquium, and associate editor of the Transactions of the Society. In 1909 he won a prize of the Royal Belgian Academy of Sciences, and in 1919 was elected a member of the National Academy of Sciences at Washington.

Wilczynski is survived by his wife and three daughters. Their loss is shared by his former students and colleagues, as well as by many other mathematicians who have come under his influence. The memory of his genial and friendly personality, his power as a teacher, and his enthusiasm for mathematics, the material evidence of his seventy-seven books and papers published in four languages, and nineteen periodicals, together with the monument of a new domain of geometry so largely created by him—these things constitute a legacy in which all of us may find a great wealth of inspiration.

ERNEST P. LANE

COLLEGIATE MATHEMATICS NEEDED IN THE SOCIAL SCIENCES[1]

The background and purpose of this report are indicated by parts of the letter which Professor E. B. Wilson, President of the Social Science Research Council in June, 1930, sent to Professor Tolley informing him of the appointment of the committee and its duties.

"You are of course aware of the request by the Advisory Committee on Social and Economic Research in Agriculture for the appointment of a committee of this Council on mathematics to determine what mathematics should be taught to students of the social sciences . . . The Committee on Problems and Policy in thinking the matter over decided that the time was not ripe for the appointment of a national committee but decided to appoint a sub-committee of the Advisory Committee on Agriculture which . . . should make a preliminary report on the subject. The only way I see of making progress is for the social scientists to know what they want, to explain to the mathematicians the

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[1] This report was prepared for the Social Science Research Council by a Committee consisting of H. R. Tolley, Director of the Giannini Foundation of Agricultural Economics, University of California; F. L. Griffin, Professor of Mathematics, Reed College; Holbrook Working, Economist of the Food Research Institute, Leland Stanford University; Charles H. Titus, Professor of Political Science, University of California at Los Angeles; and Mordecai Ezekiel, Assistant Chief Economist, the Federal Farm Board; and submitted to the Advisory Committee on Social and Economic Research in Agriculture. The report was presented and discussed (See this MONTHLY, vol. 39, 1932, p. 503) at the meeting of the Mathematical Association in Los Angeles in September, 1932.

in economics, and in the other disciplines as well, would on further consideration be included in the group for which the mathematics courses should be prerequisite.

In closing, the committee suggests (1) that further study of the problem with particular reference to the specific needs of the different social sciences and the coordination between courses in mathematics and the different social science disciplines is highly desirable, and (2) that interested institutions be encouraged to proceed with the development of courses in mathematics designed primarily to meet the needs of students in the social sciences.

SOME RELATIONS IN THE GEOMETRY OF THE TRIANGLE

By ALBERT A. BENNETT, Brown University

Let B_i $(i=1, 2, 3)$ be the vertices of a base triangle, with I as incenter, and C_i as points of contact of the inscribed circle, K, with radius r. Let R be the radius of the circumcircle, with a_i the tangent line at B_i. Let B_i' be the midpoint of the line segment B_iI. Let b_i be a side-line of the triangle $B_1B_2B_3$, b_i' a side-line of $B_1'B_2'B_3'$, and b_i'' the line parallel to b_i and b_i' through I. Let P_i be the midpoint of that arc B_jB_k of the circumcircle, not containing B_i, where here and hereafter i, j, k, denote distinct numbers of the set 1, 2, 3. Let p_i be the line P_jP_k.

Familiar or readily demonstrated theorems are the following:

1. Triangle $P_1P_2P_3$ is homothetic[1] with $C_1C_2C_3$.

2. P_i, I, B_i', B_i, are collinear. (Since P_i bisects arc P_jP_k and is therefore on B_iI.)

3. I is the orthocenter[2] of $P_1P_2P_3$, with B_i' as the foot of an altitude. (Since I and B_i are each equidistant from C_j and C_k, it follows that the line IB_i is perpendicular to C_jC_k and hence also to P_jP_k. Now b_j, b_k, b_j'', b_k'' form a rhombus with IB_i as one diagonal. Hence the midpoint B_i' of this diagonal is upon the line of the other diagonal. On the other hand the foot of the altitude is the point half way[3] between the orthocenter I and the second intersection B_i of the altitude with the circumcircle.)

4. p_i, b_j, b_k'', are concurrent, with p_i bisecting the angle between b_j and b_k''.

5. p_i, b_j', b_k', are concurrent, with p_i bisecting the angle between b_j' and b_k'.

6. P_i is equidistant[4] from I, B_j, B_k.

Hence we have readily the following new relations.

7. P_i is equidistant from the lines, b_j, b_k, b_i''.

8. P_i is equidistant from the lines, b_1', b_2', b_3'. Let this common distance be r_i'.

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[1] R. A. Johnson, *Modern Geometry* (1929), Page 194.

[2] Ibid, P. 194.

[3] Ibid, P. 163.

[4] Ibid, P. 185.

9. P_i is equidistant from the five lines, a_j, a_k, b_i, b_j'', b_k''. Let this common distance be r_i. That P_i is at the same distance r_i from b_i and b_j'' follows from the fact that it is on the angle bisector p_k. That P_i is at the same distance from b_i and a_j follows from the fact that P_i bisects arc B_jB_k, hence the line B_jP_i bisects the angle between the chord $B_jB_k=b_i$, and the tangent a_j at B_j.

10. $r+r_1+r_2+r_3=2R$.

Proof. The radius r' of the circle inscribed in $B_1'B_2'B_3'$ is half that of the circle inscribed in $B_1B_2B_3$, hence $2r'=r$. But $r_i'=r_i+r'$, and r_i' is the radius of the circle escribed to $B_1'B_2'B_3'$. Hence[1] $r_1'+r_2'+r_3'=4R'+r'$, where R' is the circumradius of $B_1'B_2'B_3'$, so that $4R'=2R$. Hence the theorem.

We may now add an intuitive proof of the following traditional theorem.[2]

In a triangle the outer common tangents to the excircles form a triangle whose incenter coincides with the circumcenter of the excenters, and the radius of whose incircle is equal to twice the circumradius plus the inradius of given triangle.

Proof. P_i is center of an escribed circle K_i' of $B_1'B_2'B_3'$, and the circumcircle of $B_1B_2B_3$ is the circumcircle of excenters. The radius of this centercircle is $2R'$. The external tangent a_i' to K_j' and K_k', other than b_i', is parallel to a_i and at a distance $r'=\frac{1}{2}r$ from it. Thus I is the incenter of the triangle a_1', a_2', a_3', and the inradius is $R+r'$ or $2R'+r'$, as desired.

ON CERTAIN PROJECTIVE TROCHOIDS

By R. M. WINGER, University of Washington

1. *Introduction.* Elsewhere[3] I have shown that the maximum axial symmetry which an algebraic curve of order m can admit is m-fold, namely symmetry with respect to m equispaced lines about a point. The equations of rational curves with this maximum symmetry have been obtained.[4] From the form of the equations it appears that for every value of m there is a finite number of such curves which are projectively distinct. We naturally ask concerning rational curves having multiple symmetry which falls short of the maximum. Some information is already available, for symmetry with respect to k lines ($k>1$) implies the existence of a dihedral collineation group G_{2k} of order $2k$ under which the curve is invariant.[4] And when the curve is rational, it has been proved that k is odd when m is odd.[1] Hence

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[1] Ibid, Page 189.

[2] This theorem (with references) is given by Johnson, page 192, but the proof is omitted as "long and dull."

[3] American Mathematical Monthly, vol. 37, 1930, p. 4. A typographical error occurs in the theorem as stated, where the number of axes is given as $2n$, but the theorem is correctly invoked on p. 5.

[4] Winger, l. c., also, American Journal of Mathematics, vol. 36, 1914, pp. 66, 57.

A rational curve of order m which has $m-1$ axes of symmetry is necessarily of even order.

We shall prove that such curves are trochoids,[2] and that there is a one-parameter family of them for every (even) value of $m(>2)$. There is also a family of rational trochoids for each even value of $m(>2)$ having $m-2$ axes of symmetry. These two families form the subject matter of this paper.

Trochoids, which are also epicycles, have occupied the attention of a host of mathematicians[3] since their invention by the ancients to account for the apparent motion of the planets, and elaborate collections of figures have been published.[4] But most of the results are metric. We shall give the right of way to the projective view, indicating however the metrical connections. We use *projective trochoid* to signify a curve that can be projected into a trochoid. We restrict the discussion to those curves which exhibit in their metrical setting the two highest orders of symmetry but results are stated for general n. Specifically, we deal with projective trochoids which are (a) rational, (b) of even[5] order $2n$, and (c) invariant under dihedral collineation groups of orders $2(2n-1)$ and $2(2n-2)$. The paper illustrates admirably the effectiveness of the group and projective attack in studying the singularities of invariant curves and particularly in resolving the higher singularities that occur in the infinite region. The two families of curves also exemplify the difference between curves having an odd and even number of axes of symmetry. I shall assume such knowledge of dihedral groups, binary and ternary, as is contained in my book on Projective Geometry, §§159, 160 and adopt the same notation and nomenclature.

I. *The Projective Trochoid C_{2n} Invariant Under a Dihedral $G_{2(2n-1)}$.*

2. *The parametric equations of the curve.* Since the curve is rational, two collineation groups are involved. For while the points of the curve are being permuted according to the transformations of the ternary group, the parameters of the points are permuted by a binary group, simply isomorphic with the first.

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[1] Winger, *Self-Projective Rational Septimics*, American Journal, vol. 47, 1925, p. 208.

[2] We use trochoids in the generic sense as including epi- and hypo-trochoids and of course epi- and hypo-cycloids which are denoted by the general term cycloids.

[3] The reader will find systematic accounts in Proctor, *The Geometry of Cycloids*, where reference is made to DeMorgan's essay in the Penny Encyclopaedia, Wieleitner, *Spezielle Ebene Kurven*, Loria, *Spezielle Algebraische und Transzendente Ebene Kurven*, Band 2. Moritz, *Cyclic-Harmonic Curves*, University of Washington Publications in Mathematics, treats some special trochoids, giving numerous figures. Instructive papers have been written by Morley, *On the Epicycloid*, American Journal of Mathematics, vol. 13, 1891, also *On Adjustable Cycloidal and Trochoidal Curves*, Ibid. vol. 16, 1894 and by Wolstenholme, *On Epicycloids and Hypocycloids*, Proceedings, London Mathematical Society, vol. 4, also by Roberts, *On Epi- and Hypo-trochoids*, Ibid., vol. 4, p. 353. For a digest of the literature and an extensive bibliography, see Wölffing, Bibliotheca Mathematica, vol. 2 (1901).

[4] H. Perigal, *Contributions to Kinematics*, for example.

[5] It is well known that algebraic trochoids are both rational and of even order. See the article by Roberts, cited in footnote 3, where these facts are proved.

Likewise the line sections of the curve must be permuted by the associated groups—the points by the ternary, the parameters by the binary group. In particular the points cut from the curve by any fixed line must be permuted among themselves.

The general parametric equations of a rational $2n$ -*ic* are

$$x_0 = f_0(t), \ x_1 = f_1(t), \ x_2 = f_2(t),$$

where f_i are polynomials in t of order $2n$. The problem is to restrict these equations so that the curve will admit the group. We take for triangle of reference the fixed triangle of the group and we select as generators of the binary group

$$S: t' = \epsilon t, \text{ and } T: t' = 1/t,$$

ϵ a primitive $(2n-1)$th root of unity. Now each side of the reference triangle is fixed under the invariant cyclic subgroup of index 2 of the ternary group, while one side (say x_2) is fixed and the other two are interchanged by the reflexions. Accordingly S must carry f_i into themselves, while T must interchange f_0 and f_1 and leave f_2 unaltered—except possibly for a constant factor. We are thus led to the canonical equations of our curve:

$$x_0 = t^{2n} + at, \ x_1 = at^{2n-1} + 1, \ x_2 = t^n, \ a \neq 0, \ \infty, \tag{1}$$

of which the line equations are

$$\begin{aligned} \xi_0 &= (n-1)at^{3n-2} - nt^{n-1} \\ \xi_1 &= -nt^{3n-1} + (n-1)at^n \\ \xi_2 &= at^{4n-2} + 2[n - a^2(n-1)]t^{2n-1} + a. \end{aligned} \tag{2}$$

We thus get a family of curves since the systems of equations (1) and (2) contain one arbitrary constant. We shall first treat the general curve of the family and then notice some particular cases.

3. *The General Curve of the Family*. From (1) it appears that x_2 is a multiple line, having n-point contact at each of the points whose parameters are 0, ∞, absorbing thus $2n-4$ points of inflexion. Again from (2) we see that the class[1] of x_2 is $2n-2$ so that it absorbs $(n-1)(2n-3)$ double lines including flexes. Or, *x_2 is equivalent to $2n-4$ flex tangents and $2n^2-7n+7$ bitangents.*

The Plücker numbers for the curve, deducting the flexes and bitangents absorbed by x_2 are

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[1] The class of a multiple line is the dual of the order of a multiple point. A general line through a multiple point of order k cuts the curve in k coincident points, and a multiple line of class k counts for k coincident tangents from a general point of the line.

$\nu=2(2n-1)$, the same as the order of the group
$d=(n-1)(2n-1)$
$\delta=4(n-1)(2n-3)-(2n^2-7n+7)=(2n-1)(3n-5)$
$\rho=6(n-1)-2(n-2)=2(2n-1)=\nu,$

where ν denotes the class, d the number of double points, δ the number of bitangents and ρ the number of flexes.

The special sets of parameters conjugate under the binary group are easily accounted for: The special pair, $0, \infty$, are the contacts of the multiple line while the two special sets of $2n-1$ parameters, namely

$$t^{2n-1}+1=0,\quad t^{2n-1}-1=0 \tag{3}$$

comprise in the aggregate the pairs of double points of the $2n-1$ involutions, viz.

$$t^2-\epsilon^{2i}=0,\quad i=0,1,\cdots,2n-2. \tag{4}$$

Thus one double point of each involution belongs in each special set (3). Moreover *the double points* (4) *lie in pairs on the axes of reflexion*, $x_0-\epsilon^i x_1=0$ *and hence name contacts of tangents from the corresponding centers.*

Since an involution has but two double points, the parameters of the $2n-2$ remaining intersections of each axis must be interchanged by an involution while the points themselves are fixed under the associated reflexion, hence all such points are nodes. Thus

The double points of the curve are distributed equally on the axes of reflexion, $n-1$ lying on each. This accounts for all of the double points.

Since all the centers of reflexion lie on the multiple line, this line counts for $2n-2$ tangents from each center. Two others, as we have seen, have contacts on the corresponding axis at the double points of an involution. We are left then with $2n-2$ tangents from each center and these must unite to form double lines, since their contacts must be conjugate pairs of an involution. Or

From each center of reflexion run $n-1$ double lines—accounting altogether for $(n-1)(2n-1)$ double lines. The residual $2(2n-1)(n-2)$ double lines must be distributed in $n-2$ general conjugate sets.

The flex equation, exclusive of $0, \infty$, each of which is an $(n-2)$-fold root is[1]

$$an(n-1)t^{4n-2}+2\left[a^2(n-1)^3-n^3\right]t^{2n-1}+an(n-1)=0. \tag{5}$$

Since all special sets of parameters have been accounted for, *the flexes must comprise a general set of conjugate points.*

The discriminant of equation (5) is a power of

$$\left[a^2(n-1)^3-n^3\right]^2-a^2n^2(n-1)^2. \tag{6}$$

[1] This is a second proof that the points $0, \infty$ absorb $n-2$ flexes each. They are however simple flexes when $n=3$.

If two flexes (5) coincide, (6) must vanish, i.e.

$$a^2(n-1)^3 - n^3 \pm an(n-1) = 0,$$

whence

$$a = \pm n/(n-1), \ \pm n^2/(n-1)^2. \tag{7}$$

Because of the symmetry however, if two flexes coincide, the whole set will coincide in pairs. The two curves corresponding to either pair of values of a in (7) are projectively equivalent: when $a = \mp n/(n-1)$, the curve has a cusp and when $a = \pm n^2/(n-1)^2$, the curve has an undulation at each of the points given by $t^{2n-1} \pm 1 = 0$.

The pencil of invariant conics

$$x_0 x_1 = \lambda x_2^2 \tag{8}$$

have contacts with each other at 0, ∞—with the multiple line as chord of contact. In addition each contains a set of points conjugate alike under the binary and the ternary group.[1] Noteworthy among these conics are the following:

(a) $n-1$ each cutting out $2n-1$ double points
(b) $n-1$, each touching the $4n-2$ tangents of the sets of double points in (a)
(c) two each with $2n-1$ contacts
(d) $n-1$ each touching $2n-1$ double lines
(e) $n-1$ each cutting out $4n-2$ contacts of the sets of double lines in (d)
(f) One cutting out the $4n-2$ flexes
(g) One touching all the flex tangents.

4. *Special Curves.*

A Curve with $2n-1$ Cusps. If $a = n/(n-1)$, the line equations of the curve become, on factoring out the cusp form:[2]

$$\xi_0 = t^{n-1}, \ \xi_1 = -t^n, \ \xi_2 = (t^{2n-1} - 1)/(n-1). \tag{9}$$

The class is reduced to $2n-1$ and all of the flexes and double lines are absorbed by the cusps. We are left however with $(n-2)(2n-1)$ double points, which lie in sets of $n-2$ on the axes of reflexion. One cusp falls on each axis, to which it is tangent, while the whole set of cusps lie on one of the invariant conics (8). The curve admits a group of the maximum order for its class, being the dual of a curve of order $2n-1$ admitting a group of double the order.[3]

A Curve with $2n-1$ Undulations, obtained when $a = n^2/(n-1)^2$. All of the flexes are absorbed but the class and number of the double points is unchanged.

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[1] The sets may be special for both groups as in (c), or the points may be a special set of the ternary group and their parameters form a general set of the binary group as in (a).

[2] When $a = -n/(n-1)$, we get a projectively equivalent curve with cusps at $t^{2n-1}+1=0$.

[3] The equations of a rational curve of order n invariant under a dihedral group of order $2n$ are

$$x_0 = t^r, \ x_1 = t^s, \ x_2 = t^n + 1, \ r + s = n, \ 1 \leqq r \neq s$$

See my first paper in the American Journal, cited on p. 578.

Each undulation tangent counts as a double line so that the number of proper bitangents is reduced to $3(n-2)(2n-1)$. Each axis of reflexion contains one undulation, whose tangent passes through the corresponding center. The undulation tangent counts for three simple tangents from a center of reflexion. There is one simple tangent from the center with its contact on the axis. The other tangents from the center, exclusive of the multiple line, unite to form bitangents, hence

From each center of reflexion run one simple tangent and one undulation tangent, each with contact on the axis—and $n-2$ bitangents.

There is one conic of the invariant pencil touching the curve at each undulation, so that the undulation tangents account for all of the common lines of the two curves.

$a=1$. The curve now has a $(2n-1)$-fold point at ξ_2 which absorbs all of the double points. Each tangent of the multiple point passes through one center of reflexion and has its contact on the corresponding axis. The remaining point of intersection of each axis is a contact of tangent from a center, the aggregate of these points comprising one of the special sets of the binary group. The other special set of $2n-1$ points are the parameters of the multiple point. The flexes and bitangents are disposed as in the general case.

5. *Metrical Specialization.*

The curve (1) may be given a metrical setting. First we interpret the x's as homogeneous Cartesian coordinates X', Y', Z', writing $x_0=X'$, $x_1=Y'$, $x_2=Z'$, where $Z'=0$ is the equation of the line at infinity.[1] Then we change to ordinary rectangular coordinates X, Y by the transformations

$$X'/Z' = X, \ Y'/Z' = Y.$$

Finally we transform to circular or absolute coordinates by the substitutions

$$x = X + iY, \ \bar{x} = X - iY, \ i = \sqrt{-1},$$

and interpret the parameter as a complex number of absolute value 1, thus letting t run around the unit circle in the complex plane. None of these operations will affect the projective properties of the curve whose equations now become

$$x = (t^{2n} + at)/t^n, \ \bar{x} = (at^{2n-1} + 1)/t^n, \ t = e^{i\theta}. \tag{10}$$

In this form the curve exhibits symmetry with respect to $2n-1$ equispaced lines, namely the axes of reflexion $x^{2n-1}-\bar{x}^{2n-1}=0$. Further it appears that the curve is a trochoid.[2] Since this curve is the only rational curve of even order admitting a dihedral group of the order here in question, we have proved:

If a rational curve of order $2n$ is invariant under a dihedral group of order

[1] Cf. *Projective Geometry*, §28.

[2] Cf. Harkness and Morley, *Theory of Functions*, §25. Also, Morley, American Journal, l. c.

$2(2n-1)$, *it is projectively equivalent to a trochoid having symmetry with respect to* $2n-1$ *lines.*

The trochoidal equations will perhaps be more familiar in rectangular coordinates. Instead of (10), let us write

$$x = bt^n + at^{1-n}, \tag{11}$$

a change which affects only the size. Then since $t=\cos\theta+i\sin\theta$, we have by De Moivre's theorem

$$x \equiv X + iY = b(\cos n\theta + i\sin n\theta) + a[\cos(1-n)\theta + i\sin(1-n)\theta].$$

Equating real and imaginary parts, we get the parametric equations in rectangular form

$$\begin{aligned} X &= b\cos n\theta + a\cos(1-n)\theta \\ Y &= b\sin n\theta + a\sin(1-n)\theta. \end{aligned} \tag{12}$$

This is the form given by Williamson[1] for an epicyclic which he identifies with a trochoid.

In particular, let $a=b$ and to avoid fractions put $\theta=2\omega$. Then by a trigonometric identity (12) reduces to

$$\begin{aligned} X &= 2b\cos(2n-1)\omega\cos\omega \\ Y &= 2b\cos(2n-1)\omega\sin\omega \end{aligned} \tag{13}$$

which leads at once to the polar equation

$$\rho = 2b\cos(2n-1)\omega,$$

a rose curve of $2n-1$ leaves.

In this metrical specialization, the multiple line is the line at infinity and the contacts fall at the circular points. In addition to their symmetry the trochoidal curves will have all of the projective properties relating to flexes, double points and double lines noted in §§3, 4. The pencil of invariant conics however are now concentric circles. In the family of trochoids (11) there is obviously one cycloidal curve: this is the curve with $2n-1$ cusps, for the condition, $h=\pm r$, that the tracing point be on the rolling circle is the same as the cusp condition, $a/b=\pm n/(n-1)$.

II. *The Projective Trochoid C_{2n} Invariant under a Dihedral $G_{2(2n-2)}$.*

6. *The General Curve of the Family.* As in §2 the canonical equations of a rational projective trochoid of order $2n$ invariant under a dihedral group of order $2(2n-2)$ are found to be

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[1] *Differential Calculus*, Seventh Edition, p. 366. From (12) may be obtained the equations of a trochoid in terms of the radii of the two circles by writing $n\theta=\phi$, $a=\mp h$, $b=R\pm r$, $(1-n)/n=\pm(R\pm r)/r$, where R is the radius of the fixed, r that of the rolling circle and h is the distance of the tracing point from the center of the rolling circle.

$$x_0 = t^{2n} + at^2,\ x_1 = at^{2n-2} + 1,\ x_2 = t^n,\ a \neq 0,\ \infty,\ n \text{ odd}^1. \tag{14}$$

Again we have a family of such curves. In this section we shall discuss the general curve of the family. The line equations are

$$\begin{aligned} \xi_0 &= (n-2)at^{3n-4} - nt^{n-2}, \\ \xi_1 &= -\,nt^{3n-2} + (n-2)at^n, \\ \xi_2 &= 2\{at^{4n-4} + [n - a^2(n-2)]t^{2n-2} + a\}, \end{aligned} \tag{15}$$

0, ∞ factoring out. Thus the class of the curve is $4n-4$.

The special pair of parameters, 0, ∞, conjugate under the binary group name cusps of which x_2, the fixed line of the ternary group, is the common tangent, having n-point contact at each. Each counts once as a cusp in reducing the class, but when $n>3$ each absorbs extra nodes, flexes and bitangents, the enumeration of which will be a special task of the immediate sequel.

Each axis of reflexion (aside from x_2) cuts out the double points of an involution, which are the parameters of contacts of simple tangents from the center of the associated reflexion. The double points of these involutions together make up the two special sets of $2n-2$ parameters of the binary group. The residual intersections of an axis must unite in pairs, forming double points. Or,

Each axis of reflexion (except x_2) carries $n-1$ nodes. There remain $(n-1)(2n-1)-2(n-1)^2=n-1$ double points. The parameters of these cannot form a special set under the binary group, for the three special sets have been accounted for. We conclude that the missing double points must fall at the two cusps—and they must be equally apportioned since the cusps are conjugate. Thus each cusp absorbs $(n-1)/2$ nodes, i.e. the *excess*[2] of nodes over the normal for each cusp is $(n-3)/2$.

The two cusps also exact a heavy toll of flexes and bitangents, which may be assigned to four causes: (a) each counts once as a cusp, (b) the tangent at each is a multiple line with n-point contact, (c) each contains latent double points and therefore latent double lines as well, (d) the two have a common tangent. We shall ascertain the total reduction in double lines as well as the number that is to be ascribed to each cause. First, two ordinary cusps on a rational curve of order $2n$ absorb together 4 flexes and $8n-13$ bitangents, as indicated by Plücker's equations. But from the flex equation

$$t^{n-1}\{n(n-2)at^{4n-4} + [(n-2)^3a^2 - n^3]t^{2n-2} + n(n-2)a\} = 0 \tag{16}$$

we infer that each cusp of our curve absorbs $n-1$ flexes, i.e. the excess of flexes in each cusp is $n-3$. Now each cusp is of class $n-2$ and as such would be equivalent to $(n-2)(n-3)/2$ double lines. But since the contacts of the tangent all coincide, $n-3$ of these double lines are flexes. To this number must be added

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[1] If n were even the order of the curve would be reduced to n.

[2] Also called the number of *latent* double points.

$(n-3)/2$ double lines—all bitangents—by way of excess, since the number of latent double lines must match the number of latent double points.[1] Thus,

Each special cusp is equivalent to one ordinary cusp, $(n-3)/2$ *nodes,* $n-3$ *flexes and* $(n-3)^2/2$ *bitangents*—and these numbers are in addition to the usual quota of flexes and bitangents.

Indeed we must expect a further reduction in the number of bitangents since x_2 is a common tangent to the cusps. Now x_2 as a $(2n-4)$-fold line of a curve of order $2n$ would be equivalent to $(n-2)(2n-5)$ double lines. Adding to this the excess of $n-3$ for both cusps, the total is $2n^2-8n+7$. But in the total are $2(n-3)$ flexes, hence

x_2 *is equivalent to* $2n^2-10n+13$ *bitangents and* $2(n-3)$ *flexes.*

Removing the $(n-3)^2$ which the two cusps would absorb if they had separate tangents, we find the number of new bitangents consumed by the double cusp tangent to be $(n-2)^2$. The mortality of double lines occasioned by the two cusps and their common tangent due to each cause we may now tabulate:

Cause	*Flexes Absorbed*	*Bitangents Absorbed*
(a)	4	$8n-13$
(b)	$2(n-3)$	$(n-3)(n-4)$
(c)	0	$n-3$
(d)	0	$(n-2)^2$
Total	$2(n-1)$	$2n(n-1)$.

Deducting the number of singularities absorbed, we obtain the Plücker numbers for the curve:

$\nu=4(n-1)$, the same as the order of the group

$d=2(n-1)^2$

$\delta=8n^2-20n+12-2n(n-1)=6(n-1)(n-2)$

$\rho=4(n-1)=\nu$.

Since all special sets of the binary group have been accounted for, *the remaining flexes belong to a general conjugate set.*

We shall consider next the distribution of the bitangents. Since the invariant cyclic subgroup of index two is of even order, it contains one reflexion (corresponding to the involution $t'=-t$) whose center is ξ_2 and whose axis is x_2. Now all of the intersections of the axis fall at the two cusps, hence all of the tangents from the center must be bitangents, for the contacts must be interchanged by the involution. Or,

$2(n-1)$ *bitangents meet at* ξ_2 *and constitute a special set of lines under the ternary group while their parameters form a general set under the binary group.*

We have seen that from each center of reflexion on x_2 run two simple tangents with contacts on the corresponding axis, while the line x_2 itself counts for $2n-4$ others. The remaining ones will in general go to the formation of bitan-

[1] Cf. Scott, American Journal of Mathematics, vol. 15, 1893, p. 230 ff. Also, Winger, *Ibid.* vol. 47, 1925, p. 210.

gents. Hence, *from each of the* $2n-2$ *centers of reflexion on* x_2 *run two simple tangents, with contacts on the corresponding axis, and* $n-1$ *bitangents.*

This accounts for $2(n-1)^2$ additional double lines. There remain $6(n-1)(n-2)-2(n-1)-2(n-1)^2=4(n-1)(n-3)$ bitangents which must be distributed in $n-3$ general conjugate sets.

The pencil of invariant conics contains several members analogous to those enumerated in Part I.

7. *Special Cases.*

The discriminant of the flex equation (16), exclusive of the parameters of the cusps, is a power of the left side of

$$[(n-2)^3a^2-n^3]^2-4a^2n^2(n-2)^2=0. \tag{17}$$

Factoring this we have

$$(n-2)^3a^2-n^3=\pm 2an(n-2),$$

whence $a=\pm n/(n-2)$, $\pm n^2/(n-2)^2$. If $a=\mp n/(n-2)$, the curve has cusps and if $a=\pm n^2/(n-2)^2$ the curve has undulations at $t^{2n-2}\pm 1=0$. The two curves of either pair are projectively equivalent, hence it will suffice to discuss one of each.

A curve with $2n-2$ *additional cusps.* $a=n/(n-2)$. The line equations (15), on removing the cusp factor, reduce to

$$\xi_0=t^{n-2},\ \xi_1=-t^n,\ \xi_2=2(t^{2n-2}-1)/(n-2). \tag{18}$$

This curve admits the maximum dihedral group for its class since it is the projective dual of a curve of order $2n-2$ admitting the maximum group, for that order.[1] It is not however the only curve of its class invariant under the same group when $n>4$.

The cusps absorb all of the flexes and all but $n-1$ of the bitangents. The $2n-2$ new cusps lie in pairs on $x_0-x_1=0$ and the axes conjugate with it.[2] The curve thus has $n-1$ double cusp tangents, which are in fact the bitangents which are left. These double cusp tangents comprise all of the tangents to the curve from the center ξ_2, while all of the tangents from the other centers of reflexion are concentrated in the multiple line x_2. The entire set of $2n$ cusps lie on a conic.

A curve with $2n-2$ *undulations.* $a=-n^2/(n-2)^2$. The undulations absorb all of the flexes and $2n-2$ bitangents. They lie in pairs on the set of $n-1$ axes conjugate with $x_0-x_1=0$. The tangents at the two undulations on each axis meet at the corresponding center and replace two of the double lines in the

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[1] See footnote, § 4.

[2] In a dihedral D_{2n}, when $n=2m$, the $2m$ axes of reflexion $x_0^{2m}-x_1^{2m}=0$ divide into two conjugate sets $x_0^m+x_1^m=0$, $x_0^m-x_1^m=0$. So also do the centers which lie one on each axis. Further the centers belonging to one set of axes lie on the other set of axes when m is odd, but on their own set when m is even. The statement in my *Projective Geometry*, p. 339 is inaccurate.

general case. One of the invariant conics, namely $(n-2)^4 x_0 x_1 - 16(n-1)^2 x_2^2 = 0$, touches the curve at each undulation so that the undulation tangents account for all of the common lines of curve and conic.

$a = \pm 1$. The curve now has a $(2n-2)$-fold point at ξ_2, which would commonly absorb $(n-1)(2n-3)$ double points and leave $n-1$. But in the multiple point are $n-1$ tacnodes which use up the remaining $n-1$ double points. The multiple point thus has only $n-1$ distinct tangents, which are the axes $x_0^{n-1} + x_1^{n-1} = 0$ or $x_0^{n-1} - x_1^{n-1} = 0$, according as $a = +1$ or -1. Each tacnodal tangent counts once as a double line and absorbs a second double line on account of excess. This leaves $2(n-1)(3n-7)$ double lines to be accounted for. All of the tangents from ξ_2 are included in the tacnodal tangents. From each center of one conjugate set of $n-1$ reflexions run one tacnodal tangent, whose parameters are the double points of an involution, and $n-1$ bitangents. From each of the centers in the other set run two simple tangents and $n-1$ bitangents besides the multiple line x_2. We have thus accounted for $2(n-1)^2$ additional bitangents. There remain $4(n-1)(n-3)$ bitangents, which belong to $n-3$ general conjugate sets.

8. *Metrical Specialization.*

As in Part I, we may write the equations of the curve (14) in circular coordinates

$$x = t^n + at^{2-n}, \quad \bar{x} = at^{n-2} + t^{-n}, \tag{19}$$

which represent a genuine trochoid with the special cusps falling at the circular points and with the line at infinity as the common cusp tangent. From the first equation (19), we obtain as before the parametric equations in rectangular coordinates

$$\begin{aligned} X &= \cos n\theta + a \cos (n-2)\theta \\ Y &= \sin n\theta - a \sin (n-2)\theta, \end{aligned} \tag{20}$$

n a positive odd integer.

Or, writing $n\theta = \phi$, (20) takes the form

$$\begin{aligned} X &= \cos \phi + a \cos [(n-2)\phi/n] \\ Y &= \sin \phi - a \sin [(n-2)\phi/n]. \end{aligned} \tag{21}$$

When $a = n/(n-2)$, (21) reduces to a cycloid with $2n-2$ cusps—in addition to the special singularities at I and J. The cusps lie on a circle with center at the center of the curve and of radius $(2n-2)/(n-2)$.

When $a = 1$, (20) becomes by a trigonometric identity

$$\begin{aligned} X &= 2 \cos (n-1)\theta \cos \theta, \\ Y &= 2 \cos (n-1)\theta \sin \theta, \end{aligned} \tag{22}$$

whence we get at once the polar equation

$$\rho = 2\cos(n-1)\theta,$$

a rose curve of $2n-2$ leaves.

Summary. It should be remarked that there are other trochoids admitting groups of lower orders. Our discussion includes however as special cases rose curves and cycloidal curves having the two highest orders of symmetry. On the other hand some of the curves are invariant under higher groups. For example when $n=a=3$, equation (19) represent the hypocycloid of class 4, commonly called the astroid, and which admits the octahedral group G_{24}. One lesson may be drawn from the present treatment: The cuspidal curves (rational cycloids) should be considered as class curves for then their parametric equations assume the simplest possible form, as exemplified in (9) and (18).

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A PROBLEM IN INTEGRATION

Professor H. S. Uhler of Yale University writes as follows:

"I am taking the liberty of asking you to propose the following problem in integration since I can not find anything satisfactory about it, and since I need the results in my work. I have run across the *indefinite* integrals whose integrands are $\cos\theta\cos(\cot\theta)d\theta$, $\sin\theta\cos(\cot\theta)d\theta$, $\cos\theta\sin(\cot\theta)d\theta$, and $\sin\theta\sin(\cot\theta)d\theta$. The limits are θ_0 and θ where neither angle needs to exceed π radians. What I desire to know is whether these integrals can be expressed in closed form in terms of any known functions. If elliptic functions, and the like, are not involved then I should like to have series expansions given together with their respective limits or regions of convergence."

Will some reader furnish the answer.

R.E.G.

A NOTE ON DETERMINANTS

By C. O. OAKLEY, Brown University

PROPOSITION. If in a determinant

$$\Delta = |a_{ij}|,\ \sum_{i=1}^{n} a_{ij} = \sigma,\ (j = 1, 2, \cdots, n)$$

and

$$\sum_{j=1}^{n} a_{ij} = \tau,\ (i = 1, 2, \cdots, n),$$

then $\sigma=\tau$, since, upon summing all elements, $n\sigma=n\tau$.

DEFINITION I. A determinant is said to be a sigma-determinant if

$$\sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = \sigma.^{1}$$

DEFINITION II. A determinant Δ is said to possess unit-property if the $2n$ determinants Δ', formed by replacing the elements of a column (row) of Δ by unity, are all equal.

THEOREM I. *A necessary and sufficient condition that a determinant* $\Delta(\neq 0)$ *possess unit-property is that it be a sigma-determinant.*

PROOF OF NECESSITY. We are given that Δ possesses unit-property.[2] That is

$$\Delta' = \begin{vmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 1 & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & & & \cdot \\ \cdot & \cdot & & \cdot \\ 1 & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & 1 & \cdots & a_{1n} \\ a_{21} & 1 & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & 1 & & a_{nn} \end{vmatrix} = \cdots = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & & \cdot \\ \cdot & & & \\ \cdot & & & \\ \cdot & \cdot & & \cdot \\ 1 & 1 & \cdots & 1 \end{vmatrix}.$$

Making use of the first two determinants in this relation and interchanging the first two columns in the second determinant, we write

$$\begin{vmatrix} 1 & a_{12} & & & \\ 1 & a_{22} & & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & & \cdot \\ 1 & \cdot & & & a_{nn} \end{vmatrix} = - \begin{vmatrix} 1 & a_{11} & & & \\ 1 & a_{21} & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ 1 & & & & a_{nn} \end{vmatrix},$$

from which we get by transposition and addition,

$$\begin{vmatrix} 1 & a_{11} + a_{12} & & & \\ 1 & a_{21} + a_{22} & & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & & \cdot \\ 1 & a_{n1} + a_{n2} & & & a_{nn} \end{vmatrix} = 0.$$

By adding the 3rd, 4th, $\cdots$, nth columns respectively to the second column this becomes[3]

$$D_{12} = \begin{vmatrix} 1 & \sum a_{1j} & & & \\ 1 & \sum a_{2j} & & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & & \cdot \\ 1 & \sum a_{nj} & & & a_{nn} \end{vmatrix} = 0. \tag{1}$$

[1] It is the determinant of a magic square in which the sums of the diagonals play no part. So far as the author is aware such determinants, the class of which overlaps those of symmetric and skew-symmetric determinants, have not been treated.

[2] It should be noted that we are really concerned here with the matrix of Δ rather than the value of Δ.

[3] The notation $D_{\mu\nu}$ shall indicate that from the set Δ', determinants have been chosen one of which involves units in the μth column and the other of which involves units in the νth column.

We now form the $n(n-1)/2$ relations (1) obtained by so treating every pair of the n determinants Δ' with columnar replacements. Each determinant $D_{\mu\nu}$ so formed will contain a column of units and a column of $\sum$'s. Laplace's development of these determinants by means of these two columns gives us the following system of $n(n-1)/2$ equations in $n(n-1)/2$ unknowns:

$$
(2)\qquad
\begin{aligned}
&\overset{12}{\alpha}_{12}A_{12} + \overset{12}{\alpha}_{13}A_{13} + \cdots + \overset{12}{\alpha}_{1n}A_{1n} + \overset{12}{\alpha}_{23}A_{23} + \cdots + \overset{12}{\alpha}_{n-1,n}\ A_{n-1,n} = 0,\\
&\overset{13}{\alpha}_{12}A_{12} + \overset{13}{\alpha}_{13}A_{13} + \cdots + \overset{13}{\alpha}_{1n}A_{1n} + \overset{13}{\alpha}_{23}A_{23} + \cdots + \overset{13}{\alpha}_{n-1,n}\ A_{n-1,n} = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\overset{n-1,n}{\alpha_{12}}A_{12} + \overset{n-1,n}{\alpha_{13}}A_{13} + \cdots + \overset{n-1,n}{\alpha_{1n}}A_{1n} + \overset{n-1,n}{\alpha_{23}}A_{23} + \cdots + \overset{n-1,n}{\alpha_{n-1,n}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad A_{n-1,n} = 0;
\end{aligned}
$$

where

$$
A_{kl} = \begin{vmatrix} 1 & \sum_j a_{kj} \\ 1 & \sum_j a_{lj} \end{vmatrix}
$$

and where $\alpha_{kl}^{\mu\nu}$ is the algebraic complement of A_{kl} in $D_{\mu\nu}$. Now the determinant H of the coefficients $\alpha_{kl}^{\mu\nu}$ in this system (2) is a power of Δ since it is a reciprocal determinant formed from Δ. Therefore H cannot vanish and hence

$$
A_{12} = A_{13} = \cdots A_{n-1,n} = 0,
$$

that is

$$
\sum a_{1j} = \sum a_{2j} = \cdots = \sum a_{nj} = \tau.
$$

This proves the theorem for rows; a similar treatment will prove it for columns. Thus Δ is a sigma-determinant.

Proof of Sufficiency. This part of the theorem follows at once from the following considerations. If in Δ we add the 2nd, 3rd, $\cdots$, nth columns respectively to the elements of the first column, we may write

$$
(3)\qquad \Delta = \begin{vmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{vmatrix} = \begin{vmatrix} \sum a_{1j} & & \\ \vdots & \ddots & \\ \sum a_{nj} & & a_{nn} \end{vmatrix} = \sigma\Delta'.
$$

Clearly we could thus factor out a σ from any column or row. This completes the proof of the theorem. Moreover the sufficiency persists even though $\Delta = 0$. For so long as $\sigma \neq 0$ it can be factored out as in (3) above; and for the case $\sigma = 0$ we can proceed as follows. Add ϵ to the elements of the main diagonal of Δ and write[1]

$$
\Delta_\epsilon = \begin{vmatrix} a_{11}+\epsilon & a_{12} & \cdots & \\ a_{21} & a_{22}+\epsilon & & \\ \vdots & & \ddots & \\ \vdots & & & a_{nn}+\epsilon \end{vmatrix} = \epsilon\Delta'_\epsilon .
$$

[1] This method was suggested to the author by Professor R. E. Gilman.

Therefore $\Delta_\epsilon' = \Delta_\epsilon/\epsilon$ and, since Δ_ϵ is a polynomial of the nth degree in ϵ, lim Δ_ϵ' exists and equals the lim Δ_ϵ/ϵ and the theorem is proved. It is of interest to note the following corollaries.

COROLLARY I. If $\Delta(=0$ or $\neq 0)$ is a sigma-determinant such that $\sum_i a_{ij} = \sigma$ and if Δ' is the value of one of the determinants formed from Δ by replacing the elements of some column (or row) by units, then $\Delta = \sigma\Delta'$.

COROLLARY II. A sigma-determinant vanishes if $\sigma = 0$.

It may happen, however, in case $\Delta = 0$, that Δ possesses unit-property but is not a sigma-determinant. Hence, in that event, necessity fails. But if $\Delta = 0$ and possesses unit-property there exists a set of multipliers $\rho_1, \rho_2, \cdots, \rho_n, v_1, v_2, \cdots, v_n$ such that if we multiply the first row of Δ by ρ_1, the second by $\rho_2 \cdots$, the nth column by v_n, then this new determinant $\rho\Delta$, say, is a sigma-determinant. We illustrate this fact by the following example. The determinant

$$\Delta = \begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ -1 & 2 & -1 \end{vmatrix} = 0$$

possesses unit-property ($\Delta' = 0$) but is not a sigma-determinant as can readily be seen. The multipliers $\rho_1 = \rho_2 = \rho_3 = v_1 = v_2 = 1$, $v_2 = 2$ give

$$2\Delta = \begin{vmatrix} 1 & 0 & 1 \\ 2 & -2 & 2 \\ -1 & 4 & -1 \end{vmatrix} = 0$$

which is a sigma-determinant ($\sigma = 2$).

It is not difficult to prove the following

THEOREM II. *The product of two sigma-determinants is a sigma-determinant.*

COROLLARY. The adjoint of a sigma-determinant is a sigma-determinant.

It is an obvious remark that a theory of what might be called semi-sigma-determinants might be developed from the point of view of rows (columns) alone; that is to say we might require that $\sum_{j=1}^n a_{ij} = \sigma$, $(i = 1, 2, \cdots, n)$ but make no restriction as to the sums $\sum_i a_{ij}$ by columns. Indeed this is essentially what has been done in the methods here used, the proposition stated at the beginning being used to tie rows and columns together; and it is this point of view that leads to some of the more interesting applications of the theory to systems of linear equations. For example consider the system

$$\sum_{j=1}^n a_{ij} x_j = b, \qquad i = 1, 2, \cdots, n; \tag{4}$$

with the condition that

$$\sum_{j=1}^n a_{ij} = \sigma, \qquad i = 1, 2, \cdots, n.$$

Nothing is said about $\sum_i a_{ij}$. Then we may state the following theorem about the solution of (4):

THEOREM III. *If* $\sum_{j=1}^{n} a_{ij} = \sigma \neq 0$, *the solution of* (4) ($\Delta \neq 0$) *is* $x_1 = x_2 = \cdots = x_n = b/\sigma$.

This follows at once from the fact that $\Delta = \sigma b \Delta'$ and hence $x_j = b\Delta'/\Delta = b/\sigma$.

PROOF BY VECTOR METHODS THAT EVERY REAL SURFACE WITH TWO SETS OF RULINGS IS A QUADRIC SURFACE

By B. F. KIMBALL, Schenectady, N. Y.

Take the two sets of rulings as the parametric curves u = constant and v = constant. Let $\boldsymbol{r}_0(u)$ and $\boldsymbol{r}_1(u)$ be the position vectors of the rulings $v=0$ and $v=1$, where u is the arc-length along the ruling $v=0$. Then the equation of the ruled surface may be written:

$$\boldsymbol{R}(u, v) = (1 - v)\boldsymbol{r}_0(u) + v\boldsymbol{r}_1(u). \tag{1}$$

Let $\boldsymbol{k}_0$ and $\boldsymbol{k}_1$ be unit direction vectors of the lines $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$ with positive sense in the direction of increasing u. Since u measures the arc-length of $\boldsymbol{r}_0$, we have:

$$\boldsymbol{r}_0' = \boldsymbol{k}_0.$$

We write

$$\boldsymbol{r}_1' = h(u)\boldsymbol{k}_1$$

where $h(u)$ is a scalar equal to the length of the vector $\boldsymbol{r}_1'$. Thus

$$\boldsymbol{R}_u = (1 - v)\boldsymbol{k}_0 + vh(u)\boldsymbol{k}_1.$$

Now since the v = constant curves are rulings, $\boldsymbol{R}(u, v_0)$ represents a straight line for constant $v = v_0$. Hence

$$\boldsymbol{R}_u \times \boldsymbol{R}_{uu} \equiv 0.$$

We have $\boldsymbol{R}_{uu} = vh'(u)\boldsymbol{k}_1$. Thus

$$\boldsymbol{R}_u \times \boldsymbol{R}_{uu} = v(1 - v)h'(u)(\boldsymbol{k}_0 \times \boldsymbol{k}_1) \equiv 0$$

for all u and v. If $\boldsymbol{k}_0 \times \boldsymbol{k}_1 = 0$, it is easily seen that the surface is a plane. If $\boldsymbol{k}_0 \times \boldsymbol{k}_1 \neq 0$, we have $h'(u) \equiv 0$. Then $\boldsymbol{r}_1' = h\boldsymbol{k}_1$ where h is constant and we may express $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$ by the relations

$$\boldsymbol{r}_0 = \boldsymbol{k}_0 u - \boldsymbol{a},$$
$$\boldsymbol{r}_1 = h\boldsymbol{k}_1 u + \boldsymbol{b},$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors. Accordingly, substituting for $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$ in equation (1) we have:

$$\boldsymbol{R}(u, v) = uv(h\boldsymbol{k}_1 - \boldsymbol{k}_0) + u\boldsymbol{k}_0 + v(\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{a}.$$

Write this equation in the form

$$\boldsymbol{R} = \boldsymbol{A}uv + \boldsymbol{B}u + \boldsymbol{C}v + \boldsymbol{D}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ are constant vectors with scalar components A_1, A_2, A_3; B_1, B_2, B_3 etc. and the scalar components of the position vector $\boldsymbol{R}$ are x, y and z, The above vector equation is equivalent to the three scalar equations

$$(2)\qquad \begin{aligned} x &= A_1uv + B_1u + C_1v + D_1 \\ y &= A_2uv + B_2u + C_2v + D_2 \\ z &= A_3uv + B_3u + C_3v + D_3. \end{aligned}$$

Eliminating u and v from these equations, one obtains a quadratic equation in x, y and z. Thus the surface in question is a quadric surface.

A Geometric Interpretation of Landen's Transformation[1]

By C. E. Rhodes, University of Cincinnati

The problem of determining the volume between a cone and a cylinder affords an excellent illustration of the use of Elliptic Integrals. In addition, it leads readily to a simple geometric interpretation of Landen's Transformation. The problem is a natural one, and can be applied in finding the volume of grain or crushed stone dumped into a cylindrical bin.

The mathematical statement of the problem is as follows: Given a right circular cylinder and a right circular cone (one nappe) with their axes parallel, and a plane perpendicular to these axes passing through the vertex R of the cone; to find the volume V inside the cylinder between the plane and the cone. For convenience, let the radius of the cylinder be unity, and let k denote the distance OR between the axis of the cylinder and the axis of the cone; ϕ, the angle between the axis of the cone and any element; and y, the variable length PR. (See figures.) The required volume is swept out by the variable triangle PQR as it revolves about the axis of the cone, P following the intersection of the cylinder and plane, and Q following the intersection of the cylinder and cone. The differential of volume is then the area of the triangle PQR multiplied by the differential of the distance through which its center of gravity moves. Denoting the angle ORP by α, we have

$$(1)\qquad dV = \frac{\operatorname{ctn}\phi}{3}\, y^3 d\alpha.$$

For the case where $k<1$, (Fig. 1) α is a convenient variable of integration, and, noting the symmetry, the required volume can be expressed as

$$(2)\qquad V = \frac{2\operatorname{ctn}\phi}{3}\int_0^{\pi} [(1 - k^2\sin^2\alpha)^{1/2} + k\cos\alpha]^3 d\alpha.$$

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[1] This paper was presented in abridged form before the Ohio Section of the Mathematical Association of America at Columbus April 7, 1932.

When this integrand function is expanded, certain terms can be dropped because their integrals vanish.

The standard Legendre Type Forms of Elliptic Integrals of the first and second kinds are

$$F(k) = \int_0^{\pi/2} \frac{dt}{(1 - k^2 \sin^2 t)^{1/2}}, \quad E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 t)^{1/2} dt.$$

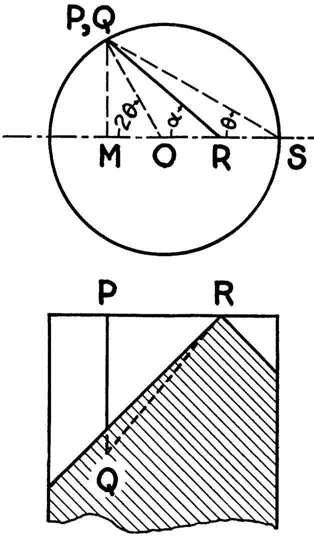

FIGURE 1

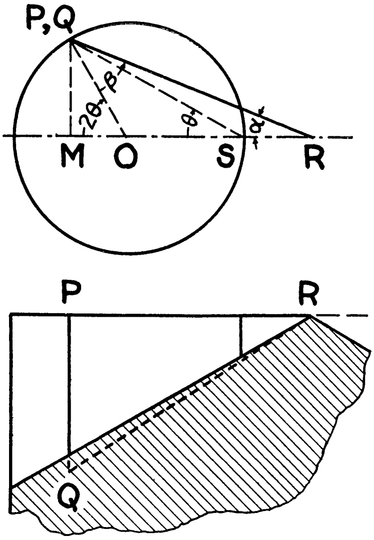

FIGURE 2

By means of the usual reductions,[1] the integral (2) can be expressed as

$$V = \frac{2 \operatorname{ctn} \phi}{9} \cdot 2[(7 + k^2)E(k) - 4(1 - k^2)F(k)] \tag{3}$$

where k is the modulus.

In considering the case shown in Fig. 2 where $k>1$, α is not a convenient variable of integration, since it does not vary monotonically as the point P moves around the cylinder. On the other hand, the angles OPR and MOP both vary monotonically. Transforming the variable of integration in (1) from α to β, the volume integral can be written

$$V = \frac{2 \operatorname{ctn} \phi}{3} \int_0^{\pi} [(k^2 - \sin^2 \beta)^{1/2} + \cos \beta]^3 \frac{\cos \beta}{(k^2 - \sin^2 \beta)^{1/2}} d\beta \tag{4}$$

Reducing (4) to the Legendre Type Forms, it becomes

[1] Hancock, *Elliptic Integrals*, pages 61, 62.

$$(5) \qquad V = \frac{2 \operatorname{ctn} \phi}{9} \cdot \frac{2}{k} \left[k^2(7 + k^2)E\left(\frac{1}{k}\right) - (3 + k^2)(k^2 - 1)F\left(\frac{1}{k}\right) \right]$$

where the modulus is $1/k$.

In using the angle MOP as the variable of integration, it is found convenient in subsequent reductions to denote it by 2θ. Making the proper substitution in (1) the volume integral turns out to be

$$(6) \qquad V = \frac{2 \operatorname{ctn} \phi}{3} \int_0^{\pi/2} (1 + k^2 + 2k \cos 2\theta)^{1/2}(1 + k \cos 2\theta)2d\theta.$$

Again applying the standard reductions, (6) becomes

$$(7) \qquad V = \frac{2 \operatorname{ctn} \phi}{9} (1 + k)[(7 + k^2)E(m) - (1 - k)^2 F(m)]$$

in which the modulus is $m = 2k^{1/2}/(1+k)$. The form (7) is valid for both cases regardless of the size of k.

Landen's Transformation as used in Elliptic Integrals is defined by the equation $\tan \alpha = \sin 2\theta/(k + \cos 2\theta)$ and transforms any Elliptic Integral with modulus k or $1/k$ into one with modulus $2k^{1/2}/(1+k)$. From Fig. 1 it is evident that

$$\tan \alpha = \frac{MP}{MO + OR} = \frac{\sin 2\theta}{k + \cos 2\theta}$$

so that the transformation of the variable of integration from α to θ is actually Landen's Transformation. It can be shown that in Fig. 2, the change from β to θ is another example of it. The plane views of the problem furnish, then, a geometric picture of Landen's Transformation. To make the picture complete, the angle θ itself can be readily constructed as the angle MSP.

This particular geometric interpretation of Landen's Transformation is given by Dr. Hancock in his monograph[1] on Elliptic Integrals. There he obtains it as a special case of a much more complicated geometric transformation due to Jacoby. Here it is obtained as a natural result of finding the volume between a cylinder and a cone, using different variables of integration.

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[1] Loc. cit. page 74.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brookly College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

La Géométrie. By Lucien Godeaux. Paris, Hermann et Cie, 1931. 181 pages. 15 francs (unbound).

In the first three chapters, about a half of the book, the author summarizes the basic ideas on geometry that have become current since the beginning of the present century: the relation of geometry to the notion of a group of transformations, the various interpretations of the non-Euclidean geometries and their interrelation, etc. Both the analytic and the synthetic approaches are used. The presentation is clear, orderly, and comprehensive, considering the space.

The last two chapters are devoted to algebraic geometry and the geometry on an algebraic variety. Little has been done to make the fundamental ideas of these subjects accessible to wider mathematical circles, and these contributions are quite welcome. The author thought it preferable to omit all bibliographical references from the text, a procedure which has its disadvantages. He compensates for the omission by a bibliography at the end of the book, in which he includes several American titles. The book may be read with profit by any student who is familiar with projective geometry.

NATHAN ALTSHILLER-COURT

Géomètres Français sous la Révolution. By Niels Nielsen. Copenhagen, Levin and Munksgaard, 1929. viii+250 pages. \$2.50 (unbound).

It would seem a priori that the times of political changes and social upheavals are not auspicious for the cultivation of so abstract and contemplative a science as mathematics. But the period of the French Revolution is a striking example to the contrary. Prof. J. L. Coolidge recently[1] called the second quarter of the nineteenth century the "heroic age" in geometry. The period of the French Revolution may with justice be called the "romantic age" of mathematics. It is nothing short of romance to think that men like Carnot and Monge, to mention only two, who were in the very midst of the political and military turmoil, could turn their minds towards mathematical creation.

The late Professor Nielsen deserved well of all those interested in the history of mathematics for having given us along with the biographies of Fourier, Lagrange, Laplace, Carnot, Legendre, Monge, the biographical sketches of their lesser contemporaries. Each biography contains also a résumé of the most important contributions of the mathematician considered. That this was a labor of love is attested by the fact that of the seventy odd mathematicians whose

[1] Bulletin of the Am. Math. Society, 1929, pp. 1–19.

biographies are given, the names of more than half are not mentioned in Cajori's "History of Mathematics." The book was awarded a prize by the Paris Académie des Sciences.

NATHAN ALTSHILLER-COURT

Die Liesche Theorie der Partiellen Differentialgleichungen Erster Ordnung. By Friedrich Engel and Karl Faber. Leipzig, B. G. Teubner, 1932. xi+367 pages. RM 28.

The purpose of this text is to set forth with rigor and elegance the theory of linear partial differential equations of the first order from Lie's original viewpoint. The work of Professor Engel in connection with the editing of Lie's works has fitted him conspicuously for this task and the resulting text is a thoroughly commendable exposition of the Lie theory.

This text is elementary in the sense that only a restricted knowledge of mathematics is presupposed. The authors state these presuppositions as follows: some familiarity with ordinary differential equations, an adequate facility in handling determinants, and the ability to extend to n dimensions the analytic geometry of ordinary space. There is however assumed, on the part of the reader, a rather high degree of mathematical maturity and, to this extent, the text is by no means an elementary one.

The painstaking care of the authors in the preparation of the subject matter is in evidence on every page. The presentation is clear and concise and the arrangement thoroughly well thought out. Although the style is a trifle monotonous and the major results are not made to stand out as conspicuously as one would wish, the book is ably written and offers an excellent approach to this important subject.

C. H. SISAM

Vector Analysis, with an Introduction to Tensor Analysis. By A. P. Wills. New York, Prentice-Hall, Inc., 1931. xxxii+286 pages. $5.00.

This book falls naturally into two main sections: Chapters I to VI on vector analysis (including dyadics), Chapters VII to X on tensor analysis.

The first section follows, in general, the methods and notation of Gibbs, the principal exception arising in the definitions of divergence and curl of a vector function at a point as the limits of integrals over surfaces about the point, divided by the volume enclosed. The gradient of a scalar function is defined in the usual manner and then also by means of a limiting surface integral; the definitions are then shown to be consistent. While these integral definitions lend themselves to simple deductions of the standard integral transformations, they have the disadvantage of requiring a proof that the definitions themselves are unambiguous—a proof not easy to present in a brief and rigorous form. The reviewer personally prefers to define the divergence and curl of a vector point function $\boldsymbol{v}$ as the first scalar and vector invariant of the derivative dyadic $\nabla \boldsymbol{v}$.

Numerous applications of vector analysis are given in this portion of the book. These include the geometry of curves and surfaces in Euclidean 3-space, kinematics, potential theory and electrodynamics.

The second section on tensor analysis will be of particular interest to students of modern physics. Beginning with coordinate systems and transformations in 3-space, the results are then generalized to a Riemannian n-space. Then follows a discussion of geodesics in R_n, the parallelism of Levi-Civita, absolute differentiation and Riemannian curvature. The last chapter is devoted to the theory of tensors regarded as multilinear forms in n base-vectors, which are invariant under coordinate transformations. This point of view, initiated by Hessenberg in his Annalen article of volume 78, contrasts with the usual treatment of tensors in which their *components* alone appear—i.e. the scalar coefficients in the above forms. This treatment is especially happy in its direct approach to covariant differentiation. The book closes with a discussion of the Riemann-Christoffel and the Ricci-Einstein tensors.

A preliminary chapter of some twenty pages gives an excellent historical introduction to vector and tensor analysis. The book is well printed, and the few misprints that occur will cause the reader no particular difficulty. The last paragraph on page 229, however, should be corrected.

LOUIS BRAND

An Introduction to the Theory of Canonical Matrices. By H. W. Turnbull and A. C. Aitken. Glasgow, Blackie & Son, 1932. xiii+192 pages. 17sh. 6 d.

As stated in the preface, this book, though intended to serve as a sequel to Turnbull's *Theory of Determinants, Matrices, and Invariants*, published in 1928, has been made self-contained, presupposing merely some knowledge of the elementary theory of determinants. The scope of the book is indicated by the following list of chapter headings: I, Definitions and Fundamental Properties of Matrices; II, Elementary Transformations. Bilinear and Quadratic Forms; III, Canonical Reduction of Equivalent Matrices; IV, Subgroups of the Group of Equivalent Transformations; V, A Rational Canonical Form for the Collineatory Group; VI, The Classical Canonical Form for the Collineatory Group; VII, Congruent and Conjunctive Transformations; VIII, Canonical Reduction by Unitary and Orthogonal Transformations; IX, The Canonical Reduction of Pencils of Matrices; X, Applications of Canonical Forms to Solution of Linear Matrix Equations; XI, Practical Applications of Canonical Reduction.

The recent applications of matrix algebra to quantum mechanics by Dirac and others, makes the appearance of a book of this kind especially timely. The physicist who wishes to obtain without prolonged preliminary study a working knowledge of the theory of matrices of finite order will find the book adapted to his needs. The mathematician will observe that though the account of the theory presented here is neither so detailed nor complete as that given by Muth's *Elementarteiler* or the *Matrices and Determinoids* of Cullis (a fact in-

dicated by the authors), the work exhibits merits of its own. Among these should be noted the systematic use of the matrix notation, the interesting historical notes that conclude each chapter, and the numerous examples that "are intended to serve not so much as exercises, many of them being quite easy, but points of relaxation, and running commentary; they will, however, be found to contain many well-known and important theorems, which the notation establishes in the minimum of space."

While one may regret the paucity of geometrical applications given by the authors, the applications of the theory contained in the last two chapters of the book are well-chosen and interestingly presented.

L. M. Blumenthal

Astronomy. By Forest Ray Moulton. New York, The Macmillan Company, 1931. xxiii+549 pages; tables xxii pages. $3.75.

This book is not a revised edition of the author's "Introduction to Astronomy," but an entirely new publication.

The subject matter has been taken up in such a way as "to unfold the subject of astronomy by proceeding step by step from the familiar earth and evening constellations out to the galaxies and super galaxies of stars."

A few pages of preliminary considerations are followed by a chapter on the constellations including stellar magnitudes and celestial coordinates. The succeeding chapters are on telescopes, the Earth, and the Moon, time, eclipses and the law of gravitation. What may be called the first part of the book is then concluded by chapters on the solar system, the planets, and meteors and comets. The last 200 pages of the text are devoted to the Sun, the evolution of the solar system, stars, nebulae, and the sidereal structure. The fact that only about one fourth of the book is on the stars is compensated for by the author's scholarly presentation of the subject matter which enabled him to include a wealth of information in a very compact form.

The questions and reference books given at the end of each chapter add greatly to the value of the book.

It is unfortunate that on the star maps the boundaries of the constellations are given in heavy solid lines which together with the rather bold designation of the stars makes it difficult to visualize the configurations. On Map I, "Path of Pole of Ecliptic" should read "Path of Pole of Equator."

The numerous diagrams and reproductions are clear and sharp.

The press work and binding are excellent, and notwithstanding the many pages and good paper the weight of the book is not excessive.

M. F. Weinrich

The Size of the Universe. By Ludwik Silberstein. Oxford University Press, 1930. viii+216 papers. $3.50.

Dr. Silberstein has in this volume collected his views and investigations con-

cerning relativistic cosmology into a systematic treatment, which may perhaps be fairly said to sum up his stand on the subject up to the time of publication—late in 1929 or early in 1930. In this book Dr. Silberstein has ignored practically all other work done in the field, with the exception of the original papers of Einstein and de Sitter, and has conducted a most bitter polemic against certain of those—notably Hubble and Weyl—whose conclusions are incompatible with his own. Although it may not seem entirely fair to criticize the author's views of two or three years ago at this comparatively late date, the reviewer would justify his criticisms (but not his procrastination!) on the ground that they refer either to matters intrinsic to the book or to work which was in the literature at the time of publication—although he does not of course deny himself the opportunity of pointing out the more recent advances which have been made on the basis of this previous work.

In Part I, following an interesting diversion concerning amorphous and projective spaces, the author discusses in some detail the metrization of space and gives a brief account of the general theory of curvature of higher dimensional manifolds. Part II begins with the assumption that the universe of space and time can in the large be considered as *homogeneous*, but not necessarily *isotropic*—any event in this idealized space-time bears the same relation to the metrical structure as any other, but not all space-time directions through them are equivalent in the same sense. The author then derives as the cases in which he is interested the universes of Einstein and de Sitter (the latter of which is completely isotropic as well as homogeneous). His assumption of complete homogeneity has thus closed to him the more general non-stationary possibilities, first realized by Friedmann in 1922, of universes which are *spatially* isotropic and homogeneous, upon which the recent developments of Lemaître, de Sitter, Tolman, Eddington, Einstein, the reviewer and others are based (for an account of which the interested reader is referred to the reviewer's report *Relativistic Cosmology* to appear in the January, 1933, number of *Reviews of Modern Physics*).

The Einstein universe he emphatically discards because of the difficulty of introducing into it a mass without causing objectionable disturbances at its polar plane. But before abandoning it Dr. Silberstein indulges in a most inexcusably bitter polemic (Part III) directed against Hubble's interesting and important paper (*Astrophysical Journal*, vol. 64 (1926), pp. 321–369) on extra-galactic nebulae, presumably because Dr. Hubble has the temerity to end his 48-page paper with a half page computation expressing the value of the density of matter in terms of the radius of the Einstein universe (with the aid of formulae from Haas's *Introduction to Theoretical Physics*, to which Dr. Silberstein reers (p. 98) as "a semi-popular book"!).

Part IV returns to a more complete discussion of the de Sitter universe, including an account of the motion of particles and light. For this purpose the author employs the original de Sitter coordinates, the inadequacy of which was recognized by Weyl in 1923, in place of the more appropriate coordinates in-

troduced by Lemaître in 1925. The last Part applies these results to a determination of the Doppler effect in light from distant objects; the general formula thus obtained depends on the perihelion distance and speed at perihelion of the star or nebula, as well as on its present distance and the "radius" $\mathfrak{R}$ of the de Sitter universe, but as these two quantities are not obtainable from the data he eliminates them with the aid of statistical assumptions to obtain an expression for the radius in terms of the mean square distance and Doppler effect of a group of objects. This statistical formula he applies to several groups of objects (stars, globular clusters and the Magellanic Clouds) in or closely associated with our galactic system and finds for the curvature radius values ranging from 1.46×10^6 to 3.09×10^7 parsecs, and to a group of 38 spiral nebulae with the result $\mathfrak{R}=7.32\times 10^8$ parsecs. Postponing for a moment discussion of these results, we continue with a summary of topics included in the remainder of Part V and in the remaining Notes (Nos. 4, 4a and 8 containing estimates of $\mathfrak{R}$ included in the above limits). The orbit of particles in the field of a spherically symmetric mass introduced into the de Sitter universe are computed and the results interpreted with the aid of the intriguing concept of a "cosmic day." The remaining notes contain, in addition to a more detailed analysis of certain points in the body of the book, a most biting rejoinder to certain criticisms by Weyl of portions of the author's work, and an account of the author's questionable attempt (*Philosophical Magazine*, vol. 9 (1930), pp. 50–57) to derive the line element describing a homogeneous universe filled with isotropic radiation—in which the author apparently considers the cosmological constant λ as variable, contrary to the general theory of relativity, as he would otherwise obtain the rejected Einstein universe!

Since the most important fact to be explained here concerns the observed "Doppler shifts" in light from distant objects, a few words concerning the history and present status of this problem may not be out of place. The original development by de Sitter of the universe which bears his name included the prediction of a residual red shift which, interpreted as a velocity, would lead to the conclusion that for a sufficiently restricted range $v\sim r^2$. Weyl showed in 1923 (*Physik. Z.* 24, pp. 230–232) that the coordinates which were employed by de Sitter covered but a portion of the universe, and found it desirable to introduce some assumption concerning the natural state of motion of matter. The assumption which he chose was the seemingly natural one that on following back the world-lines of matter they would be found to converge in the remote past—that the actual universe contains only matter which is coherent in this sense. With the aid of this assumption Weyl deduced a velocity-distance relationship which is of the form $v=cr/\mathfrak{R}$ for objects at distances r which are small compared with the radius $\mathfrak{R}$ of the de Sitter universe. That this linear law is indeed satisfied by those extra-galactic objects, the spiral nebulae, to which we can most justifiably expect to apply it was fairly widely recognized at the time Dr. Silberstein's book was written, and has since been most conclusively verified by Hubble at Mt. Wilson. Assuming that his observed red shift is in

fact that predicted by Weyl, the radius of the de Sitter universe is found to be $\mathfrak{R}=1.7\times 10^{27}$ cm. $=6\times 10^{8}$ parsecs, which is some hundreds of times the value most favored by Dr. Silberstein. The discrepancy between these results is of course to be explained by the difference in theoretical background as well as by the difference in the objects to which the theory is applied. In the first place, Silberstein has applied his statistical formula (which is equally applicable to red *and to violet* shifts) to stars and groups of stars which form a part of the relatively closely knit galactic system and finds it verified—contrary to the findings of Lundmark (*Monthly Notices R. A. S.* vol. 84 (1924), pp. 747–770) and Strömberg (*Astrophysical Journal* vol. 61 (1925), pp. 353–362), whose work he entirely ignores in this account. And in the second place he has applied it to the extra-galactic nebulae, without being able to account for the overwhelming preponderance of red shifts, to obtain results at variance with those which he obtained from his previous investigations (although an attempt to reconcile them is made by questioning Hubble's estimates of distances)—in connection with which it is interesting to note that the red shifts subsequently dealt with by Hubble and Humason are found in nebulae whose distances are estimated to extend up to 4×10^{7} parsecs, many of which must therefore lie without the limits set by Silberstein to the entire de Sitter universe!

H. P. ROBERTSON

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

1931–1932

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of the University of Illinois

The officers for 1931–1932 of our chapter were: Homer Dennis, Director; William Ted Martin, Vice Director; Lucretia Ann Mott, Recording Secretary; Harry E. Crull, Corresponding Secretary; Herbert M. Norris, Treasurer.

All officers were elected at the regular spring election, May 19, 1931. All the officers were elected by a vote of the members present at this meeting.

The University of Illinois chapter has forty-four active members and twelve new members who were initiated March 22, 1932, making a total of fifty-six members.

Those eligible to membership are undergraduate students who have an average of at least 4.5 in mathematics and have declared their intention to major or minor in mathematics and shall be

enrolled in or have taken at least one three hour course in mathematics requiring as a prerequisite Integral Calculus. Four graduate students and one member of the faculty may also be elected at the annual election.

The programs and meetings were as follows:

October 13, 1931: Business meeting.

October 27, 1931: "Real branches of algebraic curves" by Professor Brahana.

November 10, 1931: "The outline of the theory of functionals" by B. T. Darling, '34.

November 24, 1931: "Symmetric functions" by D. M. Brown. This talk was followed by a business meeting.

February 23, 1932: A short business meeting and the election of members.

March 8, 1932: Rushing party. At this meeting, Lucretia Ann Mott, '32, gave a talk on "Mathematical recreation."

March 11, 1932: Pledge luncheon.

March 22, 1932: Initiation banquet, Inman Hotel, Champaign, Illinois.

April 12, 1932: "Moebius leaves" by Dr. Wilson.

One of our biggest accomplishments of the year was the revision of our by-laws.

LUCRETIA ANN MOTT, *Recording Secretary*

Pi Mu Epsilon of Iowa State College

The officers for 1931–1932 were: C. R. Wells, Director; E. W. Carr, Vice Director; E. A. Howard, Treasurer; Marianne Preuss, Secretary; E. C. McCracken, Librarian; Professor D. L. Holl, Faculty Advisor; Dr. E. R. Smith, Head of the Department of Mathematics, Permanent Secretary.

We now have seventy-nine active members. We had two initiations the past year, one on November 18th, at which time we took in twelve new members; and the other on May 24th, when we initiated ten.

At the annual college Honor's Day in the fall, 1931, we awarded a check of $15 and a certificate to Guy E. Strong, a junior ceramics engineering student. We give this prize each year to the student having the highest scholastic average who has completed the course in calculus. He is also taken into Pi Mu Epsilon.

Pi Mu Epsilon sponsored the Mathematics Openhouse during the spring carnival, Veishea, May 5th, 6th and 7th. The main feature was the showing of a Relativity Movie.

The programs and meetings were as follows:

November 9, 1931: "Linear transformation" by E. A. Howard.

November 18, 1931: Initiation and banquet.

December 8, 1931: "Operational calculus" by Wayne Birchard and D. Paul Needham.

January 27, 1932: "Anharmonic ratios" by E. W. Carr.

March 9, 1932: "Harmonic analysers" by George Fink.

April 13, 1932: "Thermionic and Photo-electric work functions" by R. M. Bowie.

May 12, 1932: "The theory of relativity" by J. V. Atanasoff.

May 24, 1932: Initiation and banquet.

MARIANNE PREUSS, *Secretary*

LOCAL MATHEMATICS CLUBS

The Mathematics-Science Club of Drake University

Our club is a combination of Mathematics, Physics and Astronomy. The aim of the club is to give the students a chance to hear interesting topics reported on and discussed and also to promote friendship between its members.

Students having completed nine semester hours doing work on a major in one of the above mentioned subjects are eligible to membership. Selection is made by an unanimous vote of the members from those having the above qualifications. Grades are considered in the selection of

candidates but there is no specific requirement. At present we have thirty-one members including six faculty members.

The officers for 1931–1932 were: Ralph Collins, President; Guy Brunk, Vice President (first semester); Earling Jensen, Vice President (second semester); Mary Neff, Secretary-Treasurer: Martha Burton, Corresponding Secretary. The officers are elected at the April meeting.

The meetings and programs were as follows:

October 29, 1931: Picnic meeting. The names of prospective members were put before the members of the club.

November 10, 1931: "Einstein's theory of relativity" by Professor Helmick.

December 8, 1931: Initiation of six new members.

January 12, 1932: Moving picture reels showing the work in the General Electric laboratories at Schenectady were shown.

February 13, 1932: A special meeting called for the election of new members.

March 8, 1932: "Imaginaries—DeMoivre's theorem" by Professor Neff. Initiation of ten new members.

April 12, 1932: "Latest theories of the universe as presented by De Sitter and Einstein" by President Morehouse. Initiation of two new members.

May 10, 1932: Picnic meeting at the Observatory.

MARTHA ALICE BURTON, *Corresponding Secretary*

The Mathematics Club of the College of the City of Detroit

The officers for 1931–1932 were: John Malley, President; Claire G. Groteau, Secretary; Morris A. Greenberg, Treasurer. The officers are elected annually at the first meeting of the academic year.

The purpose of the society is to stimulate and promote an interest in mathematics.

The meetings and programs were as follows:

November 3, 1931: "An appreciation of Fourier's series" by Mr. Morris Rose.

December 1, 1931: "Method for approximating real roots of equations" by Dr. T. R. Running of the University of Michigan.

December 4, 1931: "Complex singularities" by Mr. Clarence Wylie.

January 20, 1932: "Magic squares" by Mr. Morris A. Greenberg.

March 17, 1932: "Some peculiar functions" by Mr. Morris Friedman.

April 26, 1932: "The trisection of the angle" by Miss Cecelia Maier.

May 10, 1932: "Some non-commutative algebras" by Mr. Edward Brushaber.

May 31, 1932: "The nine-point circle" by Miss Virginia Eyre.

CLAIRE G. CROTEAU, *Secretary*

The Napierian Club of DePauw University

The officers for 1931–1932 were: Kenneth Griffin, '32, President; Dorothy Wurst, '32, Vice President; Kenneth Smith, '32, Secretary; Harold Frey, '32, Treasurer.

The following programs were given at the regular monthly meetings:

September 28, 1931: Election of officers; election of new members.

October 21, 1931: Steak roast; election of other members.

November 5, 1931: "Life and work of Napier" by Kenneth Griffin; "Introduction to the trisection of an angle" by Frank Reid.

December 3, 1931: "Invention of calculus" by Anne Nichols; "Clippings on the trisection of an angle" by Kenneth Smith; "Japanese method of trisection" by Howard Stafford; "Report of mathematics convention" by Professor Hildebrandt.

January 7, 1932: "Magic squares" by Otto Behrens; "Pursuit problems" by W. E. Smith; "A determination of Avogadro's number by the Brownian movement" by Richard Humphreys.

February 4, 1932: Social Meeting. "Eminent American mathematicians" by Professor Edington; "Mathematical games" by Dorothy Wurst.

March 10, 1932: "Determination of Pi" by John Voliva; "Demonstration of the use of the calculating machine" by Professor Hildebrandt.
April 4, 1932: "Sixth degree curve fitting" by Professor Greenleaf.
May 5, 1932: Social meeting; election of new officers. "How to do research" by Arthur Noble; "Egyptian pyramids" by Harold Frey; "Trick problems" by W. E. Smith; "Problems" by Gordon Hiatt; "Mathematical spelling match" by Professor Arnold.

KENNETH SMITH, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON, AND WM. FITCH CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to Wm. Fitch Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 11. *Proposed by Wm. R. Ransom, Tufts College.*

This problem was offered by Mr. Francis of Exeter at a meeting of the Association of Mathematics Teachers of New England in 1914.

Circumscribed about a circle is an isosceles trapezoid, $ABCD$, in which $DC<AB$, and $AD=BC$. Two perpendiculars are drawn; DG perpendicular to AB at G, and GH perpendicular to AD at H. Show that DA, DG and DH are the arithmetic, geometric, and harmonic means, respectively, between the pair of parallel sides AB and DC.

E 12. *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

Two coplanar right triangles, AOC and BOC, have the common hypotenuse OC. Using vector methods, express the vector OC in terms of the vectors OA and OB.

E 13. *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

It is required to find two factors which, together with their product, contain each of the nine digits from one to nine just once. Each of the several solutions should be found. (Zero must not appear in any solution.)

E 14. *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

At country fairs there occasionally appears a man with five discs, each four inches in diameter, and a table on which is painted a larger red circle. He challenges anyone to place the five discs so they completely cover the red circle on the first attempt. It appears that a correct solution lacks central symmetry.

How large is the maximum red circle which may be covered by the five discs, and how must they be placed?

E 15. *Proposed by Mrs. Pearl C. Miller, Washington University.*

Prove that if two external angle-bisectors of a scalene triangle are equal, then the sines of the three interior half-angles form a geometric progression. By external angle-bisector is here meant that segment of the line bisecting the exterior angles at a vertex of a triangle, intercepted between that vertex and the opposite side of the triangle.

Advanced Problems

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3579. *Proposed by B. F. Kimball, Schenectady, N. Y.*

Given

$$F(x, a) = \sum_{s=-\infty}^{+\infty} \frac{\sin (x + sa)}{x + sa}$$

where s takes on all integral values including zero. Evaluate $F(x, a)$ for all real values of x and a. Discuss the discontinuities of the function.

3580. *Proposed by J. B. Reynolds, Lehigh University.*

If a body dropping vertically from an airplane reaches a velocity of 120 mi./hr., which is 99 percent of its terminal velocity, in 40 sec. what is the value of n on the assumption that the resistance of the air is directly proportional to the weight of the body and the nth power of its velocity?

3581. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Let a, b, c, be the edges of a trihedral angle S-abc, and a', b', c', the perpendiculars at S to the planes bc, ca, ab. If u is the axis of a cone of revolution passing through a, b, c, the line u is also the axis of a cone of revolution tangent to the planes $a'b'$, $b'c'$, $c'a'$.

3582. *Proposed by Dewey C. Duncan, University of California.*

If α and β are positive integers and $\beta > 2$, then $2^\alpha + 1$ is never divisible by $2^\beta - 1$.

3583. *Proposed by H. Grossman, New York.*

If from any point on a circle, chords be drawn to all vertices of a regular inscribed $3n$-gon, the sum of the n longest chords is equal to the sum of the $2n$ shortest ones.

3584. *Proposed by W. E. Buker, Leetsdale, Pa.*

Find the rational values of x and y for which

$$x^3 + y^3 = 2^3 + 1^3.$$

3585. *Proposed by R. E. Gaines, University of Richmond.*

If a tangent at P_1 to the cardioid $\rho = a(1+\cos\theta)$ cuts the curve again at P_2 and P_3, and if the normal at P_1 cuts the curve again at P_4, P_5, P_6, then (1) $\rho_2\rho_3/\rho_1$ is constant. Also (2) $\rho_1+\rho_2+\rho_3=\rho_4+\rho_5+\rho_6$; (3) if the chord $P_2P_3=2^{\frac{1}{2}}a$, it subtends a right angle at the origin.

3586. *Proposed by R. E. Gaines, University of Richmond.*

If while an ellipse is turned about in its plane it remains tangent to a fixed straight line at a fixed point, its foci trace a curve whose area is $2\pi a(a-b)$.

SOLUTIONS

3495. [1931, 340]. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

Prove that the $\phi(m)$ integers less than m and prime to it constitute an automorphic group, with respect to multiplication and reduction to least positive residue modulus m.

Prove that if each member of any group in (1) be multiplied by $r(r\neq m)$, a new group will be formed which will be automorphic and also isomorphic with the first group. The modulus in this case is rm.

Solution by J. D. Hill, Brown University.

Let S: $a_1, a_2, a_3, \cdots, a_{\phi(m)}$, denote the set of $\phi(m)$ integers less than m and prime to it. It will first be shown that the set S forms an Abelian group with respect to the given law of combination.

I. *The Combination Property.* Let a_i, a_j be two equal or unequal members of the set S. If their product be divided by m the positive remainder will be less than m and prime to it. Hence the product a_ia_j is congruent modulo m to some member of S.

II. *The Associative Law.* In view of the law of combination, this property is evidently satisfied.

III. *The Identical Element.* If the identical element, I, exists there must be a member x of S such that the congruence $a_ix\equiv a_i \bmod m$, is satisfied for every

element a_i of S. But this congruence is equivalent to $x\equiv 1 \bmod m$, and since S contains the element 1 it follows that $I=1$.

IV. *The Inverse Element.* Let a_i be any member of S and consider the congruence $a_ix\equiv 1 \bmod m$. Since a_i is prime to m this congruence has one and only one solution, $x\equiv x_i \bmod m$, where x_i is a certain member of S. Hence the inverse of a_i is x_i.

V. *The Commutative Law.* As in II above, this property is evidently satisfied.

The initial assertion has therefore been established. It remains to show that there exists a rearrangement of S which is isomorphic with S. Such an arrangement results immediately upon replacing in order each member of S by its inverse. For out of the relation $a_ia_j\equiv a_h \bmod m$, easily follows $a_i^{-1}a_j^{-1}\equiv a_h^{-1} \bmod m$. Hence the group S is automorphic. Moreover, the automorphism set up in this way will reduce to the identical automorphism only if the integers m have the property that each element in S is its own inverse. But the result of Problem 3488[1] proves that this can happen when and only when $m=1, 2, 3, 4, 6, 8, 12$, or 24. For the cases where $m=1, 2, 3, 4, 6$, one readily shows that no automorphism other than the identity exists, whereas in the remaining cases non-identical automorphisms do exist. For example, we have

$$\text{for } m=8,\ \begin{pmatrix} 1 & 3 & 5 & 7 \\ 1 & 5 & 7 & 3 \end{pmatrix}$$

$$\text{for } m=12,\ \begin{pmatrix} 1 & 5 & 7 & 11 \\ 1 & 7 & 11 & 5 \end{pmatrix}$$

$$\text{for } m=24,\ \begin{pmatrix} 1 & 5 & 7 & 11 & 13 & 17 & 19 & 23 \\ 1 & 11 & 13 & 23 & 17 & 19 & 5 & 7 \end{pmatrix}$$

The statement we are asked to prove in the second part of the problem is true only if the condition $r\neq m$ is replaced by the condition r prime to m. For if the set S': $ra_1, ra_2, \cdots, ra_{\phi(m)}$ constitutes a group mod mr, we must have, for each a_i and a_j, an a_k such that $ra_ira_j\equiv ra_k \bmod mr$, or $ra_ia_j\equiv a_k \bmod m$. But since a_k is prime to m, this necessitates that r be prime to m.

With r prime to m, $ra_i\equiv x_i \bmod m$, where the x's are exactly the a's in some order. If $x_ix_j\equiv x_s \bmod m$, we have $ra_ira_j\equiv ra_s \bmod m$; and since obviously $ra_ira_j\equiv ra_s \bmod r$, and r is prime to m, we have $ra_ira_j\equiv ra_s \bmod mr$. Hence the correspondence $ra_i\sim x_i$ between the elements of S' and S establishes an isomorphism.

3518. [1931, 588]. *Proposed by H. A. Simmons, Northwestern University.*

In Bôcher's *Higher Algebra*, p. 11, Exercise 2, the student is expected to make a statement which may be expressed as follows: If f is a polynomial in m variables $x_1, x_2, \cdots, x_m$ which is known not to be of higher degree than n_i in x_i, $(i=1, \cdots, m)$, and if f vanishes at the $(n_1+1)(n_2+1)\cdots(n_m+1)$ distinct points

[1] See this MONTHLY, vol. 39 (1932), p. 241.

$$(1) \qquad [x_1^{(i_1)}, x_2^{(i_2)}, \cdots, x_m^{(i_m)}],\ i_j = 1, 2, \cdots, n_j + 1;\ j = 1, \cdots, m,$$

then f vanishes identically.

We ask the question: What is the greatest number of the points (1) at which a polynomial f of the type hypothesized can vanish and yet not be identically zero?

Solution by Jeanette Fox, Student, University of Berlin.

Let

$$(1) \qquad x_i^{(1)}, x_i^{(2)}, \cdots, x_i^{(n_i+1)},$$

be n_i+1 distinct given values of the independent variable x_i; set

$$(2) \qquad g_i(x_i) = [x_i - x_i^{(1)}][x_i - x_i^{(2)}] \cdots [x_i - x_i^{(n_i+1)}];$$

and denote by $g_i'(x_i)$ its derivative with respect to x_i. Consider the polynomial of degree n_i in x_i

$$(3) \qquad \frac{g_i(x_i)}{g_i'[x_i^{(1)}][x_i - x_i^{(1)}]}.$$

For $x_i = x_i^{(1)}$, (3) has the value unity, whereas it is zero for any other value of x_i in the set (1). Hence the polynomial

$$(4) \qquad f(x_1, x_2, \cdots, x_m) = \frac{g_1(x_1)g_2(x_2)\cdots g_m(x_m)}{g_1'(x_1')g_2'(x_2')\cdots g_m'(x_m')(x_1 - x_1')(x_2 - x_2')\cdots(x_m - x_m')}$$

(where each x_i' is equal to any root of $g_i(x_i)$) has the value unity at $(x_1', x_2' \cdots x_m')$ and the value zero at any one of the remaining sets of points $[x_1^{(i_1)}, x_2^{(i_2)}, \cdots, x_m^{(i_m)}]$. Thus the polynomial defined by (4) is certainly not identically zero; it is of degree n_i in x_i; and it vanishes at

$$(5) \qquad (n_1 + 1)(n_2 + 1) \cdots (n_m + 1) - 1$$

points. Hence (5) is the maximum number of points at which a polynomial of the described degree type can vanish without being identically zero.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

The Institute for Advanced Study will begin active work in the autumn of 1933. It will be located near Princeton, New Jersey, and will consist of a series of schools, the first of which will be the school of mathematics. Through the courtesy of the authorities of Princeton University, the mathematics group will be temporarily accommodated at the new Fine Hall, which is peculiarly adapted to the purposes of an institute. Professor Albert Einstein of Berlin was appointed professor of mathematical and theoretical physics at the institute. Professor Oswald Veblen was also appointed a professor at the School of Mathematics. Dr. Walter Mayer of Berlin was made associate in mathematics and Dr. J. L. Vanderslice was appointed assistant to Professor Veblen.

The University of Oregon and Oregon State Agricultural College have been put under one board and one chancellor, and a number of changes in function have been made. One of these is to place all advanced work in science and mathematics at the State College, and Professor W. E. Milne, who has been professor of mathematics at the University since 1919, has been made head of the department of mathematics at the State College.

Dr. Jacques Hadamard, professor of mathematics at the College de France and a member of the Institute de France, plans to visit the United States in the spring of 1933.

Dr. Jan H. Oort, lecturer and observer at the University of Leyden, arrived in Delaware, Ohio, October 4, to begin photometric studies of nebulae in a cooperative plan with the Perkins Observatory of Ohio Wesleyan University. While at the Perkins Observatory Dr. Oort will give a special series of lectures on stellar statistics on Thursday afternoons, beginning October 13. The titles of the lectures are: "The Observational Data," "Methods of Determining Stellar Distributions," "General Structure of the System of Stars," "Stellar Dyanmics I," "Stellar Dynamics II," and "Galactic Rotation." The lectures and discussions are open to all interested.

Professor H. Weyl of the University of Goettingen has accepted the invitation of Swarthmore College to give the William J. Cooper Foundation Lectures for 1932–33. He will reside at Swarthmore from the middle of February until the middle of March. During this period he will give five lectures on the general subject of Mind and Nature and will take part in the honors work of the college. The lectures are open to the public.

Dr. Alois Kovarik, professor of physics at Yale University, was the recipient on July 2 of an honorary Dr. of Science from Charles University in Prague and on June 10 of the Memorial Medal from Comenius University at Bratislavia.

Dr. Leonard Carlitz has been appointed assistant professor of mathematics at Duke University.

Dr. M. R. Hestenes has been appointed research assistant at the University of Chicago.

Associate Professor V. B. Hinsch has been promoted to a professorship of mathematics at the Missouri School of Mines and Metallurgy.

Dr. V. A. Hoersch has been promoted to the rank of associate in mathematics at the University of Illinois.

E. D. McCarthy has been appointed assistant professor of mathematics at the University of Detroit (Engineering College).

Dr. David Moskovitz, of Brown University, has been appointed assistant professor of mathematics at the Carnegie Institute of Technology.

Dr. F. C. Ogg has been appointed associate professor of mathematics at Bowling Green State College, Bowling Green, Ohio.

Dr. Mina S. Rees has been promoted to an assistant professorship of mathematics at Hunter College.

Professor B. D. Roberts, of Parsons College, has been appointed professor and head of the department of mathematics at New Mexico Normal University.

Dr. Otto Struve, assistant director at the Yerkes Observatory, has been appointed director of the observatory and professor of astrophysics at the University of Chicago.

Dr. Norbert Wiener has been promoted to a professorship of mathematics at the Massachusetts Institute of Technology.

NECROLOGY

CORRIGENDA

Volume XXXIX 1932:

P. 28, A Correction to vol. XXXV, p. 11.
P. 487, line 13, for "V. H. Ivanoff" read "V. F. Ivanoff."
P. 506, line 13, for "R. M. Ashmun" read "R. N. Ashmun."
P. 507, line 9, add: "by invitation;" and, line 11, omit "by invitation."
P. 510, line 4, for "Mathias" read "Mathis."
P. 537, fifth line from bottom, for "Aitkin" read "Aitken."
P. 547, sixth line from bottom, for "1625" read "1525."
P. 555, line 15, for "Smith" read "Smyth."

The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The prize will be awarded hereafter every three years. The last award was in December, 1929, to Professor T. H. Hildebrandt. The next award will be in December, 1932, for the period 1929-1931.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

The Carus Mathematical Monographs

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THE CARUS MONOGRAPH COMMITTEE is pleased to announce that the first edition of Number Four is well advanced in sales and that each of the others has gone into a second edition; also that a German Edition of Number One is being brought out by the firm of Teubner in Leipzig and Berlin. The titles of the monographs are: (1) "Calculus of Variations" by Professor GILBERT A. BLISS; (2) "Analytic Functions of a Complex Variable" by Professor DAVID R. CURTISS; (3) "Mathematics of Statistics" by Professor HENRY L. RIETZ; "Projective Geometry," by Professor JOHN W. YOUNG.

The price of these Monographs is $1.25 per copy to institutional and individual members of the Association when ordered directly through the Secretary, one copy to each member; this is the bare cost of production. The price to all non-members of the Association and for all quantity orders for class use is $2.00 per copy, obtained only through the Open Court Publishing Company, 337 East Chicago Avenue, Chicago, Illinois, distributors to the general public of Association publications.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Seventeenth Annual Meeting of the Association, Atlantic City, N.J., Dec. 27-30, 1932.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1932 and reported to the Secretary.

ILLINOIS, Urbana, May 6-7.
INDIANA, Indianapolis, May 6-7.
IOWA, Cedar Falls, April 29-30.
KANSAS, Topeka, Feb. 13.
KENTUCKY, Lexington, May.
LOUISIANA-MISSISSIPPI, Oxford, Miss., March 11-12.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Md., Dec. 3.
MICHIGAN, Ann Arbor, March 19.
MINNESOTA, River Falls, Wis., May 7.
MISSOURI.
NEBRASKA, Omaha, May 6-7.
OHIO, Columbus, Ohio, April 7.
PHILADELPHIA, Philadelphia, Pa., Nov. 26.
ROCKY MOUNTAIN, Laramie, Wyo., April 15-16.
SOUTHEASTERN, Gainesville, Fla., Mar. 18-19.
SOUTHERN CALIFORNIA, San Diego, March 26.
TEXAS, Austin, Jan. 30.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN